SOME CLASSIFICATION PROBLEM ON WEIL BUNDLES ASSOCIATED TO MONOMIAL WEIL ALGEBRAS

JIŘÍ TOMÁŠ

Abstract. A natural $T$-function on a natural bundle $F$ is a natural operator transforming vector fields on a manifold $M$ into functions on $FM$. For a monomial Weil algebra $A$ satisfying $\dim M \geq \text{width}(A) + 1$ we determine all natural $T$-functions on $T^*TA_M$, the cotangent bundle to a Weil bundle $T^A M$.

1. The aim of this paper is the classification of all natural $T$-functions defined on the cotangent bundle to a Weil bundle $T^*TA$ corresponding to a monomial Weil algebra $A$. Roughly speaking, the concept of a monomial Weil algebra denotes an algebra of jets factorized by an ideal generated only by monomial elements. Weil algebras of this kind form a significant class of themselves, since they cover algebras of holonomic and non-holonomic velocities as well as quasivelocities, [11]. The starting point is a general result by Kolář, [4],[5], determining all natural operators $T \rightarrow TT^A$ transforming vector fields on manifolds to vector fields on a Weil bundle $T^A$. Further, partial results of our general problem are solved in [3] and [9]. We follow the basic terminology from [5].

We start from the concept of a natural $T$-function. For a natural bundle $F$, a natural $T$-function $f$ is a natural operator $f_M$ transforming vector fields on a manifold $M$ to functions on $FM$. The naturality condition reads as follows. For a local diffeomorphism $\varphi : M \rightarrow N$ between manifolds $M$, $N$ and for vector fields $X$ on $M$ and $Y$ on $N$ satisfying $T\varphi \circ X = Y \circ \varphi$ it holds $f_N(Y) \circ F\varphi = f_M(X)$. An absolute natural operator of this kind, i.e. independent of the vector field is called a natural function on $F$.

There is a related problem of the classification of all natural operators lifting vector fields on $m$-dimensional manifolds to $T^*TA$. The solution of the second problem is given by the solution of the first one as follows ([10]). Natural operators $A_M : TM \rightarrow TT^A M$ are in the canonical bijection with natural $T$-functions $g_M : T^*T^A M \rightarrow \mathbb{R}$ linear on fibers of $T^*(T^*TA M) \rightarrow T^*TA M$. Using natural equivalences $s : TT^* \rightarrow T^*T$ by Modugno-Stefani, [7] and $t : TT^* \rightarrow T^*T$ by Kolář-Radziszewski, [6], we obtain the identification of $g_M$ with natural $T$-functions $f_M : T^*TT^A M \rightarrow \mathbb{R}$ given by $f_M = g_M \circ t_{T^A M} \circ T^{-1}$. Thus we investigate natural

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$T$-functions defined on $T^*T^A \otimes ^A M$ to determine all natural operators $T \to TT^*T^A$, where $\mathbb{D}$ denotes the algebra of dual numbers.

We remind the general result by Kolár, [4], [5]. For a Weil algebra $A$, the Lie group $Aut A$ of all algebra automorphisms of $A$ has a Lie algebra $Aut A$ identified with $Der A$, the algebra of derivations of $A$. Thus every $D \in Der A$ determines a one parameter subgroup $d(t)$ and a vector field $D_M$ on $T^A M$ tangent to $(d(t))_M$. Hence we have an absolute natural operator $\lambda_D : TM \to TT^A M$ defined by $\lambda_D X = D_M$ for any vector field $X$ on $M$. For a natural bundle $F$, let $\mathcal{F}$ denote the corresponding flow operator, [5]. Further, let $L_M : A \times TT^A M \rightarrow TT^A M$ denote the natural affinor by Koszul, [4], [5]. Then the result by Kolár reads

**All natural operators** $T \to TT^A$ **are of the form** $L(c)T^A + \lambda_D$ **for some** $c \in A$ **and** $D \in Der A$.

Let $\xi : M \rightarrow TM$ be a vector field. Kolár in [3] defined an operation $\sim$ transforming a vector field on a manifold $M$ onto a function on $T^*M$ by $\tilde{\xi}(\omega) = \langle \xi(p(\omega)), \omega \rangle$, where $p$ is the cotangent bundle projection and $\omega \in T^*M$. One can immediately verify, that for a natural bundle $F$ and a natural operator $A_M : TM \rightarrow TM$ we have a natural $T$-function $A_M : T^*FM \rightarrow \mathbb{R}$ defined by $A_M(X) = A_M X$ for any vector field $X : M \rightarrow TM$.

2. In this section, we find all natural $T$-functions $f_M : T^*T^A M \rightarrow \mathbb{R}$ for any manifold $M$ for $m = \dim M \geq \text{width}(A) + 1$. For some cases of $A$, [11], all natural $T$-functions in question are of the form

$$h(L(c)T^A, \lambda_D)) \quad c \in C, \ D \in D$$

where $C$ is a basis of $A$, $D$ is a basis of Der $A$ and $h$ is any smooth function $\mathbb{R}^{\dim A + \dim Der A} \rightarrow \mathbb{R}$. Let $\mathbb{D}^A_k$ denote the algebra of jets $J^A_0(\mathbb{R}^k, \mathbb{R})$. It can be also considered as the algebra of polynomials of variables $\tau_1, \ldots, \tau_k$. By [6], any Weil algebra $A$ is obtained as the factor of $\mathbb{D}^A_k$ by an ideal $I$ of itself, i.e. $A = \mathbb{D}^A_k / I$.

