ON CERTAIN PROPERTY OF A CLOSED 2-FORM

WLODZIMIERZ BORGIEL

The goal of this paper is to show that in $\mathbb{R}^n$ or $\mathbb{C}^n$, if $\omega = \omega_1 \wedge \omega_2$ is a 2-form such that $d\omega = 0$, then there exist 1-forms $\alpha$ and $\beta$ such that $\omega = \alpha \wedge \beta$ and $d\alpha \wedge \alpha = d\beta \wedge \beta = 0$.

In order to fix the ideas, we will work initially with the $n$-dimensional space $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$, $\mathbb{R}^n_p$ the tangent space of $\mathbb{R}^n$ at $p$ and $\mathbb{R}^n_p^*$ its dual spaces. Let $\Lambda^k(\mathbb{R}^n_p)^*$ be the set of all $k$-linear alternating maps

$$\varphi : \mathbb{R}^n_p \times \cdots \times \mathbb{R}^n_p \to \mathbb{R}.$$ 

With the usual operations, $\Lambda^k(\mathbb{R}^n_p)^*$ is a vector space. Given $\varphi_1, \ldots, \varphi_k \in (\mathbb{R}^n_p)^*$, we can obtain an element $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k$ of $\Lambda^k(\mathbb{R}^n_p)^*$ by setting

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k)(v_1, v_2, \ldots, v_k) = \det(\varphi_i(v_j)),$$

where $v_1, \ldots, v_k \in \mathbb{R}^n_p$, $i, j = 1, \ldots, k$. It follows from the properties of determinates that $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k$ is in fact $k$-linear and alternate.

The set

$$\{(dx^{i_1} \wedge \cdots \wedge dx^{i_k})_p, i_1 < i_2 < \cdots < i_k, i, j \in \{1, \ldots, n\}\}$$

is a basics for $\Lambda^k(\mathbb{R}^n_p)^*$. An exterior $k$-form in $\mathbb{R}^n$ is a map $\varphi$ that associates to each $p \in \mathbb{R}^n$ an element $\varphi(p) \in \Lambda^k(\mathbb{R}^n_p)^*$ and $\varphi$ can be written as

$$\varphi(p) = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k}(p)(dx^{i_1} \wedge \cdots \wedge dx^{i_k}),$$

where $a_{i_1 \cdots i_k}$ are differentiable functions, $\varphi$ is called differential $k$-form.

**Proposition 1.** If $\varphi_1 \wedge \cdots \wedge \varphi_k = \beta_1 \wedge \cdots \wedge \beta_k = \varphi$, where $\beta_j \in (\mathbb{R}^n_p)^*$, $j = 1, \ldots, k$, are two representations of $\varphi$, then $\varphi_1 = \sum a_{i_1j}(\beta_j)$, $i = 1, \ldots, k$, with $\det(a_{ij}) = 1$.

**Proof:** Extend the $\beta_j$ into a basis $\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_n$ of $(\mathbb{R}^n_p)^*$ and write

$$\varphi_i = \sum_j a_{ij} \beta_j + \sum_r b_{ir} \beta_r, r = k + 1, \ldots, n.$$
Notice that $\beta_1 \wedge \cdots \wedge \beta_k \wedge \varphi_i = \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \varphi_i = 0$. This implies that
\[ \sum_r b_r \beta_1 \wedge \cdots \wedge \beta_k \wedge \beta_r = 0, \]
and since $\beta_1 \wedge \cdots \wedge \beta_k \wedge \beta_r$ are linearly independent, so $b_r = 0$.

Now let $v_1, \ldots, v_k$ be the vectors of $\mathbb{R}_p^n$. From definition it clearly follows that
\[ \langle \varphi_1 \wedge \cdots \wedge \varphi_k; v_1, \ldots, v_k \rangle = \det((\langle \varphi_i, v_r \rangle)) \]
\[ = \det \left( \sum_{j=1}^k a_{ij} \beta_j, v_r \right) \]
\[ = \det(a_{ij}) \sum_{r_1=1}^k \cdots \sum_{r_k=1}^k \epsilon_{r_1 \ldots r_k} \langle \beta_{r_1}, v_{r_1} \rangle \cdots \langle \beta_{r_k}, v_{r_k} \rangle \]
\[ = \det(a_{ij}) \det(\langle \beta_j, v_r \rangle) \]
\[ = \det(a_{ij}) (\beta_1 \wedge \cdots \wedge \beta_k; v_1, \ldots, v_k). \]

