ON K-CONTACT $\eta$-EINSTEIN MANIFOLDS

ABSOS ALI SHAIKH, U. C. DE AND T. Q. BINH

Abstract. The object of the present paper is to study a K-contact $\eta$-Einstein manifold satisfying certain conditions on the curvature tensor.

1. Introduction

Let $(M^n, g)$ be a contact Riemannian manifold with contact from $\eta$, associated vector field $\xi$, (1,1)-tensor field $\varphi$ and associated Riemannian metric $g$. If $\xi$ is a killing vector field, then $M^n$ is called a K-contact Riemannian manifold [1], [2]. A K-contact Riemannian manifold is called Sasakian [1], if the relation

$$ (\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X $$

holds, where $\nabla$ denotes the operator of covariant differentiation with respect of $g$.

Recently, M. C. CHAKI and M. TARAFDAR [3] studied a Sasakian manifold $M^n (n > 3)$ satisfying the relation $R(X, Y). C = 0$, where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold and $C$ is the Weyl conformal curvature tensor of type (1,3). Generalizing the result of CHAKI and TARAFDAR, N. GUHA and U. C. DE [4] proved that if a K-contact manifold with characteristic vector field $\xi$ belonging to the $k$-nullity distribution satisfies the condition $R(\xi, X). C = 0$, then $C(\xi, X)Y = 0$ for any vector fields $X, Y$. In Section 3 of the present paper we prove, without assuming that $\xi$ belongs to the $K$-nullity distribution, that a K-contact $\eta$-Einstein manifold $(M^n, g) (n > 3)$ satisfying the condition $R(X, \xi). C = 0$ is a space of constant curvature.

In [5] S. TANNO studied a K-contact manifold satisfying the condition $R(X, \xi). S = 0$, where $S$ is the Ricci tensor of type (0,2). But the condition $R(X, \xi). S = 0$ does not imply the condition $S(X, \xi). R = 0$. In Section 4 we prove that if a K-contact manifold $M^n$ still satisfies the relation $S(X, \xi). R = 0$ than it is an $\eta$-Einstein manifold.

Supported by OTKA 32058.
2. Preliminaries

In a $K$-contact Riemannian manifold the following relations hold: [1], [2], [6]

(2.1) $\phi \xi = 0$
(2.2) $\phi^2 X = -X + \eta(X)\xi$
(2.3) $g(\phi X, \phi Y) = g(X, Y)$
(2.4) $\nabla_X \xi = -\phi X$
(2.5) $g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y)$
(2.6) $R(\xi, X)\xi = -X + \eta(X)\xi$
(2.7) $S(X, \xi) = (n-1)\eta(X)$
(2.8) $(\nabla_X \phi)(Y) = R(\xi, X)Y$

for any vector fields $X, Y$.

A $K$-contact manifold $M^n$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form $S = ag + b\eta \otimes \eta$, where $a, b$ are smooth functions on $M$.

3. $K$-contact $\eta$-Einstein manifolds satisfying $R(X, \xi) \cdot C = 0$

Let us consider a $K$-contact $\eta$-Einstein manifold $M^n(n > 3)$ satisfying the relation

(3.1) $R(X, \xi) \cdot C = 0$.

In this case we have

(3.2) $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$.

Putting $X = Y = \xi$ in (3.2) and then using (2.7) and (2.1) $b$, we get

(3.3) $n - 1 = a + b$.

Also (3.2) implies that

(3.4) $r = an + b$.

From (3.3) and (3.4) we have

(3.5) $a = \frac{r}{n-1} - 1, \quad b = n - \frac{r}{n-1}$.

Again from (3.2) we obtain

(3.6) $QX = \left(\frac{r}{n-1} - 1\right) X + \left(n - \frac{r}{n-1}\right) \eta(X)\xi$,

where $Q$ denotes the Ricci operator, i.e. $g(QX, Y) = S(X, Y)$. 
By definition the Weyl conformal curvature tensor $C$ is given by

\begin{align}
C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2} [g(Y, Z)QX - g(X, Z)QY \\
&+ S(Y, Z)X - S(X, Z)Y] \\
&+ \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y].
\end{align}

Using (3.2) and (3.6) in (3.7), we get

\begin{align}
C(X, Y)Z &= R(X, Y)Z + \left[ \frac{2}{(n-1)} - \frac{r}{n(n-2)} \right] [g(Y, Z)X - g(X, Z)Y] \\
&- \left[ n \frac{n}{n-1} - \frac{r}{(n-1)(n-2)} \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\
&+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y.
\end{align}

Now (3.1) gives us by definition

\begin{align}
R(X, \xi)C(U, V)W - C(R(X, \xi)U, V)W - C(U, R(X, \xi)V)W \\
- C(U, V)R(X, \xi)W = 0, \quad \text{for all} \quad X, U, V, W.
\end{align}

Substitution of $U$ and $W$ by $\xi$ in (3.9) yields

\begin{align}
R(X, \xi)C(\xi, V)\xi - C(R(X, \xi)\xi, V)\xi - C(\xi, R(X, \xi)V)\xi \\
- C(\xi, V)R(X, \xi)\xi = 0.
\end{align}

From (3.8) we get by virtue of (2.1) (b), (2.1) (c) and (2.6),

\begin{align}
C(\xi, V)\xi = 0, \quad \text{for any vector field} \quad V.
\end{align}

Hence by virtue of (3.11) we have

\begin{align}
R(X, \xi)C(\xi, V)\xi = 0.
\end{align}

Again in view of (2.6) we get

\begin{align}
C(R(X, \xi)\xi, V)\xi = C(X, V)\xi - \eta(X)C(\xi, V)\xi
\end{align}

which implies by means of (3.11) that

\begin{align}
C(R(X, \xi)\xi, V) = C(X, V)\xi.
\end{align}

From (3.8) we obtain

\begin{align}
C(X, V)\xi = R(X, V)\xi - \eta(V)X + \eta(X)V
\end{align}

for any vector fields $X$ and $V$. By virtue of (3.13) and (3.14) we have

\begin{align}
C(R(X, \xi)\xi, V)\xi = R(X, V)\xi - \eta(V)X + \eta(X)V,
\end{align}

and by virtue of (3.11) we get

\begin{align}
C(\xi, R(X, \xi)V)\xi = 0.
\end{align}

Finally using (2.6), we have

\begin{align}
C(\xi, V)R(X, \xi)\xi = C(\xi, V)X - \eta(X)C(\xi, V)\xi,
\end{align}
from which it follows by means of (3.11) and (3.14) that
\[(3.17)\]
\[C(\xi, V)R(X, \xi)\xi = R(\xi, V)X - g(X, V)\xi + \eta(X)V.\]

Applying (3.12), (3.15), (3.16) and (3.17) in (3.10) we obtain
\[(3.18)\]
\[R(X, V)\xi + R(\xi, V)X - g(X, V)\xi - \eta(V)X + 2\eta(X)V = 0.\]

Interchanging \(X\) and \(V\) in (3.18) we have
\[(3.19)\]
\[R(V, X)\xi + R(\xi, X)V - g(X, V)\xi - \eta(X)V + 2\eta(V)X = 0.\]

Subtracting (3.19) from (3.18) and then using Bianchi’s first identity, we get
\[(3.20)\]
\[R(X, V)\xi = \eta(V)X - \eta(X)V,\]
from which it follows that
\[(3.20)\]
\[R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X.\]

In view of (1.1), (2.8) and (3.20), we obtain that the manifold is Sasakian and hence by the result of Chaki and Tarafdar [3], the manifold is a space of constant curvature 1. Thus we have the following

**Theorem 1.** A \(K\)-contact \(\eta\)-Einstein manifold \((M^n, g)\) \((n > 3)\) satisfying the condition \(R(X, \xi)\cdot C = 0\) is a space of constant curvature 1.

4. **\(K\)-contact manifolds satisfying the condition \(S(X, \xi)\cdot R = 0\)**

We consider a \(K\)-contact Riemannian manifold \(M^n\) satisfying the condition
\[(4.1)\]
\[(S(X, \xi)\cdot R(U, V)W = 0.\]

Now by definition we have
\[(4.2)\]
\[(S(X, \xi)\cdot R(U, V)W = ((X \wedge \xi)\cdot R(U, V))W = (X \wedge \xi)R(U, V)W + R(\xi(U, \wedge \xi)U, V)W + R(U, V)(X \wedge \xi)W,\]

where the endomorphism \(X \wedge \xi Y\) is defined by
\[(4.3)\]
\[(X \wedge \xi Y)Z = S(Y, Z)X - S(X, Z)Y.\]

Using the definition of (4.3) in (4.2), we get by virtue of (2.7)
\[(4.4)\]
\[S(X, \xi)\cdot R(U, V)W = (n - 1)\eta(R(U, V))X + \eta(U)R(X, V)W + \eta(V)R(U, X)W + \eta(W)R(U, V)X - S(X, R(U, V)W)\]
and by virtue of (4.1) and (4.4) we have

\[(n-1)\left[\eta(R(U, V)W)X + \eta(U)R(X, V)W + \eta(V)R(U, X)W + \eta(W)R(U, V)W\right] \]


(4.5)

Taking the inner product on both sides of (4.5) by \(\xi\), we obtain

\[(n-1)\left[\eta(R(U, V)W)\eta(X) + \eta(U)\eta(R(X, V)W) + \eta(V)\eta(R(U, X)W) + \eta(W)\eta(R(U, V)W)\right] \]

\[- S(X, R(U, V)W) - S(X, U)\eta(R(\xi, V)W) - S(X, V)\eta(R(U, \xi)W) - S(X, W)\eta(R(U, V)\xi) = 0.\]

(4.6)

Putting \(U = W = \xi\) in (4.6) and using (2.5)–(2.7) we get

\[S(X, V) = -(n-1)g(X, V) + 2(n-1)\eta(X)\eta(V)\]

which means that the manifold is \(\eta\)-Einstein.

Thus we have the following

**Theorem 2.** A \(K\)-contact Riemannian manifold \((M^n, g)\) satisfying the condition \(S(X, \xi) \cdot R = 0\) is an \(\eta\)-Einstein manifold.

From Theorem 1 and Theorem 2 we immediately have:

**Theorem 3.** A \(K\)-contact Riemannian manifold \((M^n, g)\) stisfying the conditions \(S(X, \xi) \cdot R = 0\) and \(R(X, \xi)C = 0\) is a space of constant curvature 1.

**References**


Absos Ali Shaikh and U. C. De, Department of Mathematics, University of Kalyani, Kalyani-741235, W.B., India

T. Q. Binh, Institute of Mathematics, and Informatics, Debrecen University, H–4010 Debrecen, P.O. Box 12, Hungary

E-mail address: binh@math.klte.hu