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The concept of symmetry goes well beyond that of simple geometric symmetries. From the fair organisation of the final stages of soccer competitions to the solving of equations, via the icosahedral game and Morley’s theorem, we’ll discover the multiple aspects of this concept.

The purpose of this article is to introduce the reader to the mathematical notion of symmetry, by way of a few illustrative examples.

To demonstrate the ubiquity of this concept, as a mathematician understands it, we’ll start by evoking the connection between the final stages of soccer competitions and the way we solve quartic equations.

As we move to equations of higher degree, we describe the icosahedral game and the ‘icosions’, defined in the nineteenth century by the Irish mathematician, William Hamilton.

We finish with a commentary on a theorem in geometry, proved by Frank Morley around 1899, where the symmetry of an equilateral triangle arises miraculously from an arbitrary triangle, by taking the intersection of consecutive ‘trisectors’ (the two straight lines which divide an angle into three equal parts). In 1988, I gave an algebraic formulation of this result and a proof which we shall see below.

The final stages of a soccer cup

Let’s start with the organisation of a soccer competition, for example, the
millennial European Cup. During the final stage, teams were placed in pools of four, and arrived at an order of merit within each pool. For example, one group consisted of Denmark, France, Holland, and the Czech Republic (here abbreviated to D, F, H, and C).

To arrive at an order of merit fairly, each team had to play each of the three others, which meant that games had to played on three days. For instance, when D and F met, then H and C could meet on the same day and so three days were enough for all possible configurations of matches.

In that European Cup, the matches were, FD and CH on the first day, FC and DH on the second, and FH and DC on the third. Intuitively we can see that this is a fair procedure, because none of the teams has an advantage. One checks indeed that, if we arbitrarily permute certain teams, for instance, if we interchange D and H, this amounts to a simple permutation of the first and third days.

We can visualise the symmetry which is at work by putting the letters D, F, C, H (representing the teams) at four points of the plane. The line joining two points represents a match between the corresponding teams. Each of the three days corresponds to the point of intersection of the lines representing the matches on that day. Thus the matches FD and CH are associated with the intersection of the lines FD and CH. Continuing for the two other pairs, the second day is at the intersection of the lines FC and DH and the third is at the intersection of the lines FH and DC.

The figure thus constructed, formed by four points and six lines, is called a complete quadrilateral. It is perfectly symmetrical (in an abstract sense, even if none of the usual geometric symmetries — symmetries with respect to a point or a line — are present), because each of the four points F, D, C, and H plays exactly the same role as the others, and the same is true for the points of intersection representing the three days.

Having visualised this complete quadrilateral, we can also give an algebraic formulation of the symmetry under discussion. In the following way: the quantity a, determined from the four numbers a, b, c, and d by the formula a = ab + cd only takes three values altogether when we permute a, b, c, and d. The other values are \( \beta = ac + bd \) and \( \gamma = ad + bc \).

Resolution of quartic equations by radicals

This surprising symmetry underlies the general method for solving quartic equations ‘by radicals’. The solution amounts to expressing the zeros a, b, c, and d of the polynomial \( x^4 + nx^3 + px^2 + qx + r = (x - a)(x - b)(x - c)(x - d) \) as a function of the coefficients n, p, q, and r using the extraction of roots.

To understand this assertion, we must go back in time and look at the solution of equations of degree lower than four.

Though the technique of solving quadratic equations goes back to the most ancient (Babylonians, Egyptians...), it wasn't extended to cubic equations until much later and was published only in 1545 by Girolamo Cardano in Chapters 11 to 23 of his book Ars magna sive de regulis algebraici. In fact, it wasn't realised until the 18th century that the key to the solution by radicals of the cubic equation \( x^3 + nx^2 + px + q = 0 \), with zeros a, b, and c (the ‘roots’), lay in the existence of a polynomial function \( f(a, b, c) \) of a, b, and c, which takes only two different values under the action of the six possible permutations of a, b, and c.

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### 2. SYMMETRY AND THE SOLUTION OF CUBIC AND QUARTIC EQUATIONS BY RADICALS

Solving polynomial equations by radicals requires the construction of auxiliary functions of the roots, which display symmetries when the roots are permuted.