The symbol
\[ \epsilon_{i_1 \ldots i_n} \quad (\text{or } \epsilon_{i_1 \ldots i_n}) \]
is zero unless $i_1, \ldots, i_n$ is some derangement of the first $n$ natural numbers. If the derangement is an even permutation,
\[ \epsilon_{i_1 \ldots i_n} = \epsilon_{i_1 \ldots i_n} = 1. \]
If the derangement is an odd permutation,
\[ \epsilon_{i_1 \ldots i_n} = \epsilon_{i_1 \ldots i_n} = -1. \]
Thus,
\[ \det(a_{ij}) = 1, \]
as we wished to prove.

**Proposition 2.** Let $P$, $Q$, $R$ denote three differential functions in an open set $U$ of $\mathbb{R}^3$, which do not vanish simultaneously; put
\[ \omega = Pdx^2 \wedge dx^3 + Qdx^3 \wedge dx^1 + Rdx^1 \wedge dx^2. \]
Then in the neighbourhood of each point of $U$ there exist pairs of functions $u$, $v$ satisfying
\[ du \wedge \omega = 0, \quad dv \wedge \omega = 0, \quad du \wedge dv \neq 0, \]
and if $u$, $v$ is such a pair, there exists a function $\lambda$ such that
\[ \omega = \lambda du \wedge dv. \]
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Proof. Consider the forms

\[ du \wedge \omega = \left( \frac{\partial u}{\partial x^1} dx^1 + \frac{\partial u}{\partial x^2} dx^2 + \frac{\partial u}{\partial x^3} dx^3 \right) \wedge (P dx^2 \wedge dx^3 + Q dx^3 \wedge dx^1 + R dx^1 \wedge dx^2) \]

\[ = \left( \frac{\partial u}{\partial x^1} + Q \frac{\partial u}{\partial x^2} + R \frac{\partial u}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3, \]

\[ dv \wedge \omega = \left( \frac{\partial v}{\partial x^1} dx^1 + \frac{\partial v}{\partial x^2} dx^2 + \frac{\partial v}{\partial x^3} dx^3 \right) \wedge 
\]
\[ \wedge \left( P dx^2 \wedge dx^3 + Q dx^3 \wedge dx^1 + R dx^1 \wedge dx^2 \right) \]

\[ = \left( \frac{\partial v}{\partial x^1} + Q \frac{\partial v}{\partial x^2} + R \frac{\partial v}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3, \]

and

\[ du \wedge dv = \left( \frac{\partial u}{\partial x^1} dx^1 + \frac{\partial u}{\partial x^2} dx^2 + \frac{\partial u}{\partial x^3} dx^3 \right) \wedge \left( \frac{\partial v}{\partial x^1} dx^1 + \frac{\partial v}{\partial x^2} dx^2 + \frac{\partial v}{\partial x^3} dx^3 \right) \]

\[ = \left( \frac{\partial u}{\partial x^1} \frac{\partial v}{\partial x^2} dx^3 \wedge dx^1 \wedge dx^2 + \frac{\partial u}{\partial x^2} \frac{\partial v}{\partial x^3} dx^2 \wedge dx^1 \wedge dx^2 + \frac{\partial u}{\partial x^3} \frac{\partial v}{\partial x^1} dx^3 \wedge dx^1 \wedge dx^2 \right) \]

From the conditions \( du \wedge \omega = 0 \) and \( dv \wedge \omega = 0 \) it follows that

\[ P \frac{\partial u}{\partial x^1} + Q \frac{\partial u}{\partial x^2} + R \frac{\partial u}{\partial x^3} = 0 \]

and

\[ P \frac{\partial v}{\partial x^1} + Q \frac{\partial v}{\partial x^2} + R \frac{\partial v}{\partial x^3} = 0. \]

