TORSE-FORMING VECTOR FIELDS IN T-SEMISYMMETRIC RIEMANNIAN SPACES

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Abstract. In this paper we consider torse-forming vector fields in $T$-semi-symmetric Riemannian spaces. We prove that if $T_i$- and $T_{ij}$-semisymmetric spaces admit a non-isotropic torse-forming vector field, then it is convergent; non-Einsteinian Ricci semisymmetric spaces with a harmonic Riemannian tensor do not admit non-recurrent torse-forming vector fields. Our paper generalizes earlier results by J. Kowolik and also results concerning almost Kenmotsu manifolds.

1. Introduction

This paper is concerned about certain questions of torse-forming vector fields in $T$-semisymmetric Riemannian spaces. The analysis is carried out in tensor form, locally in a class of sufficiently smooth real functions.

One of the most studied classes of special (pseudo-) Riemannian spaces $V_n$ are semisymmetric spaces, which were introduced by N.S. Sinyukov in 1954 (see [4], [13], [17]) and which generalize symmetric spaces. Semisymmetric spaces are investigated in detail by E. Boeckx, O. Kowalski and L. Vanhecke [4].

A generalization of semisymmetric spaces is Ricci semisymmetric spaces, and these are further generalized and $T$-semisymmetric spaces are introduced.

A Riemannian space $V_n$ is called $T$-semisymmetric ([12], [13]), if for a tensor $T$ the condition $R(X,Y) \circ T = 0$ holds for arbitrary vector fields $X,Y$, where $R(X,Y)$ denotes the corresponding curvature transformation and the symbol $\circ$ indicates the corresponding derivation on the algebra of all tensor fields. We can write this condition in the local transcription as

$$T_{\ldots,jk}=0$$

where \(\ldots\) denotes the covariant derivative with respect to a (possibly indefinite) metric tensor $g_{ij}$ of a Riemannian space $V_n$ and $[jk]$ denotes the alternation with respect to $j$ and $k$.

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Evidently, a $T$-semisymmetric space is semisymmetric, or Ricci semisymmetric, respectively, if $T$ is the Riemannian curvature tensor $R$, or Ricci tensor $\text{Ric}$, respectively (see [2], [3], [4], [12], [13], [17]).

The study of recurrent, convergent, concircular and torse-forming vector fields has a long history starting in 1925 by the works of H.W. Brinkmann [5], P.A. Shirokov [19] and K. Yano [22], [23]. In Riemannian spaces $V_n$ with the above vector fields there exists a metric of a special form; these spaces are now called (almost) warped products [6]. These vector fields have been used in many areas of differential geometry, for example in conformal mappings and transformations [5], [8], [22], geodesic, almost geodesic and holomorphically projective mappings and transformations (see [1], [10] – [15], [17], [18], [20], [21]), and others [1], [2], [3], [6], [7], [9], [11], [13], [16], [17], ...

In the papers [2], [3], [7], [9], [16] there were studied semisymmetric and Ricci semisymmetric spaces which contain concircular and torse-forming vector fields satisfying some other assumptions. Our work is devoted to a generalization and extension of these results.

Particularly, we extend the following

**Theorem** [J. Kowolik, Th. 1, [9]]. Let a Riemannian manifold $V_n$ ($n \geq 4$) be a Ricci-semisymmetric space whose Ricci tensor is a Codazzi tensor (i.e. $R_{ij,k} = R_{ik,j}$). If $V_n$ admits a torse-forming vector field $\xi$ then either $\xi$ is a concircular vector field or it reduces to a recurrent one.

Here, in Theorem 5, we generalize Kowolik’s result [Th. 1] for Ricci-semisymmetric spaces $V_n$ ($n > 2$) where the Ricci tensor need not be a Codazzi tensor.

Moreover, our Theorem 9 shows that under the assumptions of Kowolik’s theorem [Th. 1] we get the stronger assertion that a torse forming vector field is recurrent. This implies that the second theorem in Kowolik’s paper is contained in our Theorem 9.

