TOTALLY REAL SURFACES IN THE COMPLEX 2-SPACE

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INTRODUCTION

Let $M$ be an immersed oriented surface in the complex 2-space $\mathbb{C}^2 = (\mathbb{R}^4, \langle \cdot, \cdot \rangle, J)$, where $\mathbb{C}^2$ is identified with the real 4-space $\mathbb{R}^4$, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product and $J$ the standard almost complex structure on $\mathbb{R}^4$. A point $p$ in $M$ is called a complex point if the tangent space $T_pM$ is $J$-invariant. If there is no complex point on $M$, the surface $M$ is said to be totally real, and we obtain that $T_pM \oplus JT_pM = \mathbb{C}^2$ at each point $p \in M$. Especially, if $T_pM \perp JT_pM$ at each point $p \in M$, the surface $M$ is said to be Lagrangian.

In this article, we prove that any totally real conformal immersion from $M$ into $\mathbb{C}^2$ can be given merely by an algebraic combination of the components of a solution of a linear system of first order differential equations, which system is a specific Dirac-type equation on $M$. This equation and the combination are given by means of the Kähler angle function $\alpha : M \to (0, \pi)$ and the Lagrangian angle function $\beta : M \to \mathbb{R}/2\pi\mathbb{Z}$ for the constructed totally real immersed surface $M$ in $\mathbb{C}^2$. Moreover, the pair of $\alpha$ and $\beta$ describes the self-dual part of the generalized Gauss map of the immersed surface $M$ in the Euclidean 4-space $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$.

This representation formula for the totally real surfaces in $\mathbb{C}^2$ gives a new method of constructing surfaces in $\mathbb{R}^4$. The particular known methods are the Weierstrass-Kenmotsu formulas for surfaces with prescribed mean curvature in $\mathbb{R}^3$ and $\mathbb{R}^4$ ([Ke1, Ke2]) and their spin versions ([Ko, KL]) (cf. [AA]). The spin versions of Weierstrass-Kenmotsu formulas represent conformal immersions of surfaces by integrating a combination of the components of solutions of a similar Dirac-type equation to ours. In [HR], Hélein and Romon have given such a Weierstrass type representation formula for Lagrangian surfaces in $\mathbb{C}^2$. We note that their method does not directly imply the following known result in [CM1]: Minimal Lagrangian orientable surfaces in $\mathbb{C}^2$ can be represented as holomorphic curves by exchanging the orthogonal complex structure on $\mathbb{R}^4$, however ours implies this fact as a simple corollary.


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1. Angle functions on a surface in $\mathbb{C}^2$ and the generalized Gauss map

We consider $\mathbb{C}^2$ as the Euclidean 4-space $(\mathbb{R}^4, \langle \ , \rangle)$ with the orthonormal complex structure $J(x_1, x_2, x_3, x_4) = (-x_3, -x_4, x_1, x_2)$, that is, a complex vector $x = (x_1 + ix_3, x_2 + ix_4) \in \mathbb{C}^2$ is identified with the real vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

Let $f : M \to \mathbb{C}^2$ be a complex immersion from a Riemann surface $M$ into $\mathbb{C}^2$. For a given orthonormal basis $\{e_1, e_2\}$ of the tangent space $f^*T_pM$, we put
\[ \alpha(T_pM) = \cos^{-1}(\langle e_1, e_2 \rangle). \]
Then $\alpha(T_pM) \in [0, \pi]$ is independent of the choice of the oriented orthonormal basis of $f^*T_pM$. $\alpha(p) = \alpha(T_pM)$ is called the Kähler angle at $p \in M$. $f$ is totally real if and only if $0 < \alpha < \pi$ at all points of $M$, and in this case $\alpha : M \to (0, \pi)$ is a smooth function. $f$ is Lagrangian if and only if $\alpha \equiv \pi/2$. Regarding $e_1$ and $e_2$ as the complex column vectors in $\mathbb{C}^2$, we can obtain that $|\det(e_1, e_2)| = |\sin \alpha|$. Then, if $f$ is totally real, we can define a function $\beta : M \to \mathbb{R}/2\pi\mathbb{Z}$ by, at $p \in M$,
\[ e_1 \wedge e_2 = e^{i\beta(p)} \sin \alpha(p) e_1^C \wedge e_2^C, \]
where $e_1^C = (1, 0), e_2^C = (0, 1) \in \mathbb{C}^2$. We call $\beta$ the Lagrangian angle function for $f$.