And hence, in the neighbourhood of each point of \( U \) the vector fields \( (\frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}, \frac{\partial u}{\partial x^3}) \) and \( (\frac{\partial v}{\partial x^1}, \frac{\partial v}{\partial x^2}, \frac{\partial v}{\partial x^3}) \) are orthogonal to the vector field \( (P, Q, R) \). Thus

\[ (P, Q, R) \left( \left( \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}, \frac{\partial u}{\partial x^3} \right) \times \left( \frac{\partial v}{\partial x^1}, \frac{\partial v}{\partial x^2}, \frac{\partial v}{\partial x^3} \right) \right) = \]

\[ = \left( \left( \frac{\partial u}{\partial x^1} \frac{\partial v}{\partial x^2} - \frac{\partial u}{\partial x^2} \frac{\partial v}{\partial x^1} \right) \left( \frac{\partial v}{\partial x^2} \frac{\partial u}{\partial x^3} - \frac{\partial v}{\partial x^3} \frac{\partial u}{\partial x^2} \right) \right), \]

since from the condition \( du \wedge dv \neq 0 \) it follows that not all the coordinates of this vector field, in the neighbourhood of each point of \( U \), vanish simultaneously. Therefore, from parallely it follows that there exists a function \( \lambda \) such that

\[ \omega = \lambda du \wedge dv. \]

Now we consider a differential 3-form \( d\omega \):

\[ d\omega = d(\lambda du \wedge dv) = d\lambda \wedge du \wedge dv + \lambda (du \wedge dv) = d\lambda \wedge du \wedge dv. \]
If \( d\omega = 0 \), then
\[
\frac{\partial \lambda}{\partial x^1} \left( \frac{\partial u}{\partial x^1} \frac{\partial v}{\partial x^2} - \frac{\partial u}{\partial x^2} \frac{\partial v}{\partial x^1} \right) + \frac{\partial \lambda}{\partial x^2} \left( \frac{\partial u}{\partial x^2} \frac{\partial v}{\partial x^3} - \frac{\partial u}{\partial x^3} \frac{\partial v}{\partial x^2} \right) + \frac{\partial \lambda}{\partial x^3} \left( \frac{\partial u}{\partial x^3} \frac{\partial v}{\partial x^1} - \frac{\partial u}{\partial x^1} \frac{\partial v}{\partial x^3} \right) = 0,
\]
this means that in the neighbourhood of each point of \( U \) the vector field \( \left( \frac{\partial \lambda}{\partial x^1}, \frac{\partial \lambda}{\partial x^2}, \frac{\partial \lambda}{\partial x^3} \right) \) is orthogonal to the vector field
\[
\left( \left( \frac{\partial u}{\partial x^2} \frac{\partial v}{\partial x^3} - \frac{\partial u}{\partial x^3} \frac{\partial v}{\partial x^2} \right), \left( \frac{\partial u}{\partial x^3} \frac{\partial v}{\partial x^1} - \frac{\partial u}{\partial x^1} \frac{\partial v}{\partial x^3} \right), \left( \frac{\partial u}{\partial x^1} \frac{\partial v}{\partial x^2} - \frac{\partial u}{\partial x^2} \frac{\partial v}{\partial x^1} \right) \right).
\]
Hence and from Proposition 2 it follows that the vector field \( \left( \frac{\partial \lambda}{\partial x^1}, \frac{\partial \lambda}{\partial x^2}, \frac{\partial \lambda}{\partial x^3} \right) \) is a linear combination vector fields \( \left( \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}, \frac{\partial u}{\partial x^3} \right) \) and \( \left( \frac{\partial v}{\partial x^1}, \frac{\partial v}{\partial x^2}, \frac{\partial v}{\partial x^3} \right) \). Thus \( \lambda = au + bv \), where \( a, b \) are constants. Inversely, if \( \lambda = au + bv \), where \( a, b \) are constants, then
\[ d\omega = d\lambda \wedge du \wedge dv = (adu + bdv) \wedge du \wedge dv = adu \wedge du \wedge dv + bdv \wedge du \wedge dv = 0. \]
Now notice that if \( \lambda = au + bv \), where \( a, b \) are constants, then
\[ \omega = \lambda du \wedge dv = d \left( \frac{1}{2} au^2 + buv \right) \wedge dv. \]
Setting \( f = \frac{1}{2} au^2 + buv, g = v \) we have
\[ \omega = df \wedge dg. \]
Hence the following:

**Proposition 3.** In the neighbourhood of each point of \( U \) there exist two functions \( f, g \) such that \( \omega = df \wedge dg \) if and only if \( d\omega = 0 \).