The contravariant approach to the definition of a Weil bundle by Morimoto sets $M_A = \text{Hom}(C^\infty(M, \mathbb{R}), A)$ and was studied by many authors as Muriel, Munoz, Rodriguez, Alonso, ([1] [8]). The covariant approach (Kolár, [3], [5]) defines $T^A M$ as the space of $A$-velocities. Let $\varphi, \psi : \mathbb{R}^k \rightarrow M, \varphi(0) = \psi(0)$. Then $\varphi$ and $\psi$ are said to be $I$-equivalent iff for any $\text{germ}_x f, f : M \rightarrow \mathbb{R}$ it holds $\text{germ}(f \circ \varphi - f \circ \psi) \in I$. Classes of such an equivalence $j^A \varphi$ are said to be $A$-velocities. For a smooth map $g : M \rightarrow N$ define $T^A g(j^A \varphi) = j^A(g \circ \varphi)$. Since $T^A$ preserves products, we have $T^A \mathbb{R} = A, T^A \mathbb{R}^m = A^m$. The identification $F : M_A \rightarrow T^A M$ between those two approaches to the definition of Weil bundle is given by

$$F(j^A \varphi)(f) = j^A(f \circ \varphi) \quad \text{for any } f \in C^\infty(M, \mathbb{R})$$
We are going to construct natural T-functions defined on $T^*T^A$ from natural operators $T \rightarrow TT^A$, since there are some additional ones on $T^*T^A$, which cannot be constructed from natural operators $T \rightarrow TT^A$.

Let $p : \mathbb{D}^*_k \rightarrow A$ be the projection homomorphism of Weil algebra inducing the natural transformation $\bar{p}_M : T^*_k M \rightarrow T^A M$. There is a linear map $\iota : A \rightarrow \mathbb{D}^*_k$ such that $p \circ \iota = \text{id}_A$. By $\iota$ we construct an embedding $T^A M \rightarrow T^*_k M$. Consider any $j^A \varphi \in T^A M$ as an element of $\text{Hom}(C^\infty(M, \mathbb{R}), A)$. Then domains of $j^A \varphi \in T^A M$ can be replaced by $J^r_x(M, \mathbb{R})$. Indeed, for any $f \in C^\infty(M, \mathbb{R})$ it holds $j^A \varphi(f) = j^A((f \circ \varphi) = [\text{germ}_{x_0} f \circ \text{germ}_{x_0} \varphi]$, where $x_0 = \varphi(1), 0 \in \mathbb{R}^k$. Since any ideal $I$ in the algebra $E(k)$ of finite codimension contains the $r$-th power of the maximal ideal of $E(k)$, the last expression can be replaced by $[j^r_0((f \circ \varphi)|_J] = j^A((j^r_0 f)(x_0)), f(J)$ is an ideal of $\mathbb{D}^*_k$ corresponding to $I$.

Further, any element $j^r_0 f \in J^r_x(M, \mathbb{R})$ can be decomposed onto $f(x_0) + j^r_0 ((f(x_0)) \circ f) = f(x_0) + j^r_0 f$, where $t_y : \mathbb{R} \rightarrow \mathbb{R}$ denotes in general a translation mapping 0 onto $y$. The second expression is an element of the bundle of covelocities of type $(1, r)$, namely an element of $(T^r)^*_x M = (T^r)^*_1 x_0 M$, the bundle of covelocities of type $(k, r)$ being defined as $T^r_k M = J^r(M, \mathbb{R}^k)_0$, $[5]$.

Select any minimal set of generators $\mathcal{B}_{x_0}$ of the algebra $T^r_x M$. For any $j^r_0 f \in \mathcal{B}_{x_0}$ define $\tilde{t}_{x_0} : T^A M \rightarrow (T^r)_x^* M$ by $\tilde{t}_{x_0}(j^A \varphi) = \tilde{t}((j^A \varphi)(j^r_0 f))$. In the second step, $\tilde{t}$ can be extended onto the homomorphism $J^r_x(M, \mathbb{R}) \rightarrow \mathbb{D}^*_k$.

We extend the map $\tilde{t}_{x_0}$ to $\iota : T^A M \rightarrow T^*_k M$. For a general Weil algebra $B$ we show that any element $j^B \varphi \in T^B_0 M$ corresponds bijectively to some element $j^B \varphi_0 \in T^B_0 M$. Indeed, $j^B \varphi(j^r_0 f) = j^B((f \circ \varphi) = j^B((f \circ t^x_0 \circ t^{x}_0 \circ \varphi_0 |_J = j^B \varphi_0(j^r_0 f)$. This general property extends $\tilde{t}_{x_0}$ onto $\iota : T^A M \rightarrow T^*_k M$. We proved the following assertion

**Proposition 1.** Let $A = \mathbb{D}^*_k / I$ be a Weil algebra, $p : \mathbb{D}^*_k \rightarrow A$ the projection homomorphism with its associated natural transformation $\bar{p} : T^*_k \rightarrow T^A$ and $\iota : A \rightarrow \mathbb{D}^*_k$ a linear map satisfying $p \circ \iota = \text{id}_A$. For a manifold $M$ and $x_0 \in M$ let $\mathcal{B}_{x_0}$ be a minimal set of generators of the algebra $J^r_x(M, \mathbb{R}) = T^r_k M$. Then there is an embedding $\bar{\iota} : T^A M \rightarrow J^r_x(M, \mathbb{R})$ such that $\bar{\iota}(j^A \varphi) = ((j^A \varphi)(j^r_0 f))$ for any $j^A \varphi \in T^A_x M$ and $j^r_0 f \in \mathcal{B}_{x_0}$.