Regarding $e_1$ and $e_2$ as the real vectors in $\mathbb{R}^4$, we can define the normalization of the real wedge product $e_1 \wedge e_2$ and identify it with the real 2-subspace $\mathcal{G}(p)$ parallel to the tangent plane $f^*T_pM$ in $\mathbb{R}^4$. So we obtain the generalized Gauss map $\mathcal{G} : M \to G_{2,2}$ of the immersed surface in $\mathbb{R}^4$, where $G_{2,2}$ stands for the Grassmann manifold of oriented 2-planes in $\mathbb{R}^4$. According to the direct sum decomposition of the real wedge product space $\Lambda^2(\mathbb{R}^4)$ between the self-dual subspace $\Lambda^2_+$ and the anti-self-dual subspace $\Lambda^2_-$, $\mathcal{G}$ can be decomposed into the self-dual part $\mathcal{G}_+$ and the anti-self dual part $\mathcal{G}_-$. We consider each of the real 3-spaces $\Lambda^3_\pm$ as the Euclidean 3-subspace $\mathbb{R}^3$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ defined by $x_1 = 0$, identifying the basis $\{E^\pm_1, E^\pm_2, E^\pm_3\}$ of $\Lambda^3_\pm$ with the standard basis $\{e_2, e_3, e_4\}$ of $\mathbb{R}^3$, where
\begin{align*}
E^+_1 &= \frac{1}{2}(e_1 \wedge e_2 \mp e_3 \wedge e_4), \\
E^+_2 &= \frac{1}{2}(e_1 \wedge e_3 \pm e_2 \wedge e_4), \\
E^+_3 &= \frac{1}{2}(e_1 \wedge e_4 \pm e_2 \wedge e_3), \\
e_1 &= (1, 0, 0, 0), \ldots, e_4 = (0, 0, 0, 1) \in \mathbb{R}^4.
\end{align*}
Then $\mathcal{G}_+$ and $\mathcal{G}_-$ are maps from $M$ to the unit 2-sphere $S^2$ in the real 3-space $\mathbb{R}^3$.

**Proposition 1.** For a totally real immersed oriented surface $M$ in $\mathbb{C}^2$, the self-dual part $\mathcal{G}_+$ of the generalized Gauss map can be represented in terms of the Kähler angle function $\alpha$ and the Lagrangian angle function $\beta$ as follows:
\[ \mathcal{G}_+ = (i \cos \alpha, e^{i\beta} \sin \alpha) : M \to S^2 \subset \mathbb{C}^2. \]
This proposition follows from the framing method below.
Assume \( f : M \to \mathbb{C}^2 \) is totally real conformal immersion with the Kähler angle \( \alpha : M \to (0, \pi) \) and the Lagrangian angle \( \beta : M \to \mathbb{R}/2\pi\mathbb{Z} \). Let \( \{e_1, e_2\} \) be an oriented orthonormal tangent frame defined on a neighborhood \( U \) in the immersed surface \( M \) in \( \mathbb{R}^4 \). Then we can choose a local orthonormal normal frame \( \{e_3, e_4\} \) on \( U \) such that

\[
(1.1) \quad e_4 = \frac{1}{\sin \alpha} (J e_1 - (\cos \alpha)e_2), \quad e_4 = \frac{1}{\sin \alpha} (J e_2 + (\cos \alpha)e_1).
\]

Since the identity component \( \text{Isom}_0(\mathbb{R}^4) \) of the isometry group of \( \mathbb{R}^4 \) acts transitively also on the oriented orthonormal frame bundle on \( \mathbb{R}^4 \), we can take a smooth map \( \mathcal{E} : U \to \text{Isom}_0(\mathbb{R}^4) \) such that \( f = \mathcal{E} \cdot 0, e_a = \mathcal{E} \cdot e_a - f \) \((a = 1, 2, 3, 4)\). We call this map \( \mathcal{E} \) the adapted framing of \( f \). Making the most of the complex structure on \( \mathbb{C}^2 \) and \( M \), we will use the Lie group \( G = \mathbb{R}^4 \times (SU(2) \times SU(2)) \) instead of \( \text{Isom}_0(\mathbb{R}^4) = G/\mathbb{Z}_2 \). Identify \( \mathbb{C}^2 \) with the linear hull \( \mathbb{R} \cdot SU(2) \) of the special unitary group \( SU(2) \) by the map

\[
\mathbf{x} = (x_1^C, x_2^C) = (x_1 + i x_3, x_2 + i x_4) \mapsto \mathbf{x} = \begin{pmatrix} x_1^C & -x_2^C \\ x_2^C & x_1^C \end{pmatrix}.
\]