1) **The cubic equation:**

\[ x^3 + 3px + 2q = (x - a)(x - b)(x - c) = 0. \]

Reduced equation: \( x^3 + 2qx - p^2 = (x - \alpha)(x - \beta). \)

**Three roots:** a, b, c.

**Auxiliary functions:**

\[ \alpha = \frac{(a + b + c)^2}{3}, \quad \beta = \frac{(a + b + c)^2}{3}, \]

\[ j = (\frac{\beta}{\alpha})^{\frac{1}{3}}/2, \]

\[ \lambda = \frac{\beta}{\alpha} \] with \( \beta \) being a square root of \( -1 \), so that \( \lambda = 1 \) and \( j^2 + j + 1 = 0. \)

**Symmetry:**

The symmetry given by any permutation of a, b, and c leaves the set of auxiliary functions \( \{\alpha, \beta\} \) globally invariant.

**Solutions:**

Let \( u = \frac{\sqrt{\alpha}}{\lambda} \) and \( v = \frac{\beta}{\lambda} \) such that \( uv = -p. \)

Then

\[ a = u + v, \]
\[ b = \frac{u^2}{j^2} + jv, \]
\[ c = ju^2 + j^2v. \]

2) **The quartic equation:**

\[ x^4 + px^2 + qx + r = (x - a)(x - b)(x - c)(x - d) = 0 \]

Reduced equation: \( x^4 + px^2 - 4rx + (4pr - q^2) = (x - \alpha)(x - \beta)(x - \gamma) = 0 \)

**Four roots:** a, b, c, d.

**Three auxiliary functions:**

\[ \alpha = ab + cd, \]
\[ \beta = ac + bd, \]
\[ \gamma = ad + bc \]

**Symmetry:**

The symmetry given by any permutation of a, b, c, and d leaves the set of auxiliary functions \( \{\alpha, \beta, \gamma\} \) globally invariant.

**Solutions:**

1) ** Knowing \( ab = cd \) and \( r = (ab)(cd), \) gives the products ab and cd.
2) ** If \( ab = cd, \) the system \((a+b)+(c+d) = 0\) and \((ab+cd) = (a+b)(c+d) = r, \) gives a+b and c+d.
3) ** Knowing ab and a+b gives a and b. Similarly for c and d.
Cardano’s method amounts to writing $\alpha = [(1/3)(a + b + c)^2]^3$, where the number $j$ is the first cube root of unity, namely $(-1 + i\sqrt{3})/2$, with $i$ denoting the square root of $-1$. The cyclic permutation taking $a$ to $b$, $b$ to $c$, and $c$ to $a$ simultaneously, clearly leaves $\alpha$ invariant and the only other value obtained from the six possible permutations of $a$, $b$, and $c$ is $\beta = [1/3(a + c + b^2)]^3$, where, for example, $b$ and $c$ have been transposed. As the set consisting of the two numbers $\alpha$ and $\beta$ is invariant under all the permutations of $a$, $b$, and $c$, it is easy to express the quadratic equation whose roots are $\alpha$ and $\beta$, in terms of the coefficients of the initial equation $ax^3 + bx^2 + cx + d$: it is $x^2 + 2q/x - p^3 = (x + q + s)(x + q - s)$, where $s$ is one of the square roots of $p^3 + q^2$ and where we have also rewritten the initial equation in the equivalent form $x^3 + 3px + 2q$ (without the square term) by an appropriate translation of the roots. We introduced the factors 2 and 3 to simplify the formulas.

A simple calculation then shows that each of the roots $a$, $b$, and $c$ of the initial equation can be expressed as the sum of one of the three cube roots of $a$ and one of the three cube roots of $\beta$. These three choices are connected by the fact that their product must equal $-p$ (so there are only three pairs of choices to account for, which is reassuring, instead of the nine possibilities which we might have thought of a priori).

These formulas logically required the use of complex numbers. Indeed, even in the case where the three roots are real numbers, $p^2 + q^2$ can be negative, and then $\alpha$ and $\beta$ are necessarily complex.

The solution of cubic equations, described above, took a long time to reach its final form (we know that at least one of the special cases was being worked out by Scipione del Ferro between 1500 and 1515). On the other hand, the solution of quartic equations followed quickly, because it is also in the Ars magna (Chapter 39), where Cardano attributes it to his secretary Ludovico Ferrari, who apparently found it between 1540 and 1545 (René Descartes would publish another solution in 1637). And it is this solution which brings us back to the first symmetry we met, that of the organisation of soccer finals, the complete quadrilateral, and the expression $ab + cd$. Here again, we can start with a polynomial with no term in $x^3$ (using the same technique as before), namely $x^4 + px^2 + qx + r = (x - a)(x - b)(x - c)(x - d)$. The set of three numbers $\alpha = ab + cd$, $\beta = ac + bd$, and $\gamma = ad + bc$, is invariant under each of the 24 permutations acting on $a$, $b$, $c$, and $d$. The numbers $\alpha$, $\beta$, and $\gamma$ are thus the roots of a cubic equation whose coefficients are easily expressed as a function of $p$, $q$, and $r$. A calculation shows that the polynomial $(x - a)(x - b)(x - c)$ equals $x^3 - px^2 - 4rx + (4pr - q^2)$. It can thus be decomposed, as we have seen above, to yield $\alpha$, $\beta$, and $\gamma$. Indeed, it’s enough to find one of these roots, $\alpha$ say, to determine $a$, $b$, $c$, and $d$ (because we then know the sum $\alpha$ and the product $r$ of the two numbers $ab$ and $cd$, thus giving these numbers via a quadratic equation; all we then have to do is to exploit the equations $(a + b) + (c + d) = 0$ and $ab(c + d) + cd(a + b) = -q$ to determine $a + b$ and $c + d$ and hence, finally, $a$, $b$, $c$, and $d$).

The fundamental role of the permutations of the roots $a$, $b$, $c$ ... and of the auxiliary quantities $\alpha$, $\beta$... was brought to light by Joseph Louis Lagrange in 1770 and 1771 (published in 1772) and, to a lesser degree, by Alexandre Vandermonde in a memoir published in 1774, but written around 1770, as well as by Edward Waring in his Meditationes algebraicae of 1770 and by Francesco Malfatti. Today we rightly call those auxiliary quantities ‘Lagrange resolvents’.

The resolvents are not unique (we could equally well have put $a = (a + b - c - d)^2$ in the case of the quartic equation, which corresponds to Descartes’ method), but they are the key to all general solutions by radicals.

### Abel and Galois

Of course, mathematicians wanted to go further: Descartes certainly tried, and he wasn’t alone. The next step would clearly be that of the quintic equation. This was found to pose apparently insuperable obstacles, and since the time of Abel and Galois (who obtained their results around 1830) we have known why the search was in vain.

In all the previous cases, we were able to find a family of $n - 1$ numbers $\alpha$, $\beta$, $\gamma$... determined as polynomials of the $n$ roots $a$, $b$, $c$, and $d$... (with $n$ less than or equal to 4). This family was globally invariant under the permutations of the roots. More precisely, if we let $S_n$ denote the group of bijections of the set $\{a, b, c, d\}$ with itself, what can be done for $n$ strictly less than 5 is to define a mapping of $S_n$ onto $S_{n-1}$ which preserves compositions of permutations.

Since the beginning of the nineteenth century, we have known that this is impossible for $n$ larger than 4. The same is true for a (non-constant) composition-preserving mapping of the group $A_n$ of permutations of even order (the products of an even number of transpositions) onto the group $S_m$ when $m$ is less than $n$ and $n$ is greater than 4. This shows that Lagrange’s method can’t be extended to the case $n = 5$ or to higher values of $n$, but is, of course, not enough to show that a solution by radicals is impossible for the general equation of degree 5 or higher: other, more general, methods might succeed where
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4. THE GROUPS $A_4$ AND $A_5$

A) THE GROUP $A_4$

1) The group $A_4$ is the group of the even permutations of the four letters $(a,b,c,d)$. This group is generated by the permutations $s = \begin{bmatrix} a & b & c & d \\ a & b & d & c \end{bmatrix}$, which maps $a$ to $b$, $b$ to $a$, $c$ to $d$, and $d$ to $c$, and $t = \begin{bmatrix} a & b & c & d \\ a & c & d & b \end{bmatrix}$, which maps $a$ to $b$, $b$ to $d$, $c$ to $d$, and $d$ to $b$. They obey the rules: $s^2 = 1$, $t^2 = 1$, and $(st)^3 = 1$.

2) There's a geometric representation of the group $A_4$: it's the group of rotations preserving the regular tetrahedron $a,b,c,d$.

$s$ is represented by the symmetry with respect to the line joining the midpoints of $ab$ and $cd$. $t$ is represented by a rotation through an angle $2\pi/3$ about the axis of the tetrahedron passing through $a$.

$st = \begin{bmatrix} a & b & c & d \\ b & d & c & a \end{bmatrix}$ is the rotation through $2\pi/3$ about the axis through $c$.

The patient reader may check that the rules of simplification $s^2 = 1$, $t^2 = 1$, $(st)^3 = 1$ form a presentation of the group, that is, together with the group law, they are enough to show that there are only 12 distinct "words" formed out of the letters $s$ and $t$.

B) THE GROUP $A_5$

1) The group $A_5$ is the group of the even permutations of the five letters $(a,b,c,d,e)$. This group is generated by the permutations $u = \begin{bmatrix} a & b & c & d & e \\ b & a & c & d & e \end{bmatrix}$, which maps $a$ to $b$, $b$ to $a$, $c$ to $d$, $d$ to $c$, and $e$ to $e$, and $v = \begin{bmatrix} a & b & c & d & e \\ e & b & a & d & c \end{bmatrix}$ which maps $a$ to $e$, $b$ to $b$, $c$ to $a$, $d$ to $d$, and $e$ to $c$. These obey the rules: $u^2 = 1$, $v^3 = 1$, $(uv)^5 = 1$ (of course, $u$ and $v$ do not commute).

2) This group has 60 elements and is isomorphic to the group of rotations of a regular dodecahedron.

Thus $u$ is one of the 15 symmetries with respect to a line joining the midpoints of two edges which are symmetrically placed about the centre.

Similarly, $v$ is one of the 20 rotations through $2\pi/3$ about a line joining two vertices which are symmetrically placed about the centre.

C) PRESENTATION OF $A_5$

The patient reader may check that the simplifying rules $u^2 = 1$, $v^3 = 1$, $(uv)^5 = 1$ together with the group law, are enough to show that there are only 60 distinct words formed out of the letters $u$ and $v$. We start by putting $s = u$ and $t = k^2 uk$ (where $k = uv$), and then show that $s$ and $t$ are generators of $A_5$, that is, they obey the simplifying rules $s^2 = 1$, $t^2 = 1$, $(st)^3 = 1$. We then show, using the simplifying rules above, that every word formed from the letters $u$ and $v$ can be written in the form $k^r$, where $r$ equals 0, 1, 2, 3, 4 and $h$ is a word formed from the letters $s$ and $t$.

As there are precisely 12 distinct words $h$, we see that the group $A_5$ is given by the above relations.

D) $A_5$: A GROUP OF MATRICES

1) Let $F_5$ denote $\mathbb{Z}/5\mathbb{Z}$, the field of remainders modulo 5. In this field, $4 + 2 = 1$, $3 + 2 = 0$, $4 \times 2 = 3$, $3 \times 2 = 1$, etc.

2) We represent $u$ and $v$ as the following mappings of the projective space $P_1(F_5)$. This projective space contains the five points of $F_5$, together with a point "at infinity", denoted by $\infty$. Let us put $u(z) = -1/z$, for a point $z$ in $P_1(F_5)$. Clearly, $u^2 = 1$, that is $u^2 = 1$. Now let us put $v(z) = -1/(z+1)$ : we can check that $v^2 = 1$. We see that we have a representation of $A_5$ because $k = uv$ is given by $k(z) = z + 1$ and $k^2(z) = z^2 + 1$, so that $k^2 = 1$ since 5 equals 0 in $F_5$.

3) We give a matrix representation of the elements $u$ and $v$. If $a,b,c,d$ are elements of $F_5$ with $ad - bc = 1$, we associate the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with the mapping of $P_1(F_5)$, given by:

$$f(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}z$$

The group of mappings obtained in this way is called $PSL(2,F_5)$, for "Projective Special Linear" group of $F_5$.

Then $u, v, k$, and $t, y$ are represented by the matrices:

$$u = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \quad v = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} , \quad k^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \quad t = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$$
Lagrange had failed. Nowadays, because of Abel and Galois, we know that even such a generalisation is impossible. Many of the most celebrated mathematicians were interested in this fundamental and complex problem, among them Leonhard Euler, who returned to it several times and, above all, Karl Friederich Gauss (1801) and Louis-Augustin Cauchy (1813).

We'll stay with the case of the quintic equation. Descartes, for one, was convinced that no formula like that of Cardano could be found. In 1637 Descartes suggested a graphical solution – using the intersection of circles and cubic curves – which he had invented for the purpose. From 1799 to 1813 (the date of publication of his Riflessioni intorno alla soluzione delle equazioni algebriche generali), Paolo Ruffini published diverse attempts at proofs, each of them more refined than the last, attempting to demonstrate the impossibility of solving the general quintic equation by radicals. He had the correct idea of assigning, to each rational function of the roots, that group of permutations of the roots which left the function invariant. However, he assumed incorrectly that the radicals involved in solving the equation necessarily had to be rational functions of the roots.

In the event, it was 1824 before Niels Abel, in his Mémoire sur les équations algébriques, justified Ruffini's intuition. Abel, having at first thought, on the contrary, that he had found a general method of solution, proved the impossibility of solving the general quintic by radicals in his 1826 Mémoire sur une classe particulière d'équations résolubles algébriquement, in which he sketched a general theory which would only be fully worked out by Galois, towards 1830. Galois' work inaugurated a new era in mathematics where calculations gave way to the consideration of their potential, and concepts, such as those of abstract group or of algebraic extension, occupied the foreground.

Galois' great insight was to associate to an arbitrary equation a group of permutations which he defined in these terms:

Let an equation be given which has roots $a, b, c \ldots m$. It will always have a group of permutations of the letters $a, b, c \ldots m$ with the following properties:

1. every function of the roots which is invariant by substitutions of this group, will be rationally determined;
2. conversely, every rationally determined function of the roots will be invariant under these substitutions.

Galois then studied how this group of 'ambiguities' could be modified by adjoining auxiliary quantities thenceforth considered 'rational'. Solving an equation by radicals was then reduced to solving its Galois group.

The impossibility of reducing the quintic equation to equations of lower degree comes from the 'simplicity' of the group $A_5$ of the sixty even permutations of the five roots $a, b, c, d, e$ of the quintic. We say that an abstract group is 'simple' if there is no non-constant composition-preserving mapping of the group into a smaller group.
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The group $A_5$ is the smallest non-commutative group which is simple, and it arises very frequently in mathematics. This group can be described very economically: it is generated by two elements $u$ and $v$ satisfying the relations $u^2 = 1$, $v^3 = 1$ and $(uv)^5 = 1$, which gives us an excuse to move to Hamilton’s icosions.

Hamilton’s icosions

After discovering quaternions, William Hamilton tried, in 1857, to construct a new algebra of generalised numbers, which he called icosions. Two of them, denoted by $u$ and $v$, which Hamilton termed ‘non-commutative roots of unity’, were to satisfy $u^2 = 1$, $v^3 = 1$ and $(uv)^5 = 1$. A childishly simple calculation shows that, if $uv = vu$, then we have $v = uv^2v = u^2v$, $v^3 = uv$, $u = vu^2 = v^3 = 1$. So we can’t represent $u$ and $v$ as elements of the groups $S_n$, for $n$ less than or equal to 4. To represent $u$ and $v$ as elements of the group $A_5$ of even permutations of the five letters $a$, $b$, $c$, $d$, and $e$, it is enough to put $u = (b,a,d,c,e)$, the permutation which fixes $e$ but exchanges $a$ with $b$ and $c$ with $d$, and to put $v = (e,a,b,d,c)$, the permutation which fixes $b$ and $d$ but changes $a$ to $e = v(a)$, $c$ to $a = v(c)$, and $e$ to $c = v(e)$. The product $uv$ is then the cyclic permutation $(abcde)$ which is indeed of order 5. In fact, we could similarly represent $u$ and $v$ in altogether 120 separate (but pairwise isomorphic) ways as elements of $A_5$.

The group $A_5$ is isomorphic to the group of rotations preserving a regular icosahedron or, which amounts to the same thing, a regular dodecahedron (these are the two most interesting of the five platonic solids, whose other members are the regular tetrahedron, the cube and the regular octahedron, formed by the six centre of the faces of the cube. These are the only regular convex polyhedra which exist in our usual three-dimensional space).

To construct the isomorphism referred to above, it is enough to associate $u$ with one of the 15 rotations of order two (a symmetry whose axis is one of the 15 lines joining the midpoints of parallel edges) and to associate $v$ with one of the 20 rotations of order three (whose axis of symmetry connects one of ten pairs of two diametrically opposite vertices of the dodecahedron, or the centres of the two parallel faces of the icosahedron) in such a way that the product $uv$ is one of the members of $A_5$.

8 POINCARÉ’S MODEL

To get a geometric understanding of the group generated by two elements $u$ and $v$ and presented by the relations $u^2=1$, $v^3=1$, and $(uv)^5=1$, we start by considering the two first relations ($u^2=1$, $v^3=1$). The group thus generated is the group PSL(2, Z) which can be understood by looking at its action on an infinite tree $T$ in which three edges exit from each vertex. The third relation $(uv)^5=1$ can then be understood by identifying the tree $T$ with the universal covering of the graph in box 8 below. The universal covering, in the sense of Poincaré, is obtained by considering all the paths which follow the edges of the regular dodecahedron.

The infinite tree $T$ is represented by two models of non-Euclidean geometry: Klein’s model (A) and Poincaré’s model (B). In each model, the set of points of plane geometry lie in the interior of a disk. In Klein’s model, the “straight lines” are arcs of circles orthogonal to the edge $JJ’$ of the octahedron, and where the factor 2 arises in comparing (A) with (B).

The group PSL(2, Z) is represented by isometries of non-Euclidean geometry. A presentation of this group is given by the relations $u^2=1$ and $v^3=1$. The element $u$ is generated by symmetry about the origin $O$, and the element $v$, by the non-Euclidean rotation with centre $J$ and angle $2\pi/3$. We operate on the edge $JJ’$ by the transformations represented by the words (such as $uvuuuuuuuv…$) whose letters are the elements $u$ and $v$. This gives exactly the tree $T$ of the universal covering of the graph of Hamilton’s icosion game. In particular, each Hamiltonian path is a point of the universal covering.

We get the dodecahedron by identifying the edges of the tree $T$ which are congruent modulo 5. This congruence means that we pass from one edge to another by a non-Euclidean isometry given by an element $[a,b]$ of the group PSL(2, Z) satisfying $a = 1$ (modulo 5), $b = 0$ (modulo 5), and $c = 0$ (modulo 5). The quotient by $G$ is precisely the graph formed by the edges of the dodecahedron. The quotient group PSL(2, Z)/G is the group PSL(2, $F_5$) of box 4.
the 24 rotations of order five (whose axis of symmetry connects one of the six pairs of centres of parallel faces of the dodecahedron, or pairs of diametrically opposed vertices of the icosahedron). The 60 rotations preserving either solid can be expressed simply as products of the generators $u$ and $v$. Even though the two icosions $u$ and $v$ generate the group $A_5$ and satisfy the relations $u^2 = 1$, $v^3 = 1$ and $(uv)^5 = 1$, it’s not immediate that these relations constitute a presentation of the group, that is, it’s not immediate that every relation between $u$ and $v$ follows from these. There are two ways, algebraic or geometric, of showing this (boxes 4 and 7).

The graph of the edges of the dodecahedron, which has the same symmetries as that of the icosahedron, gave rise to Hamilton’s ‘icosahedral game’ which he also called the ‘game of non-commutative roots of unity’. This game is the first example of what is now called the search for a Hamiltonian circuit, which is a very important problem as that of the icosahedron, which has the same symmetries as the dodecahedron, or pairs of diametrically opposed vertices of the icosahedron once and once only: to finish at a vertex which is joined by an edge to the starting point. A re-

8. THE GAME OF ICOSIONS

In his book *Avec des nombres et des lignes* Sainte-Lagué brought the game of icosions, invented by the French mathematician Frank Morley was one of the first university teachers in America. At the end of the nineteenth century, while pursuing research into families of cardioids tangent to the three sides of a given triangle, he discovered the following property: the three pairs of trisectors of the angles of a triangle (that is, the straight lines which divide the interior angles into three equal parts) intersect in six points, of which three are vertices of an equilateral triangle.

The original proof is quite difficult and depends on ingenious calculations based on a profound mastery of analytic geometry. There are now many proofs of this result, as well as generalisations which produce 18, or 27 (or even more) equilateral triangles which can be constructed from the 108 points of intersection of the 18 trisectors obtained from the original ones by rotations of $\pi/3$. These proofs include ones by trigonometric calculation as well as purely geometrical ones, such as that given by Raoul Bricard in 1922.

There is a proof which is entirely different, which illustrates the result from an interesting angle, because it lets us extend the result (a priori entirely Euclidean) to the geometry of affine lines over an arbitrary field $k$. The purely algebraic result, which includes and extends the trisector property, is so general that its proof becomes a simple verification (a very general result is often easier to prove than a particular case, because the generality can reduce the number of hypotheses).

The statement is as follows:

If $G$ is the affine group of a commutative field $k$ (that is, the group of mappings $g$ of $k$ into $k$ which can be written in the form $g(x) = ax + b$, where $a = a(g)$ is non-zero), then for each triple $(f, g, h)$ of elements of $G$ such that $j = af(gh)$ is not the identity and such that $fg, gh$ and $hf$ are not translations, the following two assertions are equivalent:

a) $f^3g^3h^3 = 1$ (the identity mapping $1(x) = x$);

b) $j^3 = 1$ and $a + j\beta + j^2\gamma = 0$, where $a$ is the unique fixed point of $fg, \beta$ that of $gh$, and $\gamma$ that of $hf$.

We need to show how this very abstract property helps us better understand (and, at the same time, prove) Morley’s theorem. We shall take $k$ to be the (field of) complex numbers. Its affine group is that of the direct similarities and has the rotations as a subgroup (precisely when $a$ has modulus 1). We let $f, g$ and $h$ be the rotations about the three vertices of the triangle with each angle of rotation being two-thirds of the relevant angle of the triangle. Thus $f$ is the rotation with centre $A$ and angle $2a$ (the internal angle at $A$ being denoted by $3a$), $g$ is that with centre $B$ and angle $2b$ and $h$ that with centre $C$ and angle $2c$. The product of the cubes $f^3g^3h^3$ is 1, because, for instance, $f^3$ is the product of the two symmetries with respect to the sides of the angle at $A$, so that these symmetries simplify pairwise in the product $f^3g^3h^3$.

The equivalence above shows us that $a + j\beta + j^2\gamma = 0$, where $a$, $\beta$, and $\gamma$ are the fixed points of $fg$, $gh$, and $hf$, and where the number $j = af(gh)$ is the first cube root of unity, which we have already met in the course of this article. The relation $a + j\beta + j^2\gamma = 0$ is a well-known characterisation of equilateral triangles (it can be written in the form $(a - \beta)/((\gamma - \beta) = -j^2$, which shows that we pass from the vector $\beta\gamma$ to the vector $\beta\alpha$ by a rotation through $\pi/3$).

An old recipe, well-known to those thoroughly trained in the rigours of classical geometry, shows that the point $a$, defined by $f(g(a)) = a$, is none other than the intersection of the trisectors of the angles $A$ and $B$ ly-
Morley’s theorem states that the three meeting points $\alpha$, $\beta$, $\gamma$ of the trisectors of an arbitrary triangle $ABC$, as indicated in the diagram, form an equilateral triangle. (n red).

$\alpha$, $\beta$, $\gamma$ are the three rotations about the vertices of the given triangle, through angles which are two thirds of the angles at the vertices.

So $f$ is the rotation through $2a$ about $A$, $g$ is the rotation through $2b$ about $B$, and $h$ is the rotation through $2c$ about $C$. We shall look at the properties of these rotations. The rotation $g$ takes the point $\alpha$ to $\alpha'$, which is the reflection of $\alpha$ through $AB$. The rotation $f$ takes the point $\alpha$ to $\alpha$: so $\alpha$ is a fixed point of the product of rotations $fg$. Similarly, $\beta$ is a fixed point of the product of rotations $gh$, and $\gamma$ is a fixed point of the product of rotations $hf$.

Now we consider the product $f^3g^3h^3$. The rotation $f^3$ through an angle $6a$ about $A$ is the product $s(AC)s(AB)$ and the reflection $s(AB)$ in the side $AB$, and the reflection $s(AC)$ in the side $AC$. Similarly, $g^3$ is the product $s(BC)s(AB)$ and $h^3$ is the product $s(BC)s(AC)$. So $f^3g^3h^3 = s(AC)s(AB)s(BC)s(BC)s(AC)$. As the square of a reflection is equal to 1 this, together with the algebraic theorem, shows that the triangle is equilateral.

9. MORLEY’S THEOREM

Apology

Unfortunately, a very disturbing mistake occurred during the printing process of the recent issue 53 of the Newsletter. In the feature article 25 years of Kneser’s conjecture by Marc de Longueville, p.16 – 19, all minus-signs in the formulas and also the line breaks and signs disappeared. The Newsletter wishes to apologise sincerely for the inconvenience that this has caused for the readership; its apologies go also to the author. A corrected version of this article can be read and downloaded as a PDF-file from the following URL http://www.emis.de/newsletter/current/current9.pdf.