Notice furthermore that if \( \alpha, \beta \) are 1-forms in \( U \subset \mathbb{R}^3 \), \( \omega = \alpha \wedge \beta \) and \( d\omega = 0 \), then \( \alpha = df \) and \( \beta = dg \). From
\[
d(\alpha \wedge \beta) = \alpha \wedge \beta - \alpha \wedge d\beta
\]
\[
= \left( \frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3} \right) b_1 - \left( \frac{\partial a_3}{\partial x^3} - \frac{\partial a_1}{\partial x^2} \right) b_2 + \left( \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right) b_3 + \frac{\partial b_3}{\partial x^1} - \frac{\partial b_2}{\partial x^2} - \frac{\partial b_1}{\partial x^3} + a_1 \left( \frac{\partial b_3}{\partial x^2} - \frac{\partial b_2}{\partial x^3} \right) - a_2 \left( \frac{\partial b_3}{\partial x^1} - \frac{\partial b_1}{\partial x^2} \right) + a_3 \left( \frac{\partial b_2}{\partial x^1} - \frac{\partial b_1}{\partial x^2} \right)
\]
\[
\cdot dx^1 \wedge dx^2 \wedge dx^3
\]
\[
= \left[ \text{rot}(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) - (a_1, a_2, a_3) \cdot \text{rot}(b_1, b_2, b_3) \right]
\]
\[
\cdot dx^1 \wedge dx^2 \wedge dx^3,
\]
where $\alpha = \sum_{i=1}^{3} a_i(x)dx^i$ and $\beta = \sum_{i=1}^{3} b_i(x)dx^i$, and from Proposition 3 it follows that $d(\alpha \wedge \beta) = 0$ if and only if $\text{rot}(a_1, a_2, a_3) = 0$ and $\text{rot}(b_1, b_2, b_3) = 0$ in the neighbourhood of each point of $U$.

Now let $\alpha = \sum_{i=1}^{n} a_i(x)dx^i$ and $\beta = \sum_{i=1}^{n} b_i(x)dx^i$ be two differential forms of degree one defined in an open set $U \subset \mathbb{R}^n$. The exterior differential $d(\alpha \wedge \beta)$ of $\alpha \wedge \beta$ is defined by

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta.$$ 

Therefore

$$d(\alpha \wedge \beta) = d\left( \sum_i a_i(x)dx^i \right) \wedge \sum_j b_j(x)dx^j - \sum_i a_i(x)dx^i \wedge d\left( \sum_j b_j(x)dx^j \right)$$

$$= \sum_i da_i(x) \wedge dx^i \wedge \sum_j b_j(x)dx^j - \sum_i a_i(x)dx^i \wedge \sum_j db_j(x) \wedge dx^j$$

$$= \sum_{i_1 < i_2 < i_3} \left[ \left( \frac{\partial a_{i_3}(x)}{\partial x^{i_2}} - \frac{\partial a_{i_2}(x)}{\partial x^{i_3}} \right) b_{i_1} - \left( \frac{\partial a_{i_2}(x)}{\partial x^{i_1}} - \frac{\partial a_{i_1}(x)}{\partial x^{i_3}} \right) b_{i_3} \right] dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} +$$

$$+ \left( \frac{\partial a_{i_2}(x)}{\partial x^{i_1}} - \frac{\partial a_{i_1}(x)}{\partial x^{i_2}} \right) b_{i_3} \right] dx^{i_2} \wedge dx^{i_1} \wedge dx^{i_3} +$$

$$= \sum_{i_1 < i_2 < i_3} \left[ \left( \frac{\partial b_{i_3}(x)}{\partial x^{i_2}} - \frac{\partial b_{i_2}(x)}{\partial x^{i_3}} \right) a_{i_1} - \left( \frac{\partial b_{i_2}(x)}{\partial x^{i_1}} - \frac{\partial b_{i_1}(x)}{\partial x^{i_3}} \right) a_{i_3} \right] dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} +$$