So the standard vectors \( e_a \ (a = 1, 2, 3, 4) \) in \( \mathbb{R}^4 \) corresponds the following matrices:

\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: \mathbf{I}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} =: \mathbf{J}, \quad e_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

\( G \) acts isometrically and transitively on \( \mathbb{C}^2 \) by

\[
g \cdot \mathbf{x} = g \cdot \mathbf{x} (g = (g_1, g_2)) \in G = \mathbb{R}^4 \times (SU(2) \times SU(2)) \in G = \mathbb{R}^4 \times (SU(2) \times SU(2)).
\]

Now we can take the adapted framing \( \mathcal{E} : U \to G = \mathbb{R}^4 \times (SU(2) \times SU(2)) \) of \( f \) as follows:

\[
(1.2) \quad \mathcal{E} = (f, (e_-, e_+)) \quad \text{such that} \quad e_a = \mathcal{E} \cdot e_a e_a^* \quad (a = 1, 2, 3, 4).
\]

The complex structure \( J \in \text{Isom}_0(\mathbb{R}^4) \) corresponds the action of \( (0, \pm (\mathbf{I}, \mathbf{J}) \) \) in \( G \). We remark that \( e_{i+2} = J \cdot (\mathcal{E} \cdot e_i) = \mathcal{E} \cdot (J \cdot e_i) \) \((i = 1, 2)\). This fact implies that \( \mathcal{E}_+ \) can be written as follows and hence it is defined globally:

\[
(1.3) \quad \mathcal{E}_+ = \begin{pmatrix} e^{-i\beta/2} \cos(\alpha/2) & -ie^{-i\beta/2} \sin(\alpha/2) \\ -ie^{i\beta/2} \sin(\alpha/2) & e^{i\beta/2} \cos(\alpha/2) \end{pmatrix} T,
\]

where

\[
T := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SU(2),
\]

and moreover \( (0, (T, T)) \in G \) acts on \( \mathbb{C}^2 \) as

\[
T e_1 T^* = e_1, \quad T e_2 T^* = e_3, \quad T e_3 T^* = -e_2, \quad T e_4 T^* = e_4.
\]
Now, we can show that the generalized Gauss map \( \mathcal{G} = (\mathcal{G}_+, \mathcal{G}_-) : M \to S^2 \times S^2 \) of \( f \) is represented as

\[
\mathcal{G}_\pm = [(e_1 \wedge e_2)^\pm] = (E_\pm T^\ast) e_3 (E_\pm T^\ast)^* : M \to S^2 \subset \mathbb{R}^3 \cong \mathfrak{su}(2).
\]

Moreover, regarding \( S^2 \) as the extended complex plane \( \hat{\mathbb{C}} \) by the stereographic projection from the north pole \( e_3 \in S^2 \), we represent them as

\[
\mathcal{G}_\pm = \frac{P_\pm}{Q_\pm} : M \to \hat{\mathbb{C}}, \quad \text{where} \quad E_\pm T^* = \left( \begin{array}{c}
p_\pm \\
\mp q_\pm \\
t_\pm
\end{array} \right) \quad (|p_\pm|^2 + |q_\pm|^2 = 1).
\]

Then we can obtain \( \mathcal{G}_+ = \text{ie}^{i\beta} \cot(\alpha/2) \).

We also represent \( \mathcal{G}_\pm \) by means of the complex projective line \( \mathbb{C}P^1 \cong S^2 \) as

\[
\mathcal{G}_\pm = [P_\pm; Q_\pm] : M \to \mathbb{C}P^1.
\]

**Remark.** For a totally real immersed surface \( M \) in \( \mathbb{C}^2 \), we can define a map \( \mathcal{G}_0 : M \to S^1 \) by \( \mathcal{G}_0 = \text{e}^{i\beta} \), where \( \beta \) is the Lagrangian angle. Let \( \Omega \) be the volume form of \( S^1 \). The Maslov form \( \Phi = (\mathcal{G}_0)^* \Omega = (1/2\pi) d\beta \). The Maslov class is the first cohomology class defined by \( [\Phi] \in H^1(M; \mathbb{Z}) \).