In the following investigations, we limit ourselves to monomial Weil algebras. A Weil algebra $A = \mathbb{D}^*_k / I$ is said to be monomial if $I$ is generated only by monomials. We shall need the coordinate expression of some operators used later for the construction of natural T-functions in question. Thus we introduce coordinates on $T^A M$ and $T^*T^A M$. Consider the polynomial approach to the definition of $\mathbb{D}^*_k$. Then its elements are of the form $\frac{1}{\alpha!} x_\alpha \tau^\alpha$, where $\tau_1, \ldots, \tau_k$ are variables and $\alpha$ are multindices satisfying $0 \leq |\alpha| \leq r$. Define a linear map $\iota : A \rightarrow \mathbb{D}^*_k$ as follows. For $\tau^\alpha$, put $\iota(\tau^\alpha) = 0$ if $\tau^\alpha \in I$ and $\iota(\tau^\alpha)) = \tau^\alpha$ otherwise. As a matter of fact, $\iota : A \rightarrow \mathbb{D}^*_k$ is a zero section. Similarly as $p : \mathbb{D}^*_k \rightarrow A$, the map $\iota$ can be extended to $\bar{\iota} : A^m \rightarrow (\mathbb{D}^*_k)^m$ by components. Then it coincides with the map $\bar{\iota}$.
from Proposition 1, if we put $M = \mathbb{R}^m$, choose $x_0 \in \mathbb{R}^m$ and substitute $j_{x_0}^i x^i$ for the elements of $B_{x_0}$, where $x^i$ are canonical coordinates on $\mathbb{R}^m$. Further, define the additional coordinates on $T^*T^A M$ by $p^0_i dx^i$.

Let us define operators $T \to TT^A$ by means of $\tilde{i}$ and natural operators $T \to TT^A_k \tau$ as follows. Every natural operator $\lambda : T \to TT^A_k$ defines an operator

\[(2) \quad \Lambda : T \to TT^A \quad \text{by} \quad \Lambda = \tilde{T} \circ \lambda \circ \tilde{i}\]

which does not to have be natural and neither does the functions $\tilde{\Lambda} : T^*T^A \to \mathbb{R}$. Consider a basis of natural operators $T \to TT^A_k \tau$. The non-absolute natural operators $\lambda$ together with some of the absolute ones in this basis induce natural operators $\Lambda : T \to TT^A$, while the others will be used for the construction of the additional natural functions defined on $T^*T^A$.

By general theory, [5], searching for natural $T$-functions defined on $T^*T^A$, we are going to investigate $G\tau_{r+2}$-invariant functions defined on $(J^{r+1} T)_0 \mathbb{R}^m \times (T^*T^A)_0 \mathbb{R}^m$. Therefore we state some assertions, concerning the action of $G\tau_{r+2}$ and some of its subgroups on this space. It will be necessary to consider the coordinate expression of this action as well as that of base operators $\Lambda : T \to TT^A$ and their associated functions $\tilde{\Lambda} : T^*T^A \to \mathbb{R}$.

Denote by $\lambda^j_\beta$ a natural operator $\lambda_\alpha^\beta_{j_\beta}$ associated to a derivation of $\mathbb{D}A_j$ defined by $\tau_i \to \delta^i_j \tau^\beta$ for $j \in \{1, \ldots, k\}$ and $1 \leq |\beta| \leq r$. Then we have coordinate forms of $\lambda^j_\beta$ and $\tilde{\lambda}^j_\beta$, of the same form as those of $\Lambda^j_\beta$ and $\tilde{\Lambda}^j_\beta$. We have

\[(3) \quad \lambda^j_\beta = \frac{(\alpha + \beta)!}{\alpha!} x^\alpha p^\alpha \frac{\partial}{\partial x^{|\beta| - j}}, \quad \tilde{\lambda}^j_\beta = \frac{(\alpha + \beta)!}{\alpha!} x^\alpha p^\alpha \gamma^{|\beta| - j}\]

Let $k$ be the width of a monomial Weil algebra $A$. For $m \geq k$, define an immersion element $i \in T^A_0 \mathbb{R}^m$ by $x^i_\alpha = 0$ whenever $|\alpha| \geq 2$ and $x^i_\alpha = \delta^i_j$ for $j \in \{1, \ldots, k\}$. For general $r, k$, remind the jet group $G^r_k = \text{inv} J^r_0(\mathbb{R}^k, \mathbb{R}^k)_0$, where $\text{inv}$ indicates the invertibility of maps in question. The multiplication in $G^r_k$ is defined by the jet composition. We give the coordinate form of the action of this group on $T^*T^A$. Let $a^i_{1, \ldots, l_q}$ denote the canonical coordinates on $G^r_k$ and $\tilde{a}^i_{1, \ldots, l_q}$ indicate the inverse. Then the transformation law of the action of $G^r_k$ on $T^*T^A_0 \mathbb{R}^m$ is of the form

\[(4) \quad x^i_\alpha = a^i_{1, \ldots, l_q} x^i_{1, \ldots, l_q} \ldots x^i_{1, \ldots, l_q}\]

for all admissible multiindices $\alpha$ and their decompositions $\alpha_1, \ldots, \alpha_q$.

The jet group $G^r_k$ is identified with $\text{Aut} \mathbb{D}^r_k$, the group of automorphisms of the algebra $\mathbb{D}^r_k$, as follows. For $j^q_0 g \in G^r_k$ and $j^q_0 \varphi \in \mathbb{D}^r_k$ define

\[(5) \quad j^q_0 g(j^q_0 \varphi) = j^q_0 \varphi \circ (j^q_0 g)^{-1}\]

Let $A$ be a monomial Weil algebra of width $k$ and height $r$ and $p : \mathbb{D}^r_k \to A$ be the projection homomorphism.

In what follows, we shall consider $A$ as $\mathbb{D}^s_m / (I \cup \{\tau_{k+1}, \ldots, \tau_m\})$ for $s \geq r$, $m \geq k$ with the properly modified projection $p : \mathbb{D}^s_m \to A$. Consider a group
Let \( \text{Lemma 2.} \) \n \( G_A = \{ j_0^* \in G_m^r; p \circ j_0^* = p \} \). [1] The following lemma characterizes \( G_A \) as the stability subgroup of the immersion element \( \tilde{i} \).

**Lemma 3.** Let \( A = \mathbb{D}_m^s/I \) be a monomial Weil algebra of width \( k \), height \( r \) and \( \text{St}(i) \subseteq G_m^r \) be the stability subgroup of the immersion element \( i \in T_0^A \mathbb{R}^m \) under the canonical left action of \( G_m^r \) on \( T_0^A \mathbb{R}^m \). Then it holds \( G_A = \text{St}(i) \).