$$+ a_{i_3} \left( \frac{\partial b_{i_2}(x)}{\partial x^{i_1}} - \frac{\partial b_{i_1}(x)}{\partial x^{i_2}} \right) dx^{i_2} \wedge dx^{i_3} \wedge dx^{i_1}$$

for all $1 \leq i_1 < i_2 < i_3 \leq n$. That is in the neighbourhood of each point $U$, so $d(\alpha \wedge \beta) = 0$ if and only if

$$\text{rot}(a_1, a_2, a_3) = 0 \quad \text{and} \quad \text{rot}(b_1, b_2, b_3) = 0$$

for all $1 \leq i_1 < i_2 < i_3 \leq n$. This implies that

(1) $\text{rot } a = 0$ and $\text{rot } b = 0$,

if $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, differential vector fields, are correspondence 1-forms $\alpha$ and $\beta$ induced by the inner product.

We would like to remind that if $v$ is a differential vector field in $\mathbb{R}^n$, then the rotational rot $v$ is the $(n-2)$-form defined by

$$v \mapsto \omega \mapsto d\omega \mapsto \star(d\omega) = \text{rot } v,$$

where $v \mapsto \omega$ is the correspondence between 1-forms and vector fields induced by the inner product.

Notice that the condition (1) is equivalent to:

(2) $da = 0$ and $db = 0$. 

Hence and from Poincaré’s Lemma for 1-forms, it follows that the forms $\alpha$ and $\beta$ are locally exact, i.e., for each $p \in U$ there is a neighbourhood $V \subset U$ of $p$ and differentiable functions $f, g : V \to \mathbb{R}$ such that $df = \alpha$ and $dg = \beta$.

We will say that a connected open set $U \subset \mathbb{R}^n$ is simply-connected if every continuous closed curve in $U$ is freely homotopic to a point in $U$.

**Proposition 4.** Let $\omega$ be a closed form defined in a simply-connected domain. Then $\omega$ is exact (proof, see [2]).

From the above it follows that if the set $U$ is simply-connected, then $\alpha$ and $\beta$ are exact in $U$. Thus, the condition $d\alpha \wedge \alpha = d\beta \wedge \beta = 0$ are satisfied.

If we change all real field $\mathbb{R}^n$ into complex field $\mathbb{C}^n$ and each mapping $a_i$ and $b_i$, $i = 1, \ldots, n$, into complex holomorphic function in the previous definition, the form $\alpha$ and $\beta$ are called complex holomorphic forms.

Let $\alpha = \sum_{j=1}^n a_j(z)dz^j$ and $\beta = \sum_{j=1}^n b_j(z)dz^j$ be two differential forms of degree one defined in an open set $U \subset \mathbb{C}^n$. Here the complex space $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ by setting $z^j = x^j + iy^j$, $z^j \in \mathbb{C}$, $(x^j, y^j) \in \mathbb{R}^2$ for all $j = 1, \ldots, n$. It is convenient to introduce the complex differential form $dz^j = dx^j + idy^j$ and to write

\[
\begin{align*}
  a_j(z) &= u_j(x, y) + iv_j(x, y), \\
  b_j(z) &= w_j(x, y) + i\omega_j(x, y),
\end{align*}
\]

where $x = (x^1, \ldots, x^n)$, $y = (y^1, \ldots, y^n)$. Then the complex form

\[
\begin{align*}
  a_j(z)dz^j &= (u_j + iv_j)(dx^j + idy^j) \\
  &= (u_j dx^j - v_j dy^j) + i(u_j dy^j + v_j dx^j)
\end{align*}
\]

has $u_j dx^j - v_j dy^j$ as its real part and $u_j dy^j + v_j dx^j$ as its imaginary part, for all $j = 1, \ldots, n$. Similarly the complex form

\[
\begin{align*}
  b_j(z)dz^j &= (w_j + i\omega_j)(dx^j + idy^j) \\
  &= (w_j dx^j - \omega_j dy^j) + i(w_j dy^j + \omega_j dx^j)
\end{align*}
\]

has $w_j dx^j - \omega_j dy^j$ as its real part and $w_j dy^j + \omega_j dx^j$ as its imaginary part, for all $j = 1, \ldots, n$.