2. Representation formula for totally real surfaces in \( \mathbb{C}^2 \)

Now we give the representation formula of the totally real surfaces in \( \mathbb{C}^2 \).

**Theorem.** Let \( M \) be a Riemann surface with an isothermal coordinate \( z = x + iy \).

Given two smooth functions \( \alpha : M \to (0, \pi) \) and \( \beta : M \to \mathbb{R}/2\pi \mathbb{Z} \), put

\[
U_\pm = \frac{1}{2}(i\alpha^\pm \beta \sin \alpha), \quad V = \frac{1}{2}i\beta \cos \alpha.
\]

Let \( F = (F_1, F_2) : M \to \mathbb{C}^2 \) be a solution of the Dirac-type equation

\[
\begin{pmatrix}
0 & \nabla_z \\
-\nabla_{\overline{z}} & 0
\end{pmatrix}
\begin{pmatrix}
F_1 \\
F_2
\end{pmatrix}
= \begin{pmatrix}
U_+ & V \\
-V & U_-
\end{pmatrix}
\begin{pmatrix}
F_1 \\
F_2
\end{pmatrix},
\]

and define a smooth map \( S = (S_1, S_2) : M \to \mathbb{C}^2 \) as follows:

\[
\begin{pmatrix}
S_1 \\
S_2
\end{pmatrix}
= \begin{pmatrix}
0 & \nabla_{\overline{z}} \\
-\nabla_z & 0
\end{pmatrix}
\begin{pmatrix}
U_- & V \\
V & U_+
\end{pmatrix}
\begin{pmatrix}
F_1 \\
F_2
\end{pmatrix}.
\]

If \( S \) does not vanish on \( M \), the following functions

\[
f_1 + if_3 = \exp(i\beta/2)[\cos(\alpha/2)F_1 - i\sin(\alpha/2)\overline{F_2}],
\]

\[
f_2 + if_4 = \exp(i\beta/2)[\cos(\alpha/2)F_2 + i\sin(\alpha/2)\overline{F_1}]
\]

define a conformal immersion \( f = (f_1 + if_3, f_2 + if_4) : M \to \mathbb{C}^2 \) with the Kähler angle \( \alpha \) and the Lagrangian angle \( \beta \). The induced metric on \( M \) by \( f \) takes the form

\[
f^* ds^2 = e^{2\lambda} |dz|^2, \quad e^{2\lambda} = |S_1|^2 + |S_2|^2,
\]

and the anti-self-dual part \( \mathcal{G}_- \) of the generalized Gauss map is given by

\[
\mathcal{G}_- = [-S_2; S_1] = [-S_2/S_1] : M \to S^2 \cong \mathbb{C}P^1(\mathbb{CP}^1).
Conversely, every totally real conformal immersion \( f : M \to \mathbb{C}^2 \) with the Kähler angle \( \alpha \) and the Lagrangian angle \( \beta \) is congruent with the one constructed as above.

**Proof.** For a totally real conformal immersion \( f : M \to \mathbb{C}^2 \) with the Kähler angle \( \alpha \) and the Lagrangian angle \( \beta \), we can define the smooth map \( F = (F_1, F_2) : M \to \mathbb{C}^2 \) by

\[
F = \left( \frac{F_1}{F_2} - \frac{\bar{F}_1}{\bar{F}_2} \right) := f'E^*,
\]

where \( E_+ \) is given by (1.3).