**Proof.** The formula (4) implies that every element of \( G_m^r \) stabilizes \( i \) if and only if \( a_j^i = 0 \) for \( j \in \{1, \ldots, k\} \) and \( a_j^i = 0 \) whenever \( |\alpha| \geq 2 \), \( \tau^\alpha \not\in I \) and \( \tau^\alpha < \tau_1, \ldots, \tau_k \).

On the other hand, \( G_A = \{ j_0^* \in G_m^r; p \circ j_0^* \varphi \circ (j_0^*)^{-1} = p \circ j_0^* \varphi \forall j_0^* \varphi \in \mathbb{D}_m^s \} \). In coordinates, we have

\[
\bar{x}_\alpha = x_{1\ldots s} \bar{a}_{\alpha_1}^1 \ldots \bar{a}_{\alpha_q}^q
\]

where \( \bar{x}_\alpha \) indicates the transformed value of \( j_0^* \varphi \) (in coordinates \( x_\alpha \)) under an automorphism \( j_0^* \) (with coordinates \( a_\alpha^j \)). Substituting an \( i \)-th projection \( pr_i \) for \( \varphi \), we obtain \( \bar{x}_\alpha = \bar{a}_\alpha^i \) and consequently \( \bar{a}_j^i = a_j^i = \delta_i^j \) for \( j \in \{1, \ldots, k\} \) and \( \bar{a}_\alpha^i = a_\alpha^i = 0 \) for \( |\alpha| \geq 2 \), \( \tau^\alpha \not\in I \) and \( \tau^\alpha < \tau_1, \ldots, \tau_k \). Thus we have \( G_A \subseteq \text{St}(i) \).

The converse inclusion is immediately obtained from (6), taking into account the coordinate form of \( i \). It proves our claim.

We remind the concept of a regular \( A \)-point of a Weil bundle \( M_A \). An element \( \varphi \in M_A \) is said to be regular (a regular \( A \)-point) if and only if its image coincides with \( A \), [1]. Taking into account the identification (1), such a concept can be extended to an \( A \)-velocity \( j^A \varphi \in T^A M \). Clearly, it is regular if and only if \( \varphi \) is an immersion in \( 0 \in \mathbb{R}^k \), where \( k \) is the width of \( A \). Further, it must hold \( \dim M \geq k \).

In the case \( m = k \) the concept of regularity coincides with that of invertibility. The map \( \tilde{i} \) from Proposition 1 preserves regularity and thus \( \tilde{i} : A^k \rightarrow \mathbb{R}^k \) can be restricted to \( \text{reg}(N^k) \rightarrow G_k^r \), where \( N \) denotes the nilpotent ideal of \( A \).

Alonso in [1] proved that there is a structure of a fiber bundle on \( \text{reg} T^A M \) with the standard fiber \( G_k^r/G_A \) over a \( k \)-dimensional manifold \( M \) and therefore \( \text{reg} T^A \mathbb{R}^k \) is identified with \( G_k^r/G_A \). The elements of \( \text{reg}(T^A)_{\mathbb{R}^k} \) are left classes \( j_0^* \mathbb{R}^k \). We extend this assertion of his to \( m \)-dimensional manifolds for \( m \geq k \).

For \( \tilde{i} : A^m \rightarrow (\mathbb{D}_k^s)^m \) corresponding to a Weil algebra of width \( k \) we define a map \( \tilde{i}^* : A^m \rightarrow (\mathbb{D}_k^s)^m \) by

\[
\tilde{i}^*(x_\alpha^r, \tau^\alpha) = x_\alpha^r, \tau^\alpha + \delta_i^p \tau_p \quad p \geq k + 1
\]

Then we have a lemma, giving the decomposition of any \( j_0^* \) onto its projection from \( \tilde{i}^* \circ \tilde{p}(G_m^r) \) and the component in \( G_A \).

**Lemma 3.** Let \( A = \mathbb{D}_k^s/I \) be a Weil algebra of width \( k \) and \( j_0^* \) \( G_m^r \) such that

\[
(8) \quad j_0^* \circ \tilde{p}(G_m^r) \circ j_0^* \tilde{h}
\]
Proof. The proof of the assertion is done in coordinates and it is based on the iterated application of (4). We do it for only for $k$, since for $m \geq k$ it is almost the same. Let $c'_0$ denote the coordinates of $\tilde{j}_0^0g$, $a'_0$ the coordinates of $\tilde{i} \circ \tilde{p}(\tilde{j}_0^0g)$ and $b'_0$ the coordinates of $\tilde{j}_0^0h$ to be found. Clearly, $a'_0 = c'_0$, whenever $\tau^\gamma \not\in I$. In the first step suppose that $\alpha$ is a minimal multiindex such that $\tau^\alpha \in I$. It follows from (4), that $a'_0 = a'_0 b'_0$, if we consider the conditions for $\tilde{j}_0^0h$. The unique solution is given by the invertibility of $\tilde{j}_0^0g$ Suppose the assertion being proved for $|\alpha| \leq p$.

We prove it for $|\alpha| = p + 1$. By (4) we have $c'_0 = a'_1 + b'_0 + b'_0 + 2$, $s \geq 2$. From the regularity of $\tilde{j}_0^0g$ we obtain again the unique solution $b'_0$, which proves our claim.

In the proof of the assertion giving the main result, we need to describe the stability group of $\tilde{j}_0^{r+1}(\frac{\partial}{\partial x^{m+1}})$. The transformation laws for the action of $G_{m+1}^r$ on $(J^{r+1}T)g_{\mathbb{R}^m}$ has the coordinate expression

$$\tilde{X}_i^l = a_{l1}^{\gamma_1} \cdot \tilde{X}_j^l \cdot \tilde{a}_i^\gamma,$$

where $X_{\alpha}^l$, $|\alpha| \leq r + 1$ denote the canonical coordinates of $\tilde{j}_0^{r+1}(\frac{\partial}{\partial x^{m+1}})$. Further, any multiindex $\gamma$ including the empty one is decomposed into $\gamma_1, \gamma_2$ and the notation $\tilde{a}_i^\gamma$ denotes the system of all $\tilde{a}_{i1}^{\gamma_1} \cdot \tilde{a}_{i2}^{\gamma_2}$ for $l_1, \ldots, l_4$ forming the multiindex $\gamma$ and decompositions $\alpha_1, \ldots, \alpha_s$ forming $\alpha$. It follows, that in coordinates any element of $G_{m+1}^r$ must satisfy $a_j^0 = \delta^0_{m+1}$ and $a^0_\alpha = 0$ whenever the multiindex $\alpha$ formed by all $1, \ldots, m + 1$ contains any $m + 1$ for $|\alpha| \geq 2$. To describe the stability group of $\tilde{j}_0^{r+1}(\frac{\partial}{\partial x^{m+1}})$ by terms of Lemma 2 and Lemma 3, denote $A_{\alpha}^\beta_{\gamma_1} \cdot \gamma_2$ the Weil algebra of $\mathbb{R}_{m+1}/I$ for $I = \tau_{m+1}^{r+1} > |\alpha| \geq 1$. Thus we have proved the following lemma.

**Lemma 4.** The stability group of $\tilde{j}_0^{r+1}(\frac{\partial}{\partial x^{m+1}})$ in $G_{m+1}^r$ is of the form $\tilde{i}(A_{\alpha}^\beta_{\gamma_1} \cdot \gamma_2 \cdot i)_{m+1}^{m+1}$ and the immersion element $i \in T_{\mathbb{R}_{m+1}}A_{\alpha}^\beta_{\gamma_1} \cdot \gamma_2$ is of the form $G_{m+1} = G_{\alpha} \cdot \tilde{i}(A_{\alpha}^\beta_{\gamma_1} \cdot \gamma_2 \cdot i)_{m+1}^{m+1}$.

Let us consider the base $B$ of all $T$-functions defined on $T^*TA$ (not natural in general), constructed from the non-absolute natural operators $L_r^\alpha$ and from the absolute operators $A_{\alpha}^\beta_{\gamma_1}$ with the coordinate expression given by (3). Let $B_{\gamma_1}$ denote the subbasis of $B$ formed by natural operators $T \rightarrow TT^*$. It follows from Lemma 3, that any element $f^\gamma g \in \text{reg} TA M$ is identified with $\tilde{i}(j^\gamma g) \in G_{m+1}^{r+1}$, the only representative of the left class $j^\gamma g G_{\alpha}$ in the sense of Lemma 3. Therefore we have

$$i = (\tilde{i}(j^\gamma g)^{-1}, j^\gamma g)$$

where $l$ is the symbol for the left action of $G_{m+1}^{r+1}$ on $T_{\mathbb{R}_{m+1}}A_{\alpha}^{\beta}$ to be used also for the action of this group on $(J^{r+1}T)_{\mathbb{R}_{m+1}} \times (T^*TA)_{\mathbb{R}_{m+1}}$. Let us define a map

$$\text{Imm} : T^* \text{reg} TA_{\mathbb{R}_{m+1}} \rightarrow (T^*TA)_{\mathbb{R}_{m+1}}$$

as follows

$$\text{Imm}(w) = (\tilde{i}(q(w)))^{-1}, w),$$
Proposition 5. Let $A$ be a monomial Weil algebra and $(T^*(\text{reg } T^A))_0 \mathbb{R}^{m+1}$ be the restriction of the natural bundle $T^* T^A \mathbb{R}^{m+1} \rightarrow T^A \mathbb{R}^{m+1}$ to the opened submanifold $(T^A)_0 \mathbb{R}^{m+1}$. Then all operators from $\mathcal{B} - \mathcal{B}_0$ are $G_{m+2}$-invariant in respect to the map $\text{Imm}$.

Proof. We prove the assertion from the transformation laws of the action of $G_{m+2}$ on $(J^r+1)T_0 \mathbb{R}^{m+1} \times (T^*T^A)_0 \mathbb{R}^{m+1}$. We complete them for $p^\gamma_i$. Denote $\gamma = \alpha - \{j\}$ the multiindex from (3). Then we have

$$p_j^\gamma = \frac{(\beta + \gamma)!}{\beta! \gamma_1! \ldots \gamma_\gamma!} x^i_{j_{1-\gamma_1}} \ldots x^i_{j_{1-\gamma_\gamma}} p^{\beta_1}_{j_{1-\gamma_1}} \ldots p^{\beta_\gamma}_{j_{1-\gamma_\gamma}}$$

when the sum is made for all decompositions $\gamma_1, \ldots, \gamma_\gamma$ of multiindices $\gamma$. The formula is obtained from (3) and the standard combinatorics. To accent $\text{Imm}(w)$ as a transformed value for any $w \in T^*(\text{reg } T^A)_0 \mathbb{R}^{m+1}$, use $p^\gamma_i$ for the additional coordinates (obviously, the coordinates $\bar{x}_\alpha$ coincide with those of $i$). Then we have $\tilde{\Lambda}^\beta(j) = \tilde{\Lambda}^\beta_i(x^i_\alpha, \bar{x}_\alpha) = \beta p^\beta_i = \beta \{ \frac{(\beta+\gamma)!}{\beta! \gamma_1! \ldots \gamma_\gamma!} \} p^{\beta_1}_{j_{1-\gamma_1}} \ldots p^{\beta_\gamma}_{j_{1-\gamma_\gamma}}$ if we put $\gamma = \alpha - \{j\}$, which follows from (12). If we consider the coordinate expression of $i(A^{m+1})$ and the formula (3), we obtain that the last expression coincides with $\frac{(\beta+\gamma)!}{\gamma!} x^i_{j_{1-\gamma_1}} \ldots x^i_{j_{1-\gamma_\gamma}} = \bar{x}_\alpha \frac{(\alpha+\beta)!}{\alpha_1! \ldots \alpha_\gamma!} x^i_{j_{1-\gamma_1}} \ldots x^i_{j_{1-\gamma_\gamma}} = \tilde{\Lambda}^\beta_i(x^i_\alpha, p^\gamma_i) = \tilde{\Lambda}(\bar{w})$. It proves our claim.

The following lemma specifies a certain class of functions, among which all investigated ones must be contained.

Lemma 6. Let $m \geq k$. Then every natural $T$-function $f : T^* T^A \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ is of the form $h(L(\tau^\alpha) T^A, \tilde{\Lambda}^\beta)$ for some smooth function $h$ of the suitable type.

Proof. By general theory, we are searching for all $G_{m+2}$-invariant functions defined on $(J^r+1)T_0 \mathbb{R}^{m+1} \times (T^*T^A)_0 \mathbb{R}^{m+1}$. Let $w \in (T^*T^A)_0 \mathbb{R}^{m+1}$ and $x^i_\alpha$ denote the coordinates of $q(w)$, $q : T^* T^A \rightarrow T^A$ being the cotangent bundle projection. By a general lemma from [5], Chapter VI, the natural $T$-function must satisfy $f(J^r+1 X, w) = h(X^i_\alpha, p^\beta_i)$ for any non-zero $J^r+1 X$ of a vector field $X$ on $\mathbb{R}^{m+1}$. The coordinates used in the recent identity coincide with those defined before Lemma 2. The last expression can be considered in the form $h(L(\tau^\alpha) T^A, X^i_\alpha p^\beta_i, \tilde{\Lambda}^\beta, x^i_{j_{1-\gamma_1}} p^{\gamma_i}_{j_{1-\gamma_1}})$ for $|\beta| \geq 0$, $|\gamma| \geq 1$ and $|\delta| \geq 2$. Identify $q(w)$ with $j^A g$ for any $w \in T^*(\text{reg } T^A)_0 \mathbb{R}^{m+1}$, i.e. $q(w) = l(i(j^A g), i)$ and put $j^r+1 Y = l(i(j^A g), j^{r+1} Y)$. Then $f(J^r+1 X, w) = h(L(\tau^\alpha) T^A, X^i_\alpha p^\beta_i, \tilde{\Lambda}^\beta, 0, p^\gamma_i)$ for $|\gamma| \geq 1$ and $i \in \{1, \ldots, k\}$. Here $p^\beta_i$ indicate the transformed values of $p^\beta_i$ under the map $\text{Imm}$. The last identity follows from Proposition 5. Further, there is $J^r+1 g \in G_A \cap G_{A^{m+2}}$ such that $l(j^r+1 g, j^{r+1} \{ \frac{\partial}{\partial x^i_{j_{1-\gamma_1}}(x^i_{j_{1-\gamma_1}}) = j^{r+1} Y \}$. Then we have $f(J^r+1 X, w) =
$h(L(\tau^\alpha) T^A, 0, \tilde{X}_j^\beta, 0, p_i^\delta)$ for $i \in \{1, \ldots, k\}$. The excessive coordinates $p_i^\delta$ are annihilated by an element of $\text{Ker} \pi_r^+ \cap \hat{i}((A_m^{+1})^{m+1})$, namely by an element satisfying in coordinates $a_i^\alpha = 0$ except of $\alpha = \{i, \ldots, i\}$. Such an element stabilizes $j_0^r+1(\frac{\partial}{\partial x_\alpha})_m$ as well as $\bar{i}$, which completes the proof. 

Searching for all natural $T$-functions $T^* T^A_{m+1} \rightarrow \mathbb{R}$ among those from Lemma 6, we state the basis $\mathcal{B}$ of functions defined on $T^*_0 T^A_{m+1}$ and identify it with $\bar{\mathcal{B}}$. By general theory, [5], every natural $T$-function in question is determined by its value over $j_0^r+1(\frac{\partial}{\partial x_\alpha})_m$ on $(T^* T^A)_0 R^{m+1}$. Further, it follows from Lemma 4 and the formula (11) that the map $\text{Imm}$ stabilizes $j_0^r+1(\frac{\partial}{\partial x_\alpha})_m$ in the following sense. For any $w \in T^*_0 (\text{reg} T^A)_0 R^{m+1}$ the action of $\hat{i}(q(w))$ on $(J^r+1 T)_0 R^{m+1}$ stabilizes $j_0^r+1(\frac{\partial}{\partial x_\alpha})_m$.

Set $\mathcal{B}$ the basis of functions defined on $T^*_0 T^A_{m+1}$ obtained by the restriction of $\bar{\mathcal{B}}$ over $j_0^r+1(\frac{\partial}{\partial x_\alpha})_m$ onto $T^*_0 T^A_{m+1}$. Conversely, $\mathcal{B}$ determines $\bar{\mathcal{B}}$ by

$$\bar{\mathcal{B}}(j_0^r+1(\frac{\partial}{\partial x_\alpha})_m, w) = \mathcal{B} \circ \text{Imm}(w)$$

Analogously, we construct $\mathcal{B}_1$ from $\bar{\mathcal{B}}_1$. Moreover, for any $w \in T^*_0 (\text{reg} T^A)_0 R^{m+1}$, the values formed by $\mathcal{B}(w)$ coincide with the coordinates $p_j^\beta$ of $w$ defined before (2) for $j = 1, \ldots, k$ except that of $p_0^\beta$ for the absolute functions and $p_{m+1}^\beta$ for the non-absolute ones. Thus any base $T$-function of $\mathcal{B}$ defined on $T^*_0 (\text{reg} T^A)_0 R^{m+1}$ corresponds to some projection $p_j^\beta : T^*_0 (\text{reg} T^A)_0 R^{m+1} \rightarrow \mathbb{R}$. It follows from Lemma 4 and the fact that $L(\tau^\alpha) T^A$ are natural that all natural $T$-functions $(T^* T^A)_{m+1} \rightarrow \mathbb{R}$ from Lemma 6 are in the canonical bijection with $G_A$-invariant functions defined on $T^*_0 T^A_{m+1}$ which are of the form $h(L(\tau^\alpha) T^A)(\tilde{X}_j^\beta)$ for $\tilde{X}_j^\beta : T^*_0 T^A_{m+1} \rightarrow \mathbb{R}$. Using coordinates, we find all $G_A$-invariants of $p_j^\beta$, $j \in \{1, \ldots, k\}$, $|\beta| \geq 1$. Then we identify the functions $h(L(\tau^\alpha) T^A)(p_j^\beta)$ with $h(L(\tau^\alpha) T^A)(\tilde{X}_j^\beta)$ and by (12) we obtain all natural $T$-functions on $T^* T^A_{m+1}$.

This way we have deduced that our problem can be reduced to the problem of searching for all $G_A$-invariant functions defined on $T^*_0 T^A_{m+1}$ which can be identified with a smooth function $h : \mathbb{R}^N \rightarrow \mathbb{R}$ for a suitable integer $N$. The coordinate expression of the action of $G_A$ on $T^*_0 T^A_{m+1}$ is induced by (12) and it is of the form

$$p_j^\beta = p_j^\beta - C(\beta + \gamma, \beta) a_j^\gamma p_i^\beta \gamma \quad \text{for} \quad \tau_j \tau^\gamma \in I \quad \text{and} \quad \tau^\beta \tau^\gamma \notin I$$

where $C$ indicates the multicombinatorial number. Clearly, $T^*_0 T^A_{m+1}$ is identified with the space $R^N$ endowed with the action (14) of $G_A$. We are going to investigate $G_A \cap G_{0, m+1}$-orbits on $R^N$, since only $p_j^\beta$ depend on $B_{m+1}$ and they can be annihilated by this subgroup. For those orbits, we construct all functions
distinguishing them and then we express the corresponding invariants by terms of elements from \( \bar{B} \).

The following assertion describes an important property of \((G_A \cap \text{Ker } \pi^*_w)\)-orbits to be necessary in the proof of the main result. Denote by \( B_s \subseteq \bar{B} \) the set of all \((G_A \cap \text{Ker } \pi^*_w)\)-invariants selected from \( \bar{B} \) and denote by \( N_s \) the number of elements in \( B_s \). Clearly, \( B_1 \subseteq B_2 \subseteq \cdots \subseteq B_{r-1} \subseteq B_r \). Further, denote \( B'_s = B_s - B_1 \) and \( N'_s = N_s - N_1 \). Then we have

**Proposition 7.** Let \( w \in \mathbb{R}^N \) and \( \text{Orb}_w(w) \) be its \((G_A \cap \text{Ker } \pi^*_w)\)-orbit. Then \( B^{s+1}_s(\text{Orb}_w(w)) \) has the structure of an affine subspace of \( \mathbb{R}^{N^*_s+1} \), the modelling vector space of which being determined by the formula (14) restricted to \( B^{s+1}_s \cap G^*_s \).

**Proof.** is done directly applying the formula (14). Let \( w_1 \) and \( w_2 \) be elements of \( B^{s+1}_s(\text{Orb}_w(w)) \). Then \( w_1 \) can be achieved from \( w \) by the action of an element of \( B^{s+1}_s \cap G^*_s \). The coordinate expression of such a transformation is given by

\[
p^{\beta}_j - p^{\beta}_j - C(\beta + \gamma, \beta) a^j_\gamma p^{\gamma}_l.
\]

Analogously for \( w_1 \) and \( w_2 \), we have \( p^{\beta}_j - p^{\beta}_j - C(\beta + \gamma, \beta) a^j_\gamma p^{\gamma}_l \). Then \( p^{\beta}_j = p^{\beta}_j - (a^j_\gamma + b^j_\gamma) p^{\gamma}_l \), which follows \( \bar{w} \bar{w}_2 = \bar{w}_1 + \bar{w}_1 \bar{w}_2 \). It proves our claim.

In what follows, we construct a basis \( \bar{D} \) of natural functions from \( \bar{B} \). The construction is given by a procedure, generating step by step a base of \( G^*_s \)-invariants determining the base of natural functions. We start the procedure selecting elements of \( B_1 \) and put \( \bar{D}_1 = \bar{B}_1 \). For any \( w \in T_r^* T^A \mathbb{R}^{m+1} \), consider its orbit \( \text{Orb}_w(w) = \text{Orb}_1(w) \).

In the second step, consider \( B^2_1(\text{Orb}_1(w)) \), which is by Proposition 7 a \( k \)-dimensional affine subspace of the affine space \( \mathbb{R}^{N^*_1} \) for some \( k \leq N^*_1 \). For almost every \( G^*_s \)-orbit in the sense of density, such an affine subspace contains a unique point \( \tilde{I}^{C_2} \) satisfying \( p^j_j(\tilde{I}^{C_2}) = 0 \) for \( j \in C_2 \). The remaining components of \( I^{C_2} \) determine \( G^*_s \)-invariants \( I_1^{C_2}, \ldots, I_2^{C_2} \) identified with natural functions \( \tilde{I}^{C_2}_1, \ldots, \tilde{I}^{C_2}_2 \).

In order to express them in formulas, we notice the following property of \( B^{s+1}_s(\text{Orb}_s(w)) \) for any \( s = 1, \ldots, r - 1 \). Proposition 7 and its proof imply that if an element of \( B^{s+1}_s(\text{Orb}_s(w)) \) is stabilized by \( j_0^{s+1} g \in B^s_{m+1} \) under the canonical left action then the whole \( B^{s+1}_s(\text{Orb}_s(w)) \) is stabilized. Denote \( \text{St}_{s,m+1}^{s+1} \subseteq G_A \cap B^{s+1}_m \) the stability group of \( B^{s+1}_s(\text{Orb}_s(w)) \). One can easily deduce that \( \text{St}_{s,m+1}^{s+1} \) satisfies the stability property of this kind for almost every \( w \in \mathbb{R}^N \). Clearly, \( \text{St}_{s,m+1}^{s+1} \) is a closed and normal subgroup of \( G_A \cap B^{s+1}_m \) and thus \( H^{s+1}_{s,m+1} = G_A \cap B^{s+1}_m / \text{St}_{s,m+1}^{s+1} \) is a Lie group. It follows the existence of a section \( \sigma_{s+1,m+1} : H^{s+1}_{s,m+1} \rightarrow G_A \cap B^{s+1}_m \).

Hence for any \( w \in \mathbb{R}^N \) we have a unique \( j_0^{s+1} h \in \sigma_{s,m+1}(H^2_{s,m+1}) \simeq H^2_{1,m+1} \) such that \( B^s_1(I(j_0^{s+1} h, w)) = I^{C_2}(w) \). Thus we have a map \( \alpha_{C_2} : \mathbb{R}^N \rightarrow H^2_{1,m+1} \). Therefore,
any \( w \in T^*_s T^A \mathbb{R}^{m+1} \) is transformed onto
\[
(15) \quad l(\alpha c_2(w), w) = l_{\alpha c_2}(w) = (I^{C_2}_{j_3}(w)), \quad s = 1, \ldots, N^2_1 - k_2, j_3 \notin C_s
\]

Applying the identification (13), we obtain \( \tilde{I}_1^{C_2}, \ldots, \tilde{I}_{N^2_1-k_2}^{C_2} \) and put \( \tilde{D}_2 = \tilde{D}_1 \cup \{ \tilde{I}_{C_2}, \ldots, \tilde{I}_{N^2_1-k_2}^{C_2} \} \).

In the \((s+1)\)-th step of the procedure we come out from the basis \( \tilde{D}_s \) of natural functions and an element \( w_s = l_{\alpha c_2} \circ \cdots \circ l_{\alpha c_2}(w) \in \text{Orb}_1(w) \) instead from the second step. By Proposition 7, \( B_{s}^{+1}(\text{Orb}_s(w_s)) \) is an affine subspace of dimension \( k_{s+1} \) of \( \mathbb{R}^{N^2_1} \) for some \( k_{s+1} \). Select \( C_{s+1} \subseteq \{ 1, \ldots, N^2_1 \} \). For almost every \( w_s \in T^*_s T^A \mathbb{R}^{m+1} \) there is a unique point \( I^{C_{s+1}}_{s+1}(w_s) = I^{C_{s+1}}_{s+1}(B_s^{+1}(\text{Orb}_s(w_s))) \) such that \( pr_j \circ I^{C_{s+1}}_{s+1} = 0 \) for \( j \in C_{s+1} \). The remaining components of \( I^{C_{s+1}}_{s+1} \) determine analogously to the second step of the procedure \( G_A \)-invariants and by (13) natural functions \( \tilde{I}_{l_{s+1}}^{C_1}, \ldots, \tilde{I}_{l_{s+1}}^{C_2} \) for \( l_{s+1} \notin C_{s+1} \). Analogously to the second step, for any \( w_s \) under discussion there is a unique element \( \hat{j}_{0}^{s+1} h \in \sigma_{s+1,m+1}(H_{s,m+1}^+) \) such that \( l(\hat{j}_{0}^{s+1} h, B^{+1}_{s}(w_s)) = I^{C_{s+1}}_{s+1}(w_s) \). Hence we have a map \( \alpha_{C_s} : \mathbb{R}^N \to \sigma_{s+1,m+1}(H_{s,m+1}^+) \) such that \( l(\alpha_{C_s}(w_s), w_s) = I^{C_{s+1}}_{s+1}(w_s) = l_{\alpha c_2} \circ \cdots \circ l_{\alpha c_2}(w) = \tilde{I}_{l_{s+1}}^{C_1}, \ldots, \tilde{I}_{l_{s+1}}^{C_2} \) taking into account the identification (13). Hence we obtained the basis \( \tilde{D}_{s+1} = \tilde{D}_s \cup \{ \tilde{I}_{l_{s+1}}^{C_1}, \ldots, \tilde{I}_{l_{s+1}}^{C_2}, l_{s+1} \notin C_{s+1} \} \). We proved the main result, given by the following Proposition

Proposition 8. Let \( A = \mathbb{D}_k^m / I \) be a monomial Weil algebra of width \( k \), \( \dim M = m \geq k + 1 \). Let \( \tilde{i} : T^A M \to T^A M \) be an embedding described in Proposition 1. Consider a basis \( C \) of \( A \) and a basis \( B_0 \) of \( \text{Der}(\mathbb{D}_k^m) \). Further, let \( \tilde{B} \) be a basis of functions defined on \( T^* T^A M \) constructed from operators \( \tilde{T} \circ \lambda_D \circ \tilde{i} \) by the operation \( \sim \) defined in the very end of Section 1, \( D \in B_0 \). Then all natural \( T \)-functions \( f_M : T^* T^A M \to \mathbb{R} \) are of the form
\[
h(L_M(c) T^A M, \tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_{l_s}, \ldots, \tilde{I}_{l_s})
\]
where \( h \) is any smooth function of a suitable type, \( \tilde{I}_1, \ldots, \tilde{I}_{l_s} \) are natural functions selected directly from \( \tilde{B} \) and \( \tilde{I}_{l_s}^{C_1}, \ldots, \tilde{I}_{l_s}^{C_2} \) \( (l_s \notin C_s) \) are obtained by the procedure.

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References


Department of Mathematics, Faculty of Civil Engineering, Technical University Brno, Žižkova 17 602 00 Brno, Czech Republic

E-mail address: Tomas.Jemail.fce.vutbr.cz