Now, a computation shows that

\[
\begin{align*}
  d\alpha \wedge \beta &= d\left( \sum_{j=1}^n a_j(z)dz^j \right) \wedge \sum_{k=1}^n b_k(z)dz^k \\
  &= \left[ \left( \sum_{j=1}^n du_j \wedge dx^j - \sum_{j=1}^n dv_j \wedge dy^j \right) + i \left( \sum_{j=1}^n du_j \wedge dy^j + \sum_{j=1}^n dv_j \wedge dx^j \right) \right] \\
  &\wedge \left[ \left( \sum_{k=1}^n w_k dx^k - \sum_{k=1}^n \omega_k dy^k \right) + i \left( \sum_{k=1}^n w_k dy^k + \sum_{k=1}^n \omega_k dx^k \right) \right]
\end{align*}
\]
\[
\begin{align*}
&= \sum_{j=1}^n du_j \wedge dx^j \wedge \sum_{k=1}^n w_k dx^k + \sum_{j=1}^n dv_j \wedge dy^j \wedge \sum_{k=1}^n \omega_k dy^k + \\
&\quad - \sum_{j=1}^n dv_j \wedge dy^j \wedge i \sum_{k=1}^n w_k dx^k + \sum_{j=1}^n du_j \wedge dx^j \wedge i \sum_{k=1}^n \omega_k dx^k + \\
&\quad + i \sum_{j=1}^n dv_j \wedge dx^j \wedge \sum_{k=1}^n w_k dx^k - i \sum_{j=1}^n du_j \wedge dy^j \wedge \sum_{k=1}^n \omega_k dy^k + \\
&\quad - \sum_{j=1}^n du_j \wedge dy^j \wedge \sum_{k=1}^n w_k dy^k - \sum_{j=1}^n dv_j \wedge dx^j \wedge \sum_{k=1}^n \omega_k dx^k,
\end{align*}
\]

and

\[
\alpha \wedge d\beta = \sum_{j=1}^n a_j(z)dz^j \wedge d\left( \sum_{k=1}^n b_k(z)dz^k \right)
\]

\[
= \left[ \left( \sum_{j=1}^n du_j dx^j - \sum_{j=1}^n v_j dy^j \right) + i \left( \sum_{j=1}^n u_j dy^j + \sum_{j=1}^n v_j dx^j \right) \right]
\]

\[
\wedge \left[ \left( \sum_{k=1}^n dw_k \wedge dx^k - \sum_{k=1}^n \omega_k \wedge dy^k \right) + \\
+ i \left( \sum_{k=1}^n dw_k \wedge dy^k + \sum_{k=1}^n \omega_k \wedge dx^k \right) \right]
\]