Let \( \{\omega^1, \omega^2\} \) be the dual coframe for \( \{e_1, e_2\} \), and put \( \phi = \omega^1 + i\omega^2 \). Locally we choose the isothermal coordinate \( z = x + iy \) on \( M \) such that \( f_x = e^\lambda e_1 \) and \( f_y = e^\lambda e_2 \). Hence \( \phi = e^\lambda dz \) and the induced metric on \( M \) by \( f \) is given by \( f^*ds^2 = \phi \cdot \bar{\phi} = e^{2\lambda} |dz|^2 \). We compute that

\[
\begin{align*}
\left(f_1\omega_1 + f_2\omega_2\right)
&= E_+e_1\omega_1^* + E_+e_2\omega_2^* = E_+T^*(e_1\omega^1 + e_2\omega^2)T_E^* \\
&= \left(1 \quad 0 \right)T E_+^* \phi + \left(0 \quad 0 \right)T E_+^* \bar{\phi}, \\
\left(d\bar{f}(E_+T^*) + \bar{f}(E_+T^*)^{-1}d(E_+T^*)\right)
&= \left(\frac{P_+}{Q_+} \quad 0 \right) \phi + \left(0 \quad 0 \right) \frac{-Q_-}{P_-} \bar{\phi}, \\
&= \left|\frac{P_+ - e^\lambda - VF_1 + U_\bar{F}_2}{Q_+ - e^\lambda - VF_2 - U_\bar{F}_1} \right| \frac{\bar{\phi}}{dz}.
\end{align*}
\]

Then we obtain that

\[
\begin{align*}
(F_1)_z &= \nabla F_1 - \nabla \bar{F}_1, \\
(F_2)_z &= \nabla F_2 + \nabla \bar{F}_2, \\
\quad (F_1)_z &= P_+e^\lambda - VF_1 + U_\bar{F}_2, \\
\quad (F_2)_z &= Q_-e^\lambda - VF_2 - U_\bar{F}_1 - VF_2.
\end{align*}
\]

Put \( S_1 = Q_-e^\lambda \) and \( S_2 = -P_+e^\lambda \), then we have

\[
\begin{align*}
S_1 &= (F_2)_z + U_\bar{F}_1 + VF_2 = e^{-i\beta}(f_2 + if_3)/\cos(\alpha/2), \\
S_2 &= -(F_1)_z - VF_1 + U_\bar{F}_2 = -e^{-i\beta}((f_1 + if_3)/\cos(\alpha/2)).
\end{align*}
\]

This completes the proof of Theorem. \( \square \)

Moreover, we obtain the following
Proposition 2. The spinor representation \( S = (S_1, S_2) : M \rightarrow \mathbb{C}^2 \) of \( \mathcal{G}_+ : M \rightarrow S^2 \), which is defined by (2.4), satisfies the Dirac-type equation

\[
(2.5) \quad \begin{pmatrix} 0 & \partial_\tau \\ -\partial_\tau & 0 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} V^- & V^- \\ -V^- & V^+ \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.
\]

Remark. For a Lagrangian conformal immersed surface in \( \mathbb{C}^2 \) with the Lagrangian angle \( \beta \), we obtain that \( V \equiv 0 \) and \( U_\pm = \pm \beta_\tau \). Hence the Dirac-type equations (2.1) and (2.5) are the same as the Davey-Stewartson linear problem appeared in the Konopelchenko’s representation for surfaces in \( \mathbb{R}^4 \) ([KL]). For the explicit representation formula of Lagrangian immersed surfaces in \( \mathbb{C}^2 \), see also [A]. The Hélein-Romon’s representation formula for Lagrangian surfaces in \( \mathbb{C}^2 \) ([HR]) corresponds to the method of constructing a surface by integrating a combination of the components of a solution \( S \) of the Dirac-type equation (2.5).

Here we give a simple example.

Example (Clifford torus). The rectangular torus \( T = \mathbb{C}/(a_1 \mathbb{Z} \oplus ia_2 \mathbb{Z}) \) is conformally embedded in \( \mathbb{C}^2 \) as the product of circles by the map

\[
f(x + iy) = (ia_1 e^{2\pi i x/a_1}, ia_2 e^{2\pi i y/a_2}).
\]

(When \( a_1 = a_2 = 1 \), it is called the Clifford torus.) It is well known that this torus in \( \mathbb{R}^4 \) is flat and has parallel mean curvature vector. Moreover, it is a Lagrangian surface with the Lagrangian angle \( \beta = 2\pi(x/a_1 + y/a_2) \), and hence the Maslov class \( (1, 1) \in H^1(T; \mathbb{Z}) \cong \mathbb{Z}^2 \). So this immersion \( f \) corresponds to the solution

\[
F = (F_1, F_2) = \left( \frac{1}{\sqrt{2}} (a_2 + i a_1) \right) \left( e^{\pi i (x/a_2 - y/a_1)}, i e^{-\pi i (x/a_2 - y/a_1)} \right)
\]

of the Dirac-type equation

\[
\begin{pmatrix} 0 & \partial_\tau \\ -\partial_\tau & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} (\frac{1}{a_1} - i \frac{1}{a_2}) & 0 \\ \frac{\pi}{2} (\frac{1}{a_1} + i \frac{1}{a_2}) & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.
\]

3. CURVATURES OF TOTALLY REAL SURFACES IN \( \mathbb{C}^2 \)

Let \( f : M \rightarrow \mathbb{C}^2 \) be a totally real conformal immersion with the Kähler angle \( \alpha \) and the Lagrangian angle \( \beta \), and let \( \mathcal{E} = (f, (\mathcal{E}_-, \mathcal{E}_+)) : M \rightarrow G = \mathbb{R}^4 \times (SU(2) \times SU(2)) \) be the adapted framing of \( f \) as in (1.2) and (1.3). The Gauss-Weingarten equation of the immersed surface in \( \mathbb{R}^4 \) is given by the pull-back of the Maurer-Cartan form on the Lie group \( G \) by \( \mathcal{E} \), and hence described as follows:

\[
\mathcal{E}_-^{-1} d\mathcal{E}_- = \frac{1}{2} \begin{pmatrix} i(\omega^3_1 - \omega^2_2) & -(\omega^3_3 + \omega^2_4) + i(\omega^1_3 + \omega^3_4) \\ (\omega^3_3 + \omega^2_4) + i(\omega^1_3 + \omega^3_4) & -i(\omega^1_1 - \omega^2_2) \end{pmatrix},
\]

\[
\mathcal{E}_+^{-1} d\mathcal{E}_+ = \frac{1}{2} \begin{pmatrix} -i(\omega^3_1 + \omega^2_2) & -(\omega^3_3 - \omega^2_4) + i(\omega^1_3 - \omega^3_4) \\ (\omega^3_3 - \omega^2_4) + i(\omega^1_3 - \omega^3_4) & i(\omega^1_1 + \omega^2_2) \end{pmatrix},
\]

where \( \omega^a_3 \) are the connection forms on \( M \) defined by \( \omega^a_3 = \langle e_a, \nabla e_b \rangle \) for the Levi-Civita connection of \( (\mathbb{R}^4, \langle \cdot, \cdot \rangle) \). Moreover, from (1.1), we obtain that

\[
(3.1) \quad \omega^3_1 = \omega^1_1 \quad \text{locally} \quad \omega^3_2 = \omega^1_3 - \cot \alpha (\omega^3_3 + \omega^2_4), \quad \omega^3_3 = \omega^1_3 + \omega^3_4 - d\alpha.
\]
Hence the Gauss-Weingarten equation of the totally real immersed surface in \( \mathbb{C}^2 \) is written as the following matrix equation

\[
\begin{align*}
\mathcal{E}_-^{-1}d\mathcal{E}_- &= T^* \begin{pmatrix} i(\rho - \cot \alpha \eta) & -\bar{w} \\ \psi & -i(\rho - \cot \alpha \eta) \end{pmatrix} T, \\
\mathcal{E}_+^{-1}d\mathcal{E}_+ &= T^* \begin{pmatrix} -i\cot \alpha \eta & -\eta - (i/2)\alpha \\ \eta - (i/2)\alpha & i\cot \alpha \eta \end{pmatrix} T,
\end{align*}
\]

where \( \rho = \omega_1^2, \psi = (1/2)(\omega_1^2 - \omega_2^2) \) and \( \eta = (1/2)(\omega_1^3 + \omega_2^3) \). Combining (2.3) with (3.3), we obtain

\[
\eta = \frac{1}{2} \sin \alpha d\beta.
\]

The second fundamental form of \( f \) is given by

\[
\Pi = h^i_j \omega^i \otimes \omega^j \otimes e_3 + h^i_j \omega^i \otimes \omega^j \otimes e_4,
\]

where \( h^i_j = \omega^i(e_j) = \omega^i(e_i) \) and \( h^i_j = \omega^i(e_j) = \omega^i(e_i) \) (\( i, j = 1, 2 \)). Moreover, from the second equation in (3.1), these components satisfy

\[
h^1_{11} = h^1_{12} + d\alpha(e_1), \quad h^1_{12} = h^3_{22} + d\alpha(e_2).
\]

Put \( h^3 = (1/2)(h^1_{11} + h^3_{22}) \) and \( h^4 = (1/2)(h^1_{11} + h^3_{22}) \). The mean curvature vector \( \overline{H} \) of \( f \) is given by

\[
\overline{H} = h^3 e_3 + h^4 e_4 = \frac{1}{2}(h^3 + ih^4)(e_3 - ie_4) + \frac{1}{2}(h^3 - ih^4)(e_3 + ie_4).
\]

The 1-form \( \eta \) is also given by

\[
\eta = \frac{1}{2} \left( h^1_{11} + h^3_{22} \right) \omega^1 + \frac{1}{2} \left( h^1_{12} + h^3_{22} \right) \omega^2
\]

\[
\begin{align*}
&= h^3 \omega^1 + h^4 \omega^2 + \frac{1}{2} \left( d\alpha(e_2) \omega^1 - d\alpha(e_1) \omega^2 \right) \\
&= \frac{1}{2} \left( (h^3 - ih^4) \phi + (h^3 + ih^4) \bar{\phi} \right) + \frac{1}{2} \{ \alpha_z dz - \alpha \bar{\omega} \bar{dz} \}.
\end{align*}
\]

From (3.4) and (3.5), we obtain that \( h^3 - ih^4 = -2e^{-\lambda} U_- \). Namely, the mean curvature vector \( \overline{H} \) has the representation of

\[
\overline{H} = i e^{-2\lambda} \{ -U_- \cot(\alpha/2) f_\pm + U_- \tan(\alpha/2) f_\mp \},
\]

and the mean curvature \( H = |\overline{H}| \) is given by

\[
H = 2e^{-\lambda}|U_-|.
\]

It follows from (3.2) combined with

\[
\mathcal{E}_+ T^* = e^{-\lambda} \begin{pmatrix} -S_2 & -S_1 \\ S_1 & -S_2 \end{pmatrix}
\]

that

\[
\psi = e^{-2\lambda}(S_1 dS_2 - S_2 dS_1),
\]
\[ \rho = \frac{1}{2} \left( (\cos \alpha) d\beta - i e^{-2\lambda} \left( \overline{S}_1 dS_1 + \overline{S}_2 dS_2 - S_1 d\overline{S}_1 - S_2 d\overline{S}_2 \right) \right), \]
\[ d\rho = -\frac{1}{2} (\sin \alpha) d\alpha \wedge d\beta + i \psi \wedge \overline{\psi}. \]

We note that the \((0,1)\)-part \(\psi' dz\) of \(\psi = \psi' dz + \psi'' d\overline{z}\) coincides with \(-\left(1/2\right) (h^3 - \overline{h}^3) d\overline{z} = U \cdot d\overline{z}\). The Gauss curvature \(K\) of \(f\) is given by \(d\rho = -(1/2) K \phi \wedge \overline{\phi}\), and hence
\[ K = -e^{-2\lambda} \{ i(\alpha_z \beta_z - \alpha \overline{\beta}_z) \sin \alpha + 2(|\psi'|^2 - |\psi''|^2) \}. \]

**Proposition 3.** If a totally real immersed oriented surface \(M\) in \(\mathbb{C}^2\) is minimal, the Kähler angle \(\alpha\) and Lagrangian angle \(\beta\) satisfies the partial differential equation
\[ i \alpha_z - \beta_z \sin \alpha = 0, \]
and hence the Gauss curvature is given by
\[ K = -2e^{-2\lambda} (|\alpha_z|^2 + |\psi'|^2). \]

**Corollary.** If a totally real immersed oriented surface \(M\) in \(\mathbb{C}^2\) with either constant Kähler angle or constant Lagrangian angle is minimal, then the other angle is also constant and the map \(F = (F_1, F_2) : M \to \mathbb{C}^2\) defined as in (2.2) is holomorphic. Namely, such a surface can be represented as a holomorphic curve by exchanging the orthogonal complex structure on \(\mathbb{R}^4\).

So, this corollary implies the known result for minimal Lagrangian surfaces in \(\mathbb{C}^2\) mentioned as in Introduction (Chen-Morvan [CM1]).

**References**


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