2. **On the theory of torse-forming vector fields**

Now we will recall results concerning torse-forming vector fields and their special cases: recurrent, convergent and concircular vector fields, which have been obtained in [1], [5]–[9], [10]–[14], [16]–[23].

A vector field $\xi$ in a Riemannian space $V_n$ is called torse-forming if it satisfies $\nabla_X \xi = \varrho X + a(X)\xi$ where $X \in TM$, $a(X)$ is a linear form and $\varrho$ is a function. In the local transcription this reads

$$\xi^h = \varrho \delta^h + \xi^h a_i$$  \hspace{1cm} (2)

where $\xi^h$ and $a_i$ are the components of $\xi$ and $a$, and $\delta^h$ is the Kronecker symbol.

Throughout this paper we assume $\xi^h \neq 0$.

A torse-forming vector field $\xi$ is called

a) recurrent, if $\varrho = 0$,

b) concircular, if $a_i$ is a gradient covector (i.e. $a_i = a_j$),

c) convergent, if it is concircular, and $\varrho = \text{const} \cdot \exp(a)$. 

After a suitable normalization we can characterize concircular and convergent vector fields $\xi$ in the following form

\[ b) \quad \xi^h_i = \varrho \delta^h_i \text{ and } c) \quad \xi^h_i = \text{const} \delta^h_i \]

respectively.

A vector field $\xi$ is called isotropic if $g(\xi, \xi) = 0$, where $g$ is a metric on $V^n$.

**Lemma 1.** A non-recurrent torse-forming vector field is non-isotropic.

**Proof.** Let us suppose that $\varrho \neq 0$ and that $\xi$ is an isotropic torse-forming vector field, i.e.

\[ \xi^\alpha \xi^\beta g_{\alpha \beta} = 0, \]

where $g_{ij}$ are components of metric $g$. By covariant differentiation of the last equation we get

\[ \xi^\alpha \xi^\beta ,_i g_{\alpha \beta} = 0 \]

and using (2) we obtain

\[ \varrho \xi^\alpha g^\alpha_i = 0. \]

Therefore $\varrho = 0$, which contradicts the assumption.

A non-isotropic torse-forming vector field $\xi$ can be normalized so that

\[ \xi^\alpha \xi^\beta g_{\alpha \beta} = \pm 1 \text{ and we can write the equation (2) for a torse-forming vector field in the following form [17]}: \]

\[ \xi_{i,j} = \varrho \left( g_{ij} - \varrho \xi_i \xi_j \right), \]

where $\xi_i \equiv \xi^\alpha g_{\alpha i}$ is a locally gradient covector, i.e. $\xi_i = f'_i$ where $f$ is a function. Evidently, we have in this case:

a) if $\varrho = 0$, then $\xi$ is recurrent and convergent,

b) if $\varrho = \frac{e}{f + \text{const}}$, then $\xi$ is convergent,

c) if $\varrho$ is a function of $f$, i.e. $\varrho = g(f)$, then $\xi$ is concircular,

d) if $\varrho \neq g(f)$, then $\xi$ is neither concircular nor recurrent.

Because we are studying vector fields $\xi$ in Riemannian spaces, in what follows, we shall not distinguish contravariant ($\xi^h$) from covariant ($\xi_i \equiv \xi^\alpha g_{\alpha i}$) vectors.

From the above results it follows

**Lemma 2.** Any non-isotropic recurrent torse-forming vector field is convergent.

It is well known (see [17]) that, if a Riemannian space $V^n$ admits a non-isotropic torse-forming vector field $\xi$, then in $V^n$ there exists a coordinate system $x$, in which the metric takes the form

\[ ds^2 = e \left( dx^1 \right)^2 + F(x^1, x^2, \ldots, x^n) ds^2, \]

where $e = \pm 1$, $F(\neq 0)$ is a function, and $d\tilde{s}^2(x^2, \ldots, x^n)$ is the metric form of the associated Riemannian space $\tilde{V}^{n-1}$. In this coordinate system the vector $\xi$ has the following form:

\[ \xi^h = \delta^h_1. \]

Evidently, the following holds

a) if $F = \text{const}$, then $\xi$ is recurrent and convergent,

b) if $F = c x^{12}$, where $c$ is a constant, then $\xi$ is convergent,

c) if $F = F(x^1)$, then $\xi$ is concircular,

d) if $F \neq F(x^1)$, then $\xi$ is neither concircular nor recurrent.

In the following we shall study non-isotropic torse-forming vector fields, characterized by (4). The integrability condition arising from (4) can be written in the form

\[ \xi_\alpha R^\alpha_{ijkl} = g_{ij} \xi_k - g_{ik} \xi_j + \xi_i \alpha_{jk} \]

(5)
where $R_{ijk}^h$ is the Riemannian tensor of $V_n$, $a_{jk} \equiv -e\xi_l g_{lk}$ and

$$c_k \equiv g_{.k} + e g^2 \xi_k.$$  \hspace{1cm} (6)

**Lemma 3.** Let $\xi$ be a non-isotropic torse-forming vector field. If $c_i = 0$, then $\xi$ is convergent.

**Proof.** Let us suppose that $c_i = 0$. In view of (6), this assumption implies $\varrho = \varrho(f)$, which gives $\varrho' + e\varrho^2 = 0$. Therefore $\varrho = e f + \text{const}$ or $\varrho = 0$, which means that $\xi^h$ is convergent.

Taking the converse to Lemma 3 we get

**Lemma 4.** If a non-isotropic torse-forming vector field $\xi$ is not convergent, then $c_i \neq 0$.

### 3. Torse-forming vector fields in T-semisymmetric Riemannian spaces where T is a covector

In this section we shall be interested in $T$-semisymmetric Riemannian spaces where $T$ is a covector. In accordance with the general definition in section 1, by a $T$-semisymmetric space we understand a Riemannian space $V_n$ with a covector field $T_i$ satisfying

$$T_{i,[lm]} = 0.$$  \hspace{1cm} (7)

Using the Ricci identity we can write (7) in the form

$$R(X, Y) \circ T = 0 \text{ or } T_{\alpha} R_{\alpha}^{ijk} = 0.$$  \hspace{1cm} (8)

**Theorem 1.** Let $T (\neq 0)$ be a covector field. A non-isotropic torse-forming vector field $\xi$ in a $T$-semisymmetric space $V_n$ is convergent.

This theorem is an obvious consequence of the following more general assertion.

**Lemma 5.** Let $T (\neq 0)$ be a covector field. A non-isotropic torse-forming vector field $\xi$ in a space $V_n$ is convergent, if $R(X, \xi) \circ T = 0$ for any $X$.

**Proof.** Suppose there exists a non-isotropic torse-forming vector field $\xi$ in $V_n$ such that $R(X, \xi) \circ T = 0$ for any $X$.

With help of the conditions (5) and using properties of the Riemannian tensor these conditions can be expressed in the following form

$$g_{ij} c_k T^k - T_i c_j + \xi_a a_{jk} T^k = 0$$  \hspace{1cm} (9)

where $T^k \equiv g^{ka} T_a$.

If $c_k T^k \neq 0$, then it follows from (9) that $\text{rank} \|g_{ij}\| \leq 2$. Since $n > 2$ ($\Leftrightarrow \text{rank} \|g_{ij}\| > 2$), the formula (9) implies that $c_k T^k = 0$, and thus

$$T_i c_j = \xi_a a_{jk} T^k.$$  \hspace{1cm} (10)
where \( a \) is a non-zero function.

Next, differentiating (10) covariantly with respect to \( x^j \) and \( x^k \), and alternating in \( j \) and \( k \), we have

\[
T_{i,[jk]} = a \xi_{i,[jk]}.
\]

According to (7) and the Ricci identity we can write this equality in the form \( \xi_a R^a_{ijk} = 0 \) and in view of (5) we obtain

\[
\delta^h_j c_k - \delta^h_k c_j + \xi^h a_{jk} = 0.
\]

Since we suppose \( c_k \neq 0 \), there exists a vector \( \varepsilon^k \) such that \( c_\alpha \varepsilon^\alpha = 1 \). Contracting (11) with \( \varepsilon^k \), we find

\[
\delta^h_j - \varepsilon^h c_j + \xi^h a_{j\alpha} \varepsilon^\alpha = 0.
\]

This together with \( n > 2 \) (rank \( \| \delta^h_j \| > 2 \)) leads to a contradiction. Therefore \( c_k = 0 \) holds and we get from Lemma 3 that \( \xi \) is convergent.

### 4. Torse-forming vector fields in T-semisymmetric Riemannian spaces where T is a 2-tensor field

According to (1) by a 2-tensor T-semisymmetric (or simply \( T_{ij} \)-semisymmetric) space we mean a Riemannian space \( V_n \) with a tensor field \( T_{ij} \) satisfying

\[
R(X,Y) \circ T = 0 \quad \text{or} \quad T_{ij,[lm]} = 0.
\]

First, let us prove the following lemmas for symmetric and skew-symmetric tensors.

**Theorem 2.** Let \( T (\neq \alpha g) \) be a 2-covariant symmetric tensor field. A non-isotropic torse-forming vector field \( \xi \) in a T-semisymmetric space \( V_n \) \((n > 2)\) is convergent.

This theorem follows from the more general lemma

**Lemma 6.** Let \( T (\neq \alpha g) \) be a 2-covariant symmetric tensor field. A non-isotropic torse-forming vector field \( \xi \) in a space \( V_n \) \((n > 2)\) is convergent, if \( R(X,\xi) \circ T = 0 \) for any \( X \).

**Proof.** Let there exist a non-isotropic torse-forming vector field \( \xi \in V_n \) \((n > 2)\) with \( R(X,\xi) \circ T = 0 \) for any \( X \).

Similarly as before, using (5), the assumption \( T_{ij} = 0 \), and the properties of the Riemannian tensor, we obtain

\[
g_{ij} T_{j\alpha} e^\alpha - T_{ij} c_i + g_{ij} T_{i\alpha} e^\alpha - T_{i\alpha} c_j + \xi_\alpha \omega_{ij} = 0,
\]

where \( \omega_{ij} \) is a certain tensor, \( e^\alpha \equiv c_\alpha g^{\alpha i} \) and \( \| g^{ij} \| = \| g_{ij} \|^{-1} \).

Let us prove that there exists a function \( \mu \) such that

\[
T_{i\alpha} e^\alpha = \mu c_i.
\]
Suppose that (15) does not hold. Then we can find \(\varepsilon^i\) such that \(c_i \varepsilon^i = 0\) and \(T_{\alpha\beta} \varepsilon^\alpha \varepsilon^\beta = 1\). Contracting (14) with such an \(\varepsilon^j\) and subsequently with \(\varepsilon^i\), we obtain the following formulas
\[
g_{li} - T_{l\alpha} \varepsilon^\alpha c_i + \varepsilon_i T_{l\alpha} \varepsilon^\alpha + \xi_l \omega_{\alpha\beta} \varepsilon^\alpha \varepsilon^\beta = 0 \quad \text{and} \quad 2\varepsilon_l + \xi_l \omega_{\alpha\beta} \varepsilon^\alpha \varepsilon^\beta = 0
\]
where \(\varepsilon_i \equiv \varepsilon^\alpha g_{\alpha i}\). We can deduce that \(\text{rank} \|g_{li}\| \leq 2\). But from the assumption \(n > 2\) it follows that \(\text{rank} \|g_{li}\| > 3\), a contradiction.

Substituting (15) in (14) we get
\[
F_{li} c_j + F_{lj} c_i - \xi_l \omega_{ij} = 0
\]
(16)
where
\[
F_{ij} \equiv T_{ij} - \mu g_{ij}
\]
Let us choose \(\varphi^j\) such that \(\varphi^j c^j = 1\). When contracting (16) with such a \(\varphi^j\) and then with \(\varphi^i\), we arrive at the following formulas
\[
F_{li} + F_{lj} \varphi^j c_i - \xi_l \omega_{ij} \varphi^j = 0
\]
(18)
and
\[
F_{li} \varphi^i = \nu \xi_l
\]
where \(\nu = \frac{1}{2} \omega_{ij} \varphi^j \varphi^i\). This together with (18) leads to \(F_{li} = \xi_l \chi_i\) where \(\chi_i = -\nu c_i + \omega_{ij} \varphi^j\).

Then, according to the symmetry of the tensor \(F_{ij}\), we can write
\[
F_{ij} = \lambda \xi_i \xi_j
\]
(19)
where \(\lambda\) is a function, this implies that \(\lambda \neq 0\).

Next, differentiating (19) covariantly with respect to \(x^l\) and \(x^m\), and alternating in \(l\) and \(m\), we have
\[
F_{ij,lm} = \lambda (\xi_i,_{[lm]} \xi_j + \xi_j,_{[lm]} \xi_i)
\]
From (12) and (17) it follows \(F_{ij,lm} = 0\), which, in view of \(\xi_i \neq 0\), implies \(\xi_i,_{[lm]} = 0\). It means that \(V_n\) is \(\xi_i\)-semisymmetric and we get from Theorem 1 that \(\xi^k\) is convergent.

**Theorem 3.** Let \(T \neq 0\) be a 2-covariant skew-symmetric tensor field. A non-isotropic torse-forming vector field \(\xi\) in a \(T\)-semisymmetric space \(V_n\) \((n > 3)\) is convergent.

Similarly as above, this theorem follows from the following

**Lemma 7.** Let \(T \neq 0\) be a 2-covariant skew-symmetric tensor field. A non-isotropic torse-forming vector field \(\xi\) in a space \(V_n\) \((n > 3)\) is convergent, if \(R(X, \xi) \circ T = 0\) for any \(X\).

**Proof.** Let there exist a non-isotropic torse-forming vector field \(\xi^h\) in \(V_n\) \((n > 3)\) with \(R(X, \xi) \circ T = 0\).
Again, using (5) and the properties of the Riemannian tensor and, in addition, the assumption \( T_{ij} + T_{ji} = 0 \), we obtain
\[
g_i T_{\alpha j} e^\alpha - T_{ij} c_i - g_{ij} T_{\alpha i} e^\alpha + T_{il} c_j - \xi_i \omega_{ij} = 0,
\]
where \( \omega_{ij} \) is a certain tensor and \( e^i \equiv e_\alpha g^{\alpha i} \).

Let us prove that there exists a function \( \mu \) such that (15) is true. Suppose, on the contrary, that (15) does not hold. Then we can find \( \varepsilon^i \) such that \( T_{\alpha \beta} e^\alpha e^\beta = 1 \) and \( c_i \varepsilon^i = 0 \). Contracting (20) with such an \( \varepsilon^i \), we can deduce that \( \text{rank}\|g_{ii}\| \leq 3 \). But from the assumption \( n > 3 \) it follows that \( \text{rank}\|g_{ii}\| > 3 \), a contradiction.

Substituting (15) in (20) we get
\[
(T_{ii} - \mu g_{ii}) c_j - (T_{ij} - \mu g_{ij}) c_i - \xi_i \omega_{ij} = 0.
\]

Let us suppose that \( c_j \neq 0 \). Then there exist \( \varphi^i \) such that \( \varphi^i c_i = 1 \). Contracting (21) with \( \varphi^j \) we get
\[
T_{ii} - \mu g_{ii} = \xi_i \eta_i + \chi_i c_i
\]
where \( \eta_i \) and \( \chi_i \) are suitable vectors. Symmetrizing (22) we obtain
\[
-2\mu g_{ii} = \xi_i \eta_i + \chi_i c_i + \xi_i \eta_j + \chi_j c_i.
\]

Provided the vectors \( \xi_i, c_i, \eta_i, \chi_i \) were linearly independent, we could use (23) and verify that all they are isotropic. Since, however, \( \xi_i \) is non-isotropic, these vectors have to be linearly dependent. If \( \mu \neq 0 \), then the equality (23) implies \( \text{rank}\|g_{ii}\| \leq 3 \) which contradicts the assumption \( n > 3 \). Therefore \( \mu = 0 \) and (22) has the form
\[
T_{ii} = \xi_i \eta_i + \chi_i c_i.
\]

Using the fact that \( T_{ij} \) is skew-symmetric, we get from (24) that there is a vector \( \nu_i \) such that
\[
T_{ij} = \xi_i \nu_j - \xi_j \nu_i.
\]

Having in mind that (12) and (25) are valid, we obtain
\[
\xi_i, [im] \nu_j + \xi_i \nu_j, [im] - \xi_j, [im] \nu_i - \xi_j \nu_i, [im] = 0.
\]

We substitute \( \xi_i, [im] \equiv -\alpha_{ij} f_{im} \) and then, by (5), we have
\[
(g_{il} \nu_m - g_{im} \nu_l + \xi_i a_{im}) \nu_j + \xi_i \nu_j, [im] = \n
(g_{dj} \nu_m - g_{jm} \nu_d + \xi_j a_{jm}) \nu_i - \xi_j \nu_i, [im] = 0.
\]

From (25) and the assumption \( T_{ij} \neq 0 \) it follows that the vectors \( \xi_i \) and \( \nu_i \) cannot be collinear. Therefore there is \( \varepsilon^i \) such that \( \varepsilon^i \nu_i = 1 \) and \( \varepsilon^i \xi_i = 0 \). Contracting (26) with \( \varepsilon^j \) we get
\[
g_{il} \nu_m - g_{im} \nu_l + \xi_i b_{lm} + \nu_i c_{lm} = 0,
\]
where \( b_{lm} \) and \( c_{lm} \) are certain tensors. Contracting (27) with \( \varphi^m \) (this vector satisfies \( \varphi^m c_m = 1 \)) we find that \( \text{rank}\|g_{ii}\| \leq 3 \), a contradiction. This contradiction implies that \( c_i = 0 \), which means, by Lemma 3, that \( \xi^b \) is convergent.

For torse-forming vector fields in \( T_{ij} \)-semisymmetric Riemannian spaces an assertion which is analogous to Theorem 1 and Lemma 5 holds.
Theorem 4. Let $T \neq \alpha g$ be a 2-covariant tensor field. A non-isotropic torse-forming vector field $\xi$ in a $T$-semisymmetric space $V_n (n > 3)$ is convergent.

Analogously we show that the following is true.

Lemma 8. Let $T \neq \alpha g$ be a 2-covariant tensor field. A non-isotropic torse-forming vector field $\xi$ in a space $V_n (n > 3)$ is convergent, if $R(X, \xi) \circ T = 0$ for any $X$.

Proof. Let $T \neq \alpha g$ be a 2-covariant tensor field in $V_n (n > 3)$ with $R(X, \xi) \circ T = 0$ for any $X$. The tensor $T$ can be expressed uniquely in the form $T = U + V$ where $U$ is symmetric and $V$ is skew-symmetric. Then $U(X, Y) = \frac{1}{2}(T(X, Y) + T(Y, X))$ and $V(X, Y) = \frac{1}{2}(T(X, Y) - T(Y, X))$. From $R(X, \xi) \circ T = 0$ we get $R(X, \xi) \circ U = 0$ and $R(X, \xi) \circ V = 0$.

Further, let us suppose that there exists a non-isotropic torse-forming vector field $\xi$ in $V_n$ which is not convergent. Therefore we can use Lemma 6 and Lemma 7 and get that $U = \alpha g$ and $V = 0$. It means that $T = \alpha g$, a contradiction. This implies that the vector field $\xi$ has to be convergent.

5. Torse-forming vector fields in special $T$-semisymmetric spaces

Now, we will consider a special case of a $T$-semisymmetric space, namely, such that $T$ is the Ricci tensor. A Riemannian space $V_n$ is called Ricci-semisymmetric if the Ricci tensor $Ric$ satisfies

$$R(X, Y) \circ Ric = 0.$$ 

For non-Einsteinian spaces we have the inequality $Ric \neq \alpha g$. The following theorem follows from Theorem 2:

Theorem 5. A non-isotropic torse-forming vector field $\xi$ in a non-Einsteinian Ricci-semisymmetric space $V_n (n > 2)$ is convergent.

This theorem follows from

Lemma 9. A non-isotropic torse-forming vector field $\xi$ in a non-Einsteinian space $V_n (n > 2)$ is convergent, if $R(X, \xi) \circ Ric = 0$ for any $X$.

The structure $F^h$ in Kählerian spaces is covariantly constant, and evidently in this case $K_n$ is $F^h$-semisymmetric. Therefore we have, using Theorem 4

Theorem 6. A non-isotropic torse-forming vector field $\xi$ in a Kählerian space $K_n (n > 3)$ is convergent.

For Einsteinian spaces we have

Theorem 7. A non-isotropic torse-forming vector field $\xi$ in an Einsteinian space $V_n (n > 2)$ is concircular.
Proof. Let $V_n$ ($n > 2$) be an Einsteinian space. The Ricci tensor of this space satisfies the following equation $R_{ij} = \frac{R}{n} g_{ij}$, where $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature. Let there exist a non-istropic torse-forming vector field $\xi^h$ in $V_n$. Then the condition (5) is satisfied. By contracting of (5) with $g^{ij}$ we obtain

$$(n-2)g_{ik} = \xi^k (\frac{R}{n} + c(n-1)a^2 - c\xi^\alpha \xi_\alpha).$$

Since $\xi_k$ is a gradient vector, i.e., $\xi_k \equiv \xi^\alpha \xi_\alpha$, it follows from (28) that $\varrho = \varrho(\xi)$ which implies that $\xi^h$ is concircular. The proof of Theorem 7 is complete.

Using Theorems 5 and 7 and the property of the concircular vector field in a semisymmetric space $V_n$ ($n > 2$) with non-constant curvature [13] we have

**Theorem 8.** A non-isotropic torse-forming vector field $\xi$ in a semisymmetric space $V_n$ ($n > 3$) with non-constant curvature is convergent.


Spaces which generalize Einsteinian spaces are Riemannian spaces with a harmonic curvature tensor; they are characterized by the following formula:

$$R_{\alpha ijk} = 0 \quad (\Longleftrightarrow R_{ij,k} = R_{ik,j}).$$

These spaces are studied by many authors, for example [9], [15], [20].

We have

**Theorem 9.** A torse-forming vector field $\xi$ in a non-Einsteinian Ricci-semisymmetric space $V_n$ ($n > 2$) with harmonic curvature tensor is recurrent.

Proof. Let there exist a non-recurrent torse-forming vector field $\xi^h$ in a non-Einsteinian Ricci-semisymmetric space $V_n$ ($n > 2$) with harmonic curvature tensor. Evidently, the vector $\xi^h$ is non-isotropic. Therefore we can use Theorem 5 and get that $\xi^h$ is convergent.

For this vector formula (3c) applies in the following form:

$$\xi_{i,j} = \varrho g_{ij}, \quad \varrho \equiv \text{const} \neq 0.$$ (30)

The condition of integrability of the equation (30) has the form $\xi_\alpha R^\alpha_{ijk} = 0$. Differentiating covariantly the last formula we obtain

$$\xi_\alpha R^\alpha_{ijk,l} + \varrho R_{ijkl} = 0.$$ (31)

Contracting (31) with $g^{kl}$ and using properties of the Riemannian tensor and (29) we get:

$$\varrho R_{ij} = 0.$$ (32)

Because of $\varrho \neq 0$ ($\xi^h$ is not recurrent) we have $R_{ij} = 0$. This contradics to the fact that $V_n$ is not an Einsteinian space, and we are done.

**Remark.** T. Q. Binh, U. C. De, L. Tamássy and M. Tarafdar [2], [3] studied Ricci-semisymmetric and semisymmetric almost Kenmotsu manifolds. In Kenmotsu manifolds there exists a unit vector field $\xi$ satisfying the condition $\nabla_X \xi = \rho_X \xi$.
$X - \eta(X)\xi$, where $\eta(X) = g(X, \xi)$. By simple observation we convince ourselves that this vector field is non-isotropic and torse-forming, is not convergent and, consequently, is not recurrent. Therefore many results of [2] and [3] follow immediately from the properties of the torse-forming fields introduced in our article.

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