\[
= \sum_{j=1}^n u_j dx^j \wedge \sum_{k=1}^n dw_k \wedge dx^k + \sum_{j=1}^n v_j dy^j \wedge \sum_{k=1}^n \omega_k \wedge dy^k + \\
- \sum_{j=1}^n v_j dy^j \wedge i \sum_{k=1}^n dw_k \wedge dx^k + \sum_{j=1}^n u_j dx^j \wedge i \sum_{k=1}^n \omega_k \wedge dx^k + \\
+ i \sum_{j=1}^n v_j dx^j \wedge \sum_{k=1}^n dw_k \wedge dx^k - i \sum_{j=1}^n u_j dx^j \wedge \sum_{k=1}^n \omega_k \wedge dy^k + \\
- \sum_{j=1}^n u_j dy^j \wedge \sum_{k=1}^n dw_k \wedge dy^k - \sum_{j=1}^n v_j dx^j \wedge \sum_{k=1}^n \omega_k \wedge dx^k.
\]
In a similar way as for the real case, we can prove that
\[ d(\alpha \wedge \beta) = \sum_{j_1 < j_2 < j_3} \{(\text{rot}_x(u_{j_1}, u_{j_2}, u_{j_3}) + i \text{rot}_x(v_{j_1}, v_{j_2}, v_{j_3})) \cdot (w_{j_1}, w_{j_2}, w_{j_3}) + \]
\[ - (\text{rot}_x(v_{j_1}, v_{j_2}, v_{j_3}) + i \text{rot}_x(u_{j_1}, u_{j_2}, u_{j_3})) \cdot (\omega_{j_1}, \omega_{j_2}, \omega_{j_3}) + \]
\[ - [(u_{j_1}, u_{j_2}, u_{j_3}) \cdot (\text{rot}_x(w_{j_1}, w_{j_2}, w_{j_3}) + i \text{rot}_x(\omega_{j_1}, \omega_{j_2}, \omega_{j_3})) + \]
\[ - (v_{j_1}, v_{j_2}, v_{j_3}) \cdot (\text{rot}_x(\omega_{j_1}, \omega_{j_2}, \omega_{j_3}) + i \text{rot}_x(w_{j_1}, w_{j_2}, w_{j_3}))]) \}
\[ \cdot dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} + \]
\[ + \sum_{j_1 < j_2 < j_3} \{(\text{rot}_y(v_{j_1}, v_{j_2}, v_{j_3}) - i \text{rot}_y(u_{j_1}, u_{j_2}, u_{j_3})) \cdot (\omega_{j_1}, \omega_{j_2}, \omega_{j_3}) + \]
\[ - (\text{rot}_y(u_{j_1}, u_{j_2}, u_{j_3}) - i \text{rot}_y(v_{j_1}, v_{j_2}, v_{j_3})) \cdot (w_{j_1}, w_{j_2}, w_{j_3}) + \]
\[ - [(v_{j_1}, v_{j_2}, v_{j_3}) \cdot (\text{rot}_y(\omega_{j_1}, \omega_{j_2}, \omega_{j_3}) - i \text{rot}_y(w_{j_1}, w_{j_2}, w_{j_3})) + \]
\[ - (u_{j_1}, u_{j_2}, u_{j_3}) \cdot (\text{rot}_y(w_{j_1}, w_{j_2}, w_{j_3}) - i \text{rot}_y(\omega_{j_1}, \omega_{j_2}, \omega_{j_3}))]) \}
\[ \cdot dy^{j_1} \wedge dy^{j_2} \wedge dy^{j_3} \]
for all \(1 \leq j_1 < j_2 < j_3 \leq n\). That is in the neighbourhood of each point \(U\), so
\[ d(\alpha \wedge \beta) = 0\] if and only if
\[ \text{rot}_x(u_{j_1}, u_{j_2}, u_{j_3}) = 0, \text{rot}_x(v_{j_1}, v_{j_2}, v_{j_3}) = 0, \]
\[ \text{rot}_y(u_{j_1}, u_{j_2}, u_{j_3}) = 0, \text{rot}_y(v_{j_1}, v_{j_2}, v_{j_3}) = 0\]
and
\[ \text{rot}_x(w_{j_1}, w_{j_2}, w_{j_3}) = 0, \text{rot}_x(\omega_{j_1}, \omega_{j_2}, \omega_{j_3}) = 0, \]
\[ \text{rot}_y(w_{j_1}, w_{j_2}, w_{j_3}) = 0, \text{rot}_y(\omega_{j_1}, \omega_{j_2}, \omega_{j_3}) = 0\]
for all \(1 \leq j_1 < j_2 < j_3 \leq n\). Hence, it follows that the forms
\[ \sum_{j=1}^{n} (u_j dx^j - v_j dy^j) \text{ and } \sum_{j=1}^{n} (u_j dy^j + v_j dx^j) \]
are closed. Similarly the forms
\[ \sum_{j=1}^{n} (w_j dx^j - \omega_j dy^j) \text{ and } \sum_{j=1}^{n} (w_j dy^j + \omega_j dx^j) \]
are closed too. By Poincaré’s Lemma for 1-forms it follows that these forms are locally exact. Therefore, if \(U \subset \mathbb{C}^n\) is a simply-connected domain of these forms, then there are differential functions \(f, g : U \to \mathbb{C}\) such that \(df = \alpha\) and \(dg = \beta\). Thus the condition \(d\alpha \wedge \alpha = d\beta \wedge \beta = 0\) is satisfied.
ON CERTAIN PROPERTY OF A CLOSED 2-FORM

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Faculty of Mathematics and Information Science, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland