Vladikavkaz Mathematical Journal
2023, Volume 25, Issue 1, P. 48–63

УДК 517.982.22
DOI 10.46698/o3961-3328-9819-i

ON STABILITY OF RETRO BANACH FRAME WITH RESPECT TO b-LINEAR FUNCTIONAL IN n-BANACH SPACE

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Abstract. We introduce the notion of a retro Banach frame relative to a bounded b-linear functional in n-Banach space and see that the sum of two retro Banach frames in n-Banach space with different reconstructions operators is also a retro Banach frame in n-Banach space. Also, we define retro Banach Bessel sequence with respect to a bounded b-linear functional in n-Banach space. A necessary and sufficient condition for the stability of retro Banach frame with respect to bounded b-linear functional in n-Banach space is being obtained. Further, we prove that retro Banach frame with respect to bounded b-linear functional in n-Banach space is stable under perturbation of frame elements by positively confined sequence of scalars. In n-Banach space, some perturbation results of retro Banach frame with the help of bounded b-linear functional in n-Banach space have been studied. Finally, we give a sufficient condition for finite sum of retro Banach frames to be a retro Banach frame in n-Banach space. At the end, we discuss retro Banach frame with respect to a bounded b-linear functional in Cartesian product of two n-Banach spaces.

Keywords: frame, Banach frame, retro Banach frame, stability, n-Banach space, b-linear functional.

AMS Subject Classification: 42C15, 46C07, 46M05, 47A80.

For citation: *Ghosh, P. and Samanta, T. K.* On Stability of Retro Banach Frame with Respect to b-Linear Functional in n-Banach Space, *Vladikavkaz Math. J.*, 2023, vol. 25, no. 1, pp. 48–63. DOI: 10.46698/o3961-3328-9819-i.

1. Introduction and Preliminaries

In 1946, D. Gabor [1] first initiated a technique for rebuilding signals using a family of elementary signals. In 1952, Duffin and Schaeffer [2] abstracted the fundamental notion of Gabor method for studying signal processing and they gave the formal definition of frame for Hilbert space. Later on, in 1986, it was reintroduced, developed and popularized by Daubechies et al. [3].

Frame for Hilbert space was defined as a sequence of basis-like elements in Hilbert space. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is called a frame for a separable Hilbert space $(H, \langle \cdot\,, \cdot \rangle)$, if there exist positive constants $0 < A \leqslant B < \infty$ such that

$$A\,\|f\|^2 \leqslant \sum_{i=1}^{\infty} \left|\langle f, f_i\rangle\right|^2 \leqslant B\,\|f\|^2 \quad (\forall\, f \in H).$$

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But, in Banach space, due to the absence of inner product, frame was completely defined as a sequence of bounded linear functionals from the dual space of the Banach space. Before the notion of Banach frame was be formalized, it emerged in the fundamental work of Feichtinger and Groching [4, 5] related to the atomic decomposition for Banach spaces. Grochenig [6] introduced Banach frame in more general way in Banach space. Thereafter, further development of Banach frame was done by Casazza et al. [7]. P. K. Jain et al. [8] introduced and studied retro Banach frame and it was further developed by Vashisht [9]. Stability theorems for Banach frames were studied by Christensen and Heil [10] and P. K. Jain et al. [11]. S. Gahler [12] was the first to introduce the notion of linear 2-normed space. A generalization of a linear 2-normed space for $n \geqslant 2$ was developed by H. Gunawan and Mashadi [13]. P. Ghosh and T. K. Samanta [14–16] have studied the frames in n-Hilbert spaces and in their tensor products.

In this paper, we present the retro Banach frame relative to bounded b-linear functional in n-Banach space. A sufficient condition for the stability of retro Banach frame associated to $(a_2, \ldots, a_n)$ in n-Banach space under some perturbations is discussed. We establish that retro Banach frame associated to $(a_2, \ldots, a_n)$ is stable under perturbation of frame elements by positively confined sequence of scalars. Also, we consider the finite sum of retro Banach frame associated to $(a_2, \ldots, a_n)$ and establish a sufficient condition for the finite sum to be a retro Banach frame associated to $(a_2, \ldots, a_n)$ in n-Banach space. Finally, retro Banach frame associated to $(a_2, \ldots, a_n)$ in Cartesian product of two n-Banach spaces is presented.

Throughout this paper, E is considered to be a separable Banach space and E^*, it's dual space. By $\mathscr{B}(E)$ we denotes the space of all bounded linear operators on E. Let E_d be a sequence space, which is a Banach space and for which the co-ordinate functionals are continuous. Let $\{g_i\}_{i \in I} \subseteq E^*$ and $S : E_d \to E$ be a bounded linear operator. Then the pair $(\{g_i\}, S)$ is said to be a Banach frame for E with respect to E_d if

(i) $\{g_i(f)\} \in E_d$ $(\forall\, f \in E)$;

(ii) there exist $B \geqslant A > 0$ such that $A\|f\|_E \leqslant \|\{g_i(f)\}\|_{E_d} \leqslant B\|f\|_E$ $(\forall\, f \in E)$;

(iii) $S(\{g_i(f)\}) = f$ $(\forall\, f \in E)$.

The constants A, B are called Banach frame bounds and S is called the reconstruction operator.

Let E_d^* be a Banach space of scalar-valued sequences associated with E^* indexed by $\mathbb{N}$. Let $\{x_k\} \subseteq E$ and $T : E_d^* \to E^*$ be given. The pair $(\{x_k\}, T)$ is called a retro Banach frame for E^* with respect to E_d^* if

(i) $\{f(x_k)\} \in E_d^*$ for each $f \in E^*$;

(ii) there exist positive constants A and B with $0 < A \leqslant B < \infty$ such that $A\|f\|_{E^*} \leqslant \|\{f(x_k)\}\|_{E_d^*} \leqslant B\|f\|_{E^*}$, $f \in E^*$;

(iii) T is a bounded linear operator such that $T(\{f(x_k)\}) = f$, $f \in E^*$.

The constants A and B are called frame bounds. The operator T is called the reconstruction operator or pre-frame operator.

A n-norm on a linear space X (over the field $\mathbb{K}$ of real or complex numbers) is a function

$$(x_1, x_2, \ldots, x_n) \longmapsto \|x_1, x_2, \ldots, x_n\|, \quad x_1, x_2, \ldots, x_n \in X,$$

from X^n to the set $\mathbb{R}$ of all real numbers such that

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;

(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutations of $x_1, x_2, \ldots, x_n$;

(iii) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha|\|x_1, x_2, \ldots, x_n\|$;

(iv) $\|x + y, x_2, \ldots, x_n\| \leqslant \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$, for every $x_1, x_2, \ldots, x_n \in X$ and $\alpha \in \mathbb{K}$. A linear space X, together with a n-norm $\|\cdot, \ldots, \cdot\|$, is called a linear n-normed space. A sequence $\{x_k\}$ in linear n-normed space X is said to be convergent in X if there exists $x \in X$ such that

$$\lim_{k \to \infty} \|x_k - x, e_2, \ldots, e_n\| = 0 \quad (\forall\, e_2, \ldots, e_n \in X),$$

and it is called a Cauchy sequence if

$$\lim_{l, k \to \infty} \|x_l - x_k, e_2, \ldots, e_n\| = 0 \quad (\forall\, e_2, \ldots, e_n \in X).$$

The space X is said to be complete or n-Banach space if every Cauchy sequence in this space is convergent in X.

2. Main Results

In this section, the notion of retro Banach frame in n-Banach space X is introduced and some stability theorems for retro Banach frame relative to bounded b-linear functional in n-Banach space have been derived.

Now, we first define a bounded b-linear functional. Let $(X, \|\cdot, \ldots, \cdot\|)$ be a linear n-normed space and $a_2, \ldots, a_n$ be fixed elements in X. Let W be a subspace of X and $\langle a_i \rangle$ denote the subspaces of X generated by a_i, for $i = 2, 3, \ldots, n$. Then a map $T : W \times \langle a_2 \rangle \times \ldots \times \langle a_n \rangle \to \mathbb{K}$ is called a b-linear functional defined on $W \times \langle a_2 \rangle \times \ldots \times \langle a_n \rangle$, if for every $x, y \in W$ and $k \in \mathbb{K}$, the following conditions hold:

(*i*) $T(x + y, a_2, \ldots, a_n) = T(x, a_2, \ldots, a_n) + T(y, a_2, \ldots, a_n)$;

(*ii*) $T(kx, a_2, \ldots, a_n) = k\, T(x, a_2, \ldots, a_n)$.

A b-linear functional is said to be bounded if there exists a real number $M > 0$ such that

$$|T(x, a_2, \ldots, a_n)| \leqslant M\, \|x, a_2, \ldots, a_n\| \quad (\forall\, x \in W).$$

The norm of the bounded b-linear functional T is defined by

$$\|T\| = \inf \big\{ M > 0 : |T(x, a_2, \ldots, a_n)| \leqslant M\, \|x, a_2, \ldots, a_n\| \ (\forall\, x \in W) \big\}.$$

The norm of T can be expressed by any one of the following equivalent formula:

(*i*) $\|T\| = \sup \big\{ |T(x, a_2, \ldots, a_n)| \ \|x, a_2, \ldots, a_n\| \leqslant 1 \big\}$;

(*ii*) $\|T\| = \sup \big\{ |T(x, a_2, \ldots, a_n)| \ \|x, a_2, \ldots, a_n\| = 1 \big\}$;

(*iii*) $\|T\| = \sup \Big\{ \frac{|T(x, a_2, \ldots, a_n)|}{\|x, a_2, \ldots, a_n\|} \ \|x, a_2, \ldots, a_n\| \neq 0 \Big\}$.

For more details on bounded b-linear functional defined on $X \times \langle a_2 \rangle \times \ldots \times \langle a_n \rangle$ one can go through the paper [17]. For the remaining part of this paper, X denotes the n-Banach space with respect to the n-norm $\|\cdot, \ldots, \cdot\|$ and X_F^* denotes the Banach space of all bounded b-linear functional defined on $X \times \langle a_2 \rangle \times \ldots \times \langle a_n \rangle$ with respect to the norm given by above.

Definition 1. Let X be a n-Banach space and X_d^* be a Banach space of scalar-valued sequences associated to X_F^* indexed by $\mathbb{N}$. Let $\{x_k\} \subseteq X$ and $S : X_d^* \to X_F^*$ be given. Then the pair $(\{x_k\}, S)$ is said to be a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^* if

(*i*) $\{T(x_k, a_2, \ldots, a_n)\} \in X_d^*$ for each $T \in X_F^*$;

(*ii*) there exist constants $0 < A \leqslant B < \infty$ such that

$$A\, \|T\|_{X_F^*} \leqslant \|\{T(x_k, a_2, \ldots, a_n)\}\|_{X_d^*} \leqslant B\, \|T\|_{X_F^*} \quad (\forall\, T \in X_F^*); \tag{1}$$

(*iii*) S is a bounded linear operator such that

$$S(\{T(x_k, a_2, \ldots, a_n)\}) = T \quad (\forall\, T \in X_F^*).$$

The constants A, B are called frame bounds. If $A = B$, then $(\{x_k\}, S)$ is called tight retro Banach frame associated to $(a_2, \ldots, a_n)$ and for $A = B = 1$, it is called normalized tight retro Banach frame associated to $(a_2, \ldots, a_n)$. The inequality (1) is called the frame inequality for the retro Banach frame associated to $(a_2, \ldots, a_n)$. The operator $S : X_d^* \to X_F^*$ is called the reconstruction operator or the pre-frame operator.

Definition 2. A sequence $\{x_k\} \subseteq X$ is said to be a retro Banach Bessel sequence associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^* if
(*i*) $\{T(x_k, a_2, \ldots, a_n)\} \in X_d^*$ for each $T \in X_F^*$;
(*ii*) there exists a constant $B > 0$ such that

$$\|\{T(x_k, a_2, \ldots, a_n)\}\|_{X_d^*} \leqslant B\,\|T\|_{X_F^*} \quad (\forall\, T \in X_F^*)$$

The constant B is called a retro Banach Bessel bound for the retro Banach Bessel sequence $\{x_k\}$ associated to $(a_2, \ldots, a_n)$. Let X_B denotes the set of all retro Banach Bessel sequence associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*. For $\{x_k\} \in X_B$, define

$$\mathscr{R}_A : X_F^* \to X_d^* \quad \text{by} \quad \mathscr{R}_A(T) = \{T(x_k, a_2, \ldots, a_n)\} \quad (\forall\, T \in X_F^*).$$

Then it is easy to verify that $\mathscr{R}_A$ is a bounded linear operator. The operator $\mathscr{R}_A$ is called the analysis operator.

Next, we verify that scalar combinations of two retro Banach frames associated to $(a_2, \ldots, a_n)$ becomes a retro Banach frame associated to $(a_2, \ldots, a_n)$.

Theorem 1. *Let* $(\{x_k\}, S)$ *and* $(\{y_k\}, S)$ *be two retro Banach frames associated to* $(a_2, \ldots, a_n)$ *for* X_F^* *with respect to* X_d^* *having bounds* A, B *and* C, D, *respectively. Then for any scalars* α, β, $\big(\{\alpha x_k + \beta y_k\}, \frac{1}{\alpha+\beta} S\big)$ *is a retro Banach frame associated to* $(a_2, \ldots, a_n)$ *for* X_F^* *with respect to* X_d^*.

$\lhd$ For each $T \in X_F^*$, we have

$$\big\|\{T(\alpha x_k + \beta y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} = \big\|\{\alpha T(x_k, a_2, \ldots, a_n) + \beta T(y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*}$$
$$\leqslant |\alpha|\, \big\|\{T(x_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} + |\beta|\, \big\|\{T(y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \leqslant \big(|\alpha| B + |\beta| D\big) \big\|T\big\|_{X_F^*}.$$

On the other hand,

$$\big\|\{T(\alpha x_k + \beta y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \geqslant |\alpha|\, \big\|\{T(x_k, a_2, \ldots, a_n)\}\big\|_{X_d^*}$$
$$- |\beta|\, \big\|\{T(y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \geqslant \big(|\alpha| A - |\beta| C\big) \|T\|_{X_F^*}, \quad T \in X_F^*.$$

Also, for $T \in X_F^*$, we have

$$S\big(\{T(x_k, a_2, \ldots, a_n)\}\big) = T \quad \text{and} \quad S\big(\{T(y_k, a_2, \ldots, a_n)\}\big) = T.$$

Then for $T \in X_F^*$, we have

$$\frac{1}{\alpha + \beta}\, S\big(\{T(\alpha x_k + \beta y_k, a_2, \ldots, a_n)\}\big) = \frac{1}{\alpha + \beta} \Big[\alpha S\big(\{T(x_k, a_2, \ldots, a_n)\}\big)$$
$$+ \beta S\big(\{T(y_k, a_2, \ldots, a_n)\}\big)\Big] = \frac{1}{\alpha + \beta}\, (\alpha T + \beta T) = T.$$

Hence, the family $\big(\{\alpha x_k + \beta y_k\}, \frac{1}{\alpha+\beta} S\big)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^* having bounds $(|\alpha|A - |\beta|C)$ and $(|\alpha|B + |\beta|D)$. ▷

In the next theorem, we will see that the sum of two retro Banach frames associated to $(a_2, \ldots, a_n)$ with different reconstructions operators is also a retro Banach frame associated to $(a_2, \ldots, a_n)$.

Theorem 2. *Let $(\{x_k\}, S)$ and $(\{y_k\}, P)$ be two retro Banach frames associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^* having bounds A, B and C, D, respectively. Let $R : X_d^* \to X_d^*$ be a linear homeomorphism such that*

$$R(\{T(x_k, a_2, \ldots, a_n)\}) = \{T(y_k, a_2, \ldots, a_n)\}, \quad T \in X_F^*.$$

Then there exists a reconstruction operator $Q : X_d^* \to X_F^*$ such that the family $(\{x_k + y_k\}, Q)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*.

◁ Let U, V be the corresponding coefficient mappings for the retro Banach Bessel sequences $\{x_k\}$ and $\{y_k\}$, respectively and I denotes the identity mapping on X_d^*. Now, for each $T \in X_F^*$, we have

$$\begin{aligned}\big\|\{T(x_k + y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} &= \big\|\{T(x_k, a_2, \ldots, a_n)\} + \{T(y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \\ &= \big\|\{T(x_k, a_2, \ldots, a_n)\} + R(\{T(x_k, a_2, \ldots, a_n)\})\big\|_{X_d^*} \\ &\leqslant \|I + R\| \big\|\{T(x_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \leqslant B\,\|I + R\| \|T\|_{X_F^*}.\end{aligned}$$

Similarly, for each $T \in X_F^*$, we have

$$\big\|\{T(x_k + y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \leqslant D\,\big\|I + R^{-1}\big\| \|T\|_{X_F^*}.$$

Thus, for each $T \in X_F^*$, we get

$$\big\|\{T(x_k + y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \leqslant \min\big\{B\,\|I + R\|, D\,\big\|I + R^{-1}\big\|\big\} \|T\|_{X_F^*}.$$

On the other hand, for each $T \in X_F^*$, we have

$$\begin{aligned}&\big\|\{T(x_k + y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \\ &\geqslant \big\|\{T(x_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} - \big\|\{T(y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \geqslant A\,\|I - R\| \|T\|_{X_F^*}.\end{aligned}$$

Also, for each $T \in X_F^*$, we have

$$\big\|\{T(x_k + y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \geqslant C\,\big\|R^{-1} - I\big\| \|T\|_{X_F^*}.$$

Therefore, for each $T \in X_F^*$, we get

$$\big\|\{T(x_k + y_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \geqslant \max\big\{A\,\|I - R\|, C\,\big\|R^{-1} - I\big\|\big\} \|T\|_{X_F^*}.$$

Now, for $T \in X_F^*$, we have

$$\begin{aligned}R(\{T(x_k + y_k, a_2, \ldots, a_n)\}) &= R(\{T(x_k, a_2, \ldots, a_n)\}) + R(\{T(y_k, a_2, \ldots, a_n)\}) \\ &= (I + R)\{T(y_k, a_2, \ldots, a_n)\} = (I + R)P^{-1}T.\end{aligned}$$

Therefore, if we take $Q = ((I + R)P^{-1})^{-1}$, then $Q : X_d^* \to X_F^*$ is a bounded linear operator such that

$$Q(\{T(x_k + y_k, a_2, \ldots, a_n)\}) = T \quad (\forall\, T \in X_F^*).$$

Hence, $(\{x_k+y_k\},Q)$ is a retro Banach frame associated to $(a_2,\dots,a_n)$ for X_F^* with respect to X_d^*. $\rhd$

Now, we start with a necessary and sufficient condition for the stability of a retro Banach frame associated to $(a_2,\dots,a_n)$.

Theorem 3. *Let $(\{x_k\},S)$ be a retro Banach frame associated to $(a_2,\dots,a_n)$ for X_F^* with respect to X_d^* having bounds A, B. Let $\{y_k\}$ be a sequence in X such that $\{T(y_k,a_2,\dots,a_n)\}\in X_d^*$, $T\in X_F^*$. Suppose $R: X_d^*\to X_d^*$ be a bounded linear operator such that*

$$R(\{T(y_k,a_2,\dots,a_n)\})=\{T(x_k,a_2,\dots,a_n)\},\quad T\in X_F^*.$$

Then there exists a bounded linear operator $P: X_d^*\to X_F^*$ such that $(\{y_k\},P)$ is a retro Banach frame associated to $(a_2,\dots,a_n)$ for X_F^* with respect to X_d^* if and only if there exists a constant $K>1$ such that

$$\begin{aligned}&\big\|\{T(x_k-y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\\ &\leqslant K\min\big\{\|\{T(x_k,a_2,\dots,a_n)\}\|_{X_d^*},\|\{T(y_k,a_2,\dots,a_n)\}\|_{X_d^*}\big\}.\end{aligned}\tag{2}$$

$\lhd$ First we suppose that $(\{y_k\},P)$ is a retro Banach frame associated to $(a_2,\dots,a_n)$. Then there exist constants $C,D>0$ such that

$$A\,\|T\|_{X_F^*}\leqslant\big\|\{T(x_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\leqslant B\,\|T\|_{X_F^*}\quad(\forall\,T\in X_F^*),\tag{3}$$

$$C\,\|T\|_{X_F^*}\leqslant\big\|\{T(y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\leqslant D\,\|T\|_{X_F^*}\quad(\forall\,T\in X_F^*).\tag{4}$$

Therefore, for each $T\in X_F^*$, we have

$$\begin{aligned}\big\|\{T(x_k-y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}&\leqslant\big\|\{T(x_k,a_2,\dots,a_n)\}\big\|_{X_d^*}+\big\|\{T(y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\\ &\overset{(4)}{\leqslant}\big\|\{T(x_k,a_2,\dots,a_n)\}\big\|_{X_d^*}+D\,\|T\|_{X_F^*}\overset{(3)}{\leqslant}\left(1+\frac{D}{A}\right)\big\|\{T(x_k,a_2,\dots,a_n)\}\big\|_{X_d^*}.\end{aligned}$$

Similarly, it can be shown that

$$\big\|\{T(x_k-y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\leqslant\left(1+\frac{B}{C}\right)\big\|\{T(y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\quad(\forall\,T\in X_F^*).$$

Thus, for each $T\in X_F^*$, we get

$$\big\|\{T(x_k-y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\leqslant K\min\big\{\|\{T(x_k,a_2,\dots,a_n)\}\|_{X_d^*},\|\{T(y_k,a_2,\dots,a_n)\}\|_{X_d^*}\big\},$$

where $K=\max\big\{\big(1+\frac{D}{A}\big),\big(1+\frac{B}{C}\big)\big\}$.

Conversely, suppose that there exists $K>1$ such that (2) holds. Now, for each $T\in X_F^*$, we have

$$\begin{aligned}A\,\|T\|_{X_F^*}\leqslant\big\|\{T(x_k,a_2,\dots,a_n)\}\big\|_{X_d^*}&\leqslant\big\|\{T(x_k-y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\\ &+\big\|\{T(y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}\overset{(2)}{\leqslant}(K+1)\big\|\{T(y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}.\end{aligned}$$

This implies that

$$\frac{A}{(K+1)}\,\|T\|_{X_F^*}\leqslant\big\|\{T(y_k,a_2,\dots,a_n)\}\big\|_{X_d^*}.$$

On the other hand, for each $T \in X_F^*$, we have

$$\left\|\{T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \leqslant \left\|\{T(x_k - y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \left\|\{T(x_k, a_2, \ldots, a_n)\}\right\|_{X_d^*}$$
$$\leqslant (K+1)\left\|\{T(x_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \leqslant B(K+1)\,\|T\|_{X_F^*}.$$

Now, take $P = SR$. Then $P : X_d^* \to X_F^*$ is a bounded linear operator such that

$$P(\{T(y_k, a_2, \ldots, a_n)\}) = SR(\{T(y_k, a_2, \ldots, a_n)\}) = S\{T(x_k, a_2, \ldots, a_n)\} = T, \quad T \in X_F^*.$$

Thus, $(\{y_k\}, P)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*. This completes the proof. $\rhd$

The Theorem 3 shows that the stability of retro Banach frame associated to $(a_2, \ldots, a_n)$ depends on the value of K. For large value of K, the retro Banach frame inequality is lost. Therefore, to get optimal frame bounds, we still need to modify the stability conditions. In the following theorem, we give a sufficient conditions for the stability of a retro Banach frame associated to $(a_2, \ldots, a_n)$.

Theorem 4. *Let $(\{x_k\}, S)$ be a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*. Let $\{y_k\} \subseteq X$ be such that $\{T(y_k, a_2, \ldots, a_n)\} \in X_d^*$, $T \in X_F^*$ and let $U : X_F^* \to X_d^*$ be the coefficient mapping given by*

$$U(T) = \{T(x_k, a_2, \ldots, a_n)\}, \quad T \in X_F^*.$$

If there exist positive constants α, β (< 1) and μ such that

(i) $$\max\left\{\frac{\|S\|[(2+\alpha-\beta)\|U\|+\mu]}{(1-\beta)}, \beta\right\} < 1;$$

(ii) $$\left\|\{T(x_k - y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \leqslant \alpha\left\|\{T(x_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \beta\left\|\{T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \mu\,\|T\|_{X_F^*}, \quad T \in X_F^*,$$

then there exists a reconstruction operator $P : X_d^* \to X_F^*$ such that $(\{y_k\}, P)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*.

$\lhd$ Let $V : X_F^* \to X_d^*$ be an operator defined by

$$V(T) = \{T(y_k, a_2, \ldots, a_n)\}, \quad T \in X_F^*.$$

Using the operators U and V, condition (ii) can be written as

$$\|UT - VT\|_{X_d^*} \leqslant \alpha\,\|UT\|_{X_d^*} + \beta\,\|VT\|_{X_d^*} + \mu\,\|T\|_{X_F^*}, \quad T \in X_F^*.$$

Thus, for $T \in X_F^*$, we have

$$\|\{T(y_k, a_2, \ldots, a_n)\}\|_{X_d^*} = \|VT\|_{X_d^*} \leqslant \|UT - VT\|_{X_d^*} + \|UT\|_{X_d^*} \leqslant \frac{(1+\alpha)\|U\|+\mu}{1-\beta}\|T\|_{X_F^*}.$$

Therefore, V is a bounded linear operator such that

$$\|UT - VT\|_{X_d^*} \leqslant \frac{(2+\alpha-\beta)\|U\|+\mu}{1-\beta}\,\|T\|_{X_F^*} \quad (\forall\, T \in X_F^*).$$

Now,

$$\|I_F - SV\| \leqslant \|S\|\|U - V\| \leqslant \frac{\|S\|[(2+\alpha-\beta)\|U\|+\mu]}{1-\beta} < 1.$$

This shows that SV is an invertible operator with satisfying

$$\left\|(SV)^{-1}\right\| \leqslant \frac{1}{1-\frac{\|S\|[(2+\alpha-\beta)\|U\|+\mu]}{1-\beta}}.$$

Now, take $P=(SV)^{-1}S$. Then $PV=I_F$, where I_F is the identity operator on X_F^*. Thus, $P: X_d^* \to X_F^*$ is a bounded linear operator such that

$$P(\{T(y_k,a_2,\ldots,a_n)\})=T, \quad T\in X_F^*.$$

Now, for $T\in X_F^*$, we have

$$\|T\|_{X_F^*}=\|PVT\|_{X_F^*}\leqslant \|P\|\|VT\|_{X_d^*}\leqslant \frac{\|S\|}{1-\frac{\|S\|[(2+\alpha-\beta)\|U\|+\mu]}{1-\beta}}\|VT\|_{X_d^*}.$$

This implies that

$$\|S\|^{-1}\left(1-\frac{[(2+\alpha-\beta)\|U\|+\mu]\|S\|}{1-\beta}\right)\|T\|_{X_F^*}\leqslant \left\|\{T(y_k,a_2,\ldots,a_n)\}\right\|_{X_d^*} \quad (\forall\, T\in X_F^*).$$

Hence, $(\{y_k\},P)$ is a retro Banach frame associated to $(a_2,\ldots,a_n)$ for X_F^* with respect to X_d^*. This completes the proof. $\rhd$

Next, we give a stability condition of a retro Banach frame associated to $(a_2,\ldots, a_n)$ by using a given retro Banach Bessel sequence associated to $(a_2,\ldots,a_n)$.

Theorem 5. *Let $(\{x_k\},S)$ be a retro Banach frame associated to $(a_2,\ldots,a_n)$ for X_F^* with respect to X_d^* having bounds A,B. Let $\{y_k\}$ be a sequence in X such that $(\{T(y_k,a_2,\ldots,a_n)\}\in X_d^*)$, $T\in X_F^*$ and for some constant $K>0$*

$$\left\|\{T(y_k,a_2,\ldots,a_n)\}\right\|_{X_d^*}\leqslant K\|T\|_{X_F^*} \quad (\forall\, T\in X_F^*).$$

Then for any non-zero constant λ with $|\lambda|<\frac{\|S\|^{-1}}{K}$, there exists a reconstruction operator $P: X_d^*\to X_F^*$ such that $(\{x_k+\lambda y_k\},P)$ is a retro Banach frame associated to $(a_2,\ldots,a_n)$ for X_F^* with respect to X_d^* having frame bounds $(\|S\|^{-1}-|\lambda|K)$ and $(B+|\lambda|K)$.

$\lhd$ Let $V: X_F^*\to X_d^*$ be a bounded linear operator defined by

$$V(T)=\{T(y_k,a_2,\ldots,a_n)\}, \quad T\in X_F^*,$$

and $U: X_F^*\to X_d^*$ be a bounded linear operator given by

$$U(T)=\{T(x_k,a_2,\ldots,a_n)\}, \quad T\in X_F^*.$$

Then it is easy to verify that $\{T(x_k+\lambda y_k,a_2,\ldots,a_n)\}\in X_d^*$, for all $T\in X_F^*$. Now, for each $T\in X_F^*$, we have

$$\|UT+\lambda VT\|_{X_d^*}=\left\|\{T(x_k+\lambda y_k,a_2,\ldots,a_n)\}\right\|_{X_d^*}$$
$$\leqslant \left\|\{T(x_k,a_2,\ldots,a_n)\}\right\|_{X_d^*}+|\lambda|\left\|\{T(y_k,a_2,\ldots,a_n)\}\right\|_{X_d^*}\leqslant (B+|\lambda|K)\|T\|_{X_F^*}.$$

On the other hand, for each $T\in X_F^*$, we have

$$\left(\|S\|^{-1}-|\lambda|K\right)\|T\|_{X_F^*}\leqslant \left\|\{T(x_k,a_2,\ldots,a_n)\}\right\|_{X_d^*}-|\lambda|\left\|\{T(y_k,a_2,\ldots,a_n)\}\right\|_{X_d^*}$$
$$\leqslant \left\|\{T(x_k+\lambda y_k,a_2,\ldots,a_n)\}\right\|_{X_d^*}.$$

Define, $L : X_F^* \to X_d^*$ by $L(T) = \{T(x_k + \lambda y_k, a_2, \dots, a_n)\}$, $T \in X_F^*$. Then L is a bounded linear operator such that

$$\|UT - LT\|_{X_d^*} = \big\|\{T(x_k, a_2, \dots, a_n)\} - \{T(x_k + \lambda y_k, a_2, \dots, a_n)\}\big\|_{X_d^*}$$
$$= \big\|\{\lambda T(y_k, a_2, \dots, a_n)\}\big\|_{X_d^*} \leqslant |\lambda| K \|T\|_{X_F^*}, \quad T \in X_F^*.$$

This verifies that $\|U - L\| \leqslant |\lambda| K$. Since $SU = I_F$, I_F is the identity operator on X_F^*, we have

$$\|I_F - SL\| = \|SU - SL\| \leqslant \|S\|\|U - L\| < 1.$$

Thus SL is invertible. Take $P = (SL)^{-1}S$. Then $P : X_d^* \to X_F^*$ is a bounded linear operator such that

$$P(\{T(x_k + \lambda y_k, a_2, \dots, a_n)\}) = T, \quad T \in X_F^*.$$

Hence, $(\{x_k + \lambda y_k\}, P)$ is a retro Banach frame associated to $(a_2, \dots, a_n)$ for X_F^* with respect to X_d^* having frame bounds $(\|S\|^{-1} - |\lambda| K)$ and $(B + |\lambda| K)$. ▷

Theorem 6. *Let $(\{x_k\}, S)$ be a retro Banach frame associated to $(a_2, \dots, a_n)$ for X_F^* with respect to X_d^*. Let $\{y_k\} \subseteq X$ and $\{\alpha_k\} \subseteq \mathbb{R}$ be any positively confined sequence such that $\{T(\alpha_k y_k, a_2, \dots, a_n)\} \in X_d^*$, $T \in X_F^*$. If $V : X_F^* \to X_d^*$ defined by*

$$V(T) = \{T(y_k, a_2, \dots, a_n)\}, \quad T \in X_F^*,$$

such that $\|V\| < \frac{\|S\|^{-1}}{\sup_{1 \leqslant k < \infty} \alpha_k}$, then there exists a reconstruction operator $P : X_d^* \to X_F^*$ such that $(\{x_k + \alpha_k y_k\}, P)$ is a retro Banach frame associated to $(a_2, \dots, a_n)$ for X_F^* with respect to X_d^*.

◁ Let $U : X_F^* \to X_d^*$ be a bounded linear operator defined by

$$U(T) = \{T(x_k, a_2, \dots, a_n)\}, \quad T \in X_F^*.$$

It is easy to verify that $\{T(x_k + \alpha_k y_k, a_2, \dots, a_n)\} \in X_d^*$, for all $T \in X_F^*$. Now, for each $T \in X_F^*$, we have

$$\big\|\{T(x_k + \alpha_k y_k, a_2, \dots, a_n)\}\big\|_{X_d^*} \leqslant \big\|\{T(x_k, a_2, \dots, a_n)\}\big\|_{X_d^*} + \big\|\{\alpha_k T(y_k, a_2, \dots, a_n)\}\big\|_{X_d^*}$$
$$\leqslant \big\|\{T(x_k, a_2, \dots, a_n)\}\big\|_{X_d^*} + \Big(\sup_{1 \leqslant k < \infty} \alpha_k\Big)\big\|\{T(y_k, a_2, \dots, a_n)\}\big\|_{X_d^*}$$
$$\leqslant \Big[\|U\| + \|V\|\Big(\sup_{1 \leqslant k < \infty} \alpha_k\Big)\Big]\|T\|_{X_F^*}.$$

On the other hand, for each $T \in X_F^*$, we have

$$\big\|\{T(x_k + \alpha_k y_k, a_2, \dots, a_n)\}\big\|_{X_d^*} \geqslant \big\|\{T(x_k, a_2, \dots, a_n)\}\big\|_{X_d^*} - \big\|\{\alpha_k T(y_k, a_2, \dots, a_n)\}\big\|_{X_d^*}$$
$$\geqslant \Big[\|S\|^{-1} - \|V\|\Big(\sup_{1 \leqslant k < \infty} \alpha_k\Big)\Big]\|T\|_{X_F^*}.$$

Define, $L : X_F^* \to X_d^*$ by

$$L(T) = \{T(x_k + \alpha_k y_k, a_2, \dots, a_n)\}, \quad T \in X_F^*.$$

Following the lines of proof of the Theorem 5, L is a bounded linear operator on X_F^* such that $\|U-L\| \leqslant \sup_{1\leqslant k<\infty} \alpha_k \|V\|$ and SL is invertible. Take $P=(SL)^{-1}S$. Then $P : X_d^* \to X_F^*$ is a bounded linear operator such that

$$P(\{T(x_k+\alpha_k y_k, a_2, \ldots, a_n)\}) = T, \quad T \in X_F^*.$$

Hence, $(\{x_k+\alpha_k y_k\}, P)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*. $\rhd$

In the next theorem, we establish that retro Banach frame associated to $(a_2, \ldots, a_n)$ is stable under perturbation of frame elements by positively confined sequence of scalars.

Theorem 7. *Let $(\{x_k\}, S)$ be a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*. Let $\{y_k\} \subseteq X$ be such that $\{T(y_k, a_2, \ldots, a_n)\} \in X_d^*$, $T \in X_F^*$. Let $R : X_d^* \to X_d^*$ be a bounded linear operator such that*

$$R(\{T(y_k, a_2, \ldots, a_n)\}) = \{T(x_k, a_2, \ldots, a_n)\}, \quad T \in X_F^*.$$

Suppose $\{\alpha_k\}$ and $\{\beta_k\}$ are two positively confined sequences in $\mathbb{R}$. If there exist constants λ, $\mu (0 \leqslant \lambda,\ \mu < 1)$ and γ such that

(i) $\gamma < (1-\lambda)\|S\|^{-1}\left(\inf_{1\leqslant k<\infty} \alpha_k\right)$;

(ii) $\left\|\{\alpha_k T(x_k, a_2, \ldots, a_n)\} - \{\beta_k T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \leqslant \lambda \left\|\{\alpha_k T(x_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \mu \left\|\{\beta_k T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \gamma \|T\|_{X_F^*}, \quad T \in X_F^*.$

Then there exists a reconstruction operator $P : X_d^* \to X_F^*$ such that $(\{y_k\}, P)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*.

$\lhd$ Let $U : X_F^* \to X_d^*$ be a bounded linear operator defined by

$$U(T) = \{T(x_k, a_2, \ldots, a_n)\}, \quad T \in X_F^*.$$

Since the operator $SU : X_F^* \to X_F^*$ is an identity operator, for $T \in X_F^*$,

$$\|T\|_{X_F^*} = \|SU(T)\|_{X_F^*} \leqslant \|S\| \left\|\{T(x_k, a_2, \ldots, a_n)\}\right\|_{X_d^*}.$$

Now, for each $T \in X_F^*$, we have

$$\begin{aligned} &\left\|\{\beta_k T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \\ &\leqslant \left\|\{\alpha_k T(x_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \left\|\{\alpha_k T(x_k, a_2, \ldots, a_n)\} - \{\beta_k T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \\ &\leqslant (1+\lambda) \left\|\{\alpha_k T(x_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \mu \left\|\{\beta_k T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} + \gamma \|T\|_{X_F^*}. \end{aligned}$$

Thus,

$$(1-\mu) \left\|\{\beta_k T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \leqslant \left[(1+\lambda) \|U\| \left(\sup_{1\leqslant k<\infty} \alpha_k\right) + \gamma\right] \|T\|_{X_F^*}.$$

This implies that

$$(1-\mu)\left(\inf_{1\leqslant k<\infty} \beta_k\right) \left\|\{T(y_k, a_2, \ldots, a_n)\}\right\|_{X_d^*} \leqslant \left[(1+\lambda) \|U\| \left(\sup_{1\leqslant k<\infty} \alpha_k\right) + \gamma\right] \|T\|_{X_F^*}.$$

On the other hand, by condition (ii), we get

$$\begin{aligned}(1+\mu)\left\|\{\beta_k T(y_k,a_2,\dots,a_n)\}\right\|_{X_d^*} &\geqslant (1-\lambda)\left\|\{\alpha_k T(x_k,a_2,\dots,a_n)\}\right\|_{X_d^*} - \gamma\,\|T\|_{X_F^*}\\ &\geqslant \left[(1-\lambda)\,\|S\|^{-1}\Big(\inf_{1\leqslant k<\infty}\alpha_k\Big)-\gamma\right]\|T\|_{X_F^*}, \quad T\in X_F^*.\end{aligned}$$

Therefore, for each $T \in X_F^*$, we have

$$\begin{aligned}(1+\mu)\Big(\sup_{1\leqslant k<\infty}\beta_k\Big)\left\|\{T(y_k,a_2,\dots,a_n)\}\right\|_{X_d^*} &\geqslant (1+\mu)\left\|\{\beta_k T(y_k,a_2,\dots,a_n)\}\right\|_{X_d^*}\\ &\geqslant \left[(1-\lambda)\,\|S\|^{-1}\Big(\inf_{1\leqslant k<\infty}\alpha_k\Big)-\gamma\right]\|T\|_{X_F^*}.\end{aligned}$$

Thus, for each $T \in X_F^*$, we have

$$\begin{aligned}\frac{(1-\lambda)\|S\|^{-1}\Big(\inf\limits_{1\leqslant k<\infty}\alpha_k\Big)-\gamma}{(1+\mu)\Big(\sup\limits_{1\leqslant k<\infty}\beta_k\Big)}\,\|T\|_{X_F^*} &\leqslant \left\|\{T(y_k,a_2,\dots,a_n)\}\right\|_{X_d^*}\\ &\leqslant \frac{(1+\lambda)\|U\|\Big(\sup\limits_{1\leqslant k<\infty}\alpha_k\Big)+\gamma}{(1-\mu)\Big(\inf\limits_{1\leqslant k<\infty}\beta_k\Big)}\,\|T\|_{X_F^*}.\end{aligned}$$

Now, take $P = SV$. Then $P : X_d^* \to X_F^*$ is a bounded linear operator such that

$$P\big(\{T(y_k,a_2,\dots,a_n)\}\big) = T, \quad T \in X_F^*.$$

Hence, $(\{y_k\}, P)$ is a retro Banach frame associated to $(a_2,\dots,a_n)$ for X_F^* with respect to X_d^*. ▷

Definition 3. A sequence $\{x_k\}$ in X is said to be total over X_F^* if

$$\big\{T \in X_F^* : T(x_k,a_2,\dots,a_n) = 0\ (\forall\, k)\big\} = \{\theta\},$$

where $\theta \in X_F^*$ is the null operator.

In the following two theorems, some sufficient condition will be describe under which the finite sum of retro Banach frame associated to $(a_2,\dots,a_n)$ is again a retro Banach frame associated to $(a_2,\dots,a_n)$.

Theorem 8. *For $i \in E_m = \{1,2,\dots,m\}$, let $(\{x_{k,i}\}, S_i)$ be retro Banach frames associated to $(a_2,\dots, a_n)$ for X_F^* with respect to X_d^*. Then there exists a reconstruction operator $P : X_d^* \to X_F^*$ such that $(\{\sum_{i=1}^m x_{k,i}\}, P)$ is a tight retro Banach frame associated to $(a_2,\dots,a_n)$ for X_F^* with respect to X_d^*, provided*

$$\left\|\{T(x_{k,j},a_2,\dots,a_n)\}\right\|_{X_d^*} \leqslant \left\|\left\{T\Big(\sum_{i=1}^m x_{k,i},a_2,\dots,a_n\Big)\right\}\right\|_{X_d^*},$$

for $T \in X_F^*$, for some $j \in E_m$.

◁ For $T \in X_F^*$, we have

$$\|T\|_{X_F^*} = \big\|S_i(\{T(x_{k,j}, a_2, \dots, a_n)\})\big\|_{X_F^*} \leqslant \|S_i\| \big\|\{T(x_{k,j}, a_2, \dots, a_n)\}\big\|_{X_d^*}$$
$$\leqslant \|S_i\| \left\| \left\{ T\left(\sum_{i=1}^{m} x_{k,i}, a_2, \dots, a_n \right) \right\} \right\|_{X_d^*}.$$

Thus, $\{T(\sum_{i=1}^m x_{k,i}, a_2, \dots, a_n)\}$ is total over X_F^*. Therefore, by Remark 7.1 in [18], there exists an associated Banach space

$$X_{d_1} = \left\{ \left\{ T\left(\sum_{i=1}^{m} x_{k,i}, a_2, \dots, a_n \right) \right\} : \ T \in X_F^* \right\}$$

equipped with the norm

$$\left\| \left\{ T\left(\sum_{i=1}^{m} x_{k,i}, a_2, \dots, a_n \right) \right\} \right\|_{X_{d_1}} = \|T\|_{X_F^*}, \quad T \in X_F^*,$$

and a bounded linear operator $P : X_d^* \to X_F^*$ defined by

$$P\left(\left\{ T\left(\sum_{i=1}^{m} x_{k,i}, a_2, \dots, a_n \right) \right\} \right) = T, \quad T \in X_F^*,$$

such that $(\{\sum_{i=1}^m x_{k,i}\}, P)$ is a tight retro Banach frame associated to $(a_2, \dots, a_n)$ for X_F^* with respect to X_d^*. ▷

Theorem 9. *Let $(\{x_{k,i}\}, S_i)$, $i \in E_m = \{1, 2, \dots, m\}$ be retro Banach frames associated to $(a_2, \dots, a_n)$ for X_F^* with respect to X_d^*. Let $\{y_{k,i}\} \subseteq X$ be such that $\{T(y_{k,i}, a_2, \dots, a_n)\} \in X_d^*$, $T \in X_F^*$. Suppose $R : X_d^* \to X_d^*$ be a bounded linear operator such that*

$$R\left(\left\{ T\left(\sum_{i=1}^{m} y_{k,i}, a_2, \dots, a_n \right) \right\} \right) = \big\{T(x_{k,p}, a_2, \dots, a_n)\big\}, \quad T \in X_F^*,$$

for some $p \in E_m$ and for each $i \in E_m$, let $U_i : X_F^* \to X_d^*$ be an operator defined by

$$U_i(T) = \big\{T(x_{k,i}, a_2, \dots, a_n)\big\}, \quad T \in X_F^*.$$

If there exist constants $\alpha, \beta > 0$ such that

(i) $\alpha \sum\limits_{i \in E_m} \|U_i\| + m\beta < \|S_j\|^{-1} - \sum\limits_{\substack{i \in E_m, \\ i \neq j}} \|U_i\|$, *for some $j \in E_m$;*

(ii) $\big\|\{T(x_{k,i} - y_{k,i}, a_2, \dots, a_n)\}\big\|_{X_d^*} \leqslant \alpha \big\|\{T(x_{k,i}, a_2, \dots, a_n)\}\big\|_{X_d^*} + \beta \|T\|_{X_F^*}$, $T \in X_F^*$, $i \in E_m$,

then there exists a bounded linear operator $P : X_d^* \to X_F^*$ such that the family $(\{\sum_{i \in E_m} y_{k,i}\}, P)$ is a retro Banach frame associated to $(a_2, \dots, a_n)$ for X_F^* with respect to X_d^*.

◁ For each $i \in E_m$, $S_i U_i$ is an identity operator on X_F^*. Therefore, for each $T \in X_F^*$, we have

$$\|T\|_{X_F^*} = \|S_i U_i T\|_{X_F^*} \leqslant \|S_i\| \, \big\|\{T(x_{k,i}, a_2, \dots, a_n)\}\big\|_{X_d^*}. \tag{5}$$

Also, for $T \in X_F^*$, we have

$$\left\| \left\{ T\left(\sum_{i\in E_m} x_{k,i}, a_2, \ldots, a_n \right) \right\} \right\|_{X_d^*} = \left\| \sum_{i\in E_m} \{T(x_{k,i}, a_2, \ldots, a_n)\} \right\|_{X_d^*} \leqslant \sum_{i\in E_m} \|U_i\| \|T\|_{X_F^*}. \tag{6}$$

Now, for each $T \in X_F^*$, we have

$$\left\| \left\{ T\left(\sum_{i\in E_m} y_{k,i}, a_2, \ldots, a_n \right) \right\} \right\|_{X_d^*} = \left\| \sum_{i\in E_m} \{T(x_{k,i}, a_2, \ldots, a_n) - T(x_{k,i} - y_{k,i}, a_2, \ldots, a_n)\} \right\|_{X_d^*}$$

$$\geqslant \left\| \sum_{i\in E_m} \{T(x_{k,i}, a_2, \ldots, a_n)\} \right\|_{X_d^*} - \left\| \sum_{i\in E_m} \{T(x_{k,i} - y_{k,i}, a_2, \ldots, a_n)\} \right\|_{X_d^*}$$

$$\geqslant \left\| \{T(x_{k,j}, a_2, \ldots, a_n)\} + \sum_{\substack{i\in E_m,\\ i\neq j}} \{T(x_{k,i}, a_2, \ldots, a_n)\} \right\|_{X_d^*} - \left\| \sum_{i\in E_m} \{T(x_{k,i} - y_{k,i}, a_2, \ldots, a_n)\} \right\|_{X_d^*}$$

$$\overset{(5),(6)}{\geqslant} \|S_j\|^{-1} - \left[\alpha \sum_{i\in E_m} \|U_i\| + \sum_{\substack{i\in E_m,\\ i\neq j}} \|U_i\| + m\beta \right] \|T\|_{X_F^*}.$$

On the other hand, using (6), for each $T \in X_F^*$, we get

$$\left\| \left\{ T\left(\sum_{i\in E_m} y_{k,i}, a_2, \ldots, a_n \right) \right\} \right\|_{X_d^*} \leqslant \left((1+\alpha) \sum_{i\in E_m} \|U_i\| + \beta \right) \|T\|_{X_F^*}.$$

Now, we take $P = S_p R$, where p is fixed. Then $P : X_d^* \to X_F^*$ is a bounded linear operator such that

$$P\left(\left\{ T\left(\sum_{i\in E_m} y_{k,i}, a_2, \ldots, a_n \right) \right\} \right) = T, \quad T \in X_F^*.$$

Hence, $(\{\sum_{i\in E_m} y_{k,i}\}, P)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^*. $\triangleright$

We end this section by discussing retro Banach frame associated to $(a_2, \ldots, a_n)$ in Cartesian product of two n-Banach spaces.

Let $(X, \|\cdot, \ldots, \cdot\|_X)$ and $(Y, \|\cdot, \ldots, \cdot\|_Y)$ be two n-Banach spaces. Then the Cartesian product of X and Y is denoted by $X \oplus Y$ and defined to be an n-Banach space with respect to the n-norm

$$\|x_1 \oplus y_1, x_2 \oplus y_2, \ldots, x_n \oplus y_n\| = \|x_1, x_2, \ldots, x_n\|_X + \|y_1, y_2, \ldots, y_n\|_Y,$$

for all $x_1 \oplus y_1, x_2 \oplus y_2, \ldots, x_n \oplus y_n \in X \oplus Y$, and $x_1, x_2, \ldots, x_n \in X$; $y_1, y_2, \ldots, y_n \in Y$. Consider Y_G^* as the Banach space of all bounded b-linear functional defined on $Y \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$ and $Z_{F\oplus G}^*$ as the Banach space of all bounded b-linear functional defined on $X \oplus Y \times \langle a_2 \oplus b_2 \rangle \times \ldots \times \langle a_n \oplus b_n \rangle$, where $b_2, \ldots, b_n \in Y$ and $a_2 \oplus b_2, \ldots, a_n \oplus b_n \in X \oplus Y$ are fixed elements. Now, if $T \in X_F^*$ and $U \in Y_G^*$, for all $x \oplus y \in X \oplus Y$, we define $T \oplus U \in Z_{F\oplus G}^*$ by

$$(T \oplus U)(x \oplus y, a_2 \oplus b_2, \ldots, a_n \oplus b_n) = T(x, a_2, \ldots, a_n) \oplus U(y, b_2, \ldots, b_n) \ \ (\forall\, x \in X,\ \forall\, y \in Y).$$

Let us consider Y_d^* and Z_d^* as the Banach spaces of scalar-valued sequences associated with Y_G^* and $Z_{F\oplus G}^*$, respectively.

Theorem 10. *Let* $(\{x_k\}, S_X)$ *be a retro Banach frame associated to* $(a_2, \ldots, a_n)$ *for* X_F^* *with respect to* X_d^* *having bounds* A, B *and* $(\{y_k\}, S_Y)$ *be a retro Banach frame associated to* $(b_2, \ldots, b_n)$ *for* Y_G^* *with respect to* Y_d^* *having bounds* C, D*. Then* $(\{x_k \oplus y_k\}, S_X \oplus S_Y)$ *is a retro Banach frame associated to* $(a_2 \oplus b_2, \ldots, a_n \oplus b_n)$ *for* $Z_{F\oplus G}^*$ *with respect to* Z_d^**.*

$\lhd$ Since $(\{x_k\}, S_X)$ is a retro Banach frame associated to $(a_2, \ldots, a_n)$ for X_F^* with respect to X_d^* and $(\{y_k\}, S_Y)$ is a retro Banach frame associated to $(b_2, \ldots, b_n)$ for Y_G^* with respect to Y_d^*, we have

$$A\|T\|_{X_F^*} \leqslant \big\|\{T(x_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} \leqslant B\|T\|_{X_F^*} \quad (\forall\, T \in X_F^*), \tag{7}$$

$$C\|R\|_{Y_G^*} \leqslant \big\|\{R(y_k, b_2, \ldots, b_n)\}\big\|_{Y_d^*} \leqslant D\|R\|_{X_F^*} \quad (\forall\, R \in Y_G^*). \tag{8}$$

Adding (7) and (8), we get

$$A\|T\|_{X_F^*} + C\|R\|_{Y_G^*} \leqslant \big\|\{T(x_k, a_2, \ldots, a_n)\}\big\|_{X_d^*} + \big\|\{R(y_k, b_2, \ldots, b_n)\}\big\|_{Y_d^*} \leqslant B\|T\|_{X_F^*} + D\|R\|_{Y_G^*}.$$

This implies that

$$\min(A, C)\big\{\|T\|_{X_F^*} + \|R\|_{Y_G^*}\big\} \leqslant \big\|\{T(x_k, a_2, \ldots, a_n)\} \oplus \{R(y_k, b_2, \ldots, b_n)\}\big\|_{Z_d^*} \leqslant \max(B, D)\big\{\|T\|_{X_F^*} + \|R\|_{Y_G^*}\big\}.$$

Thus,

$$\min(A, C)\|T \oplus R\|_{Z_{F\oplus G}^*} \leqslant \|\{(T \oplus R)(x_k \oplus y_k, a_2 \oplus b_2, \ldots, a_n \oplus b_n)\}\|_{Z_d^*} \leqslant \max(B, D)\|T \oplus R\|_{Z_{F\oplus G}^*} \quad (\forall\, T \oplus R \in Z_{F\oplus G}^*).$$

Also, we have

$$S_X\big(\{T(x_k, a_2, \ldots, a_n)\}\big) = T, \ \ T \in X_F^*, \quad \text{and} \quad S_Y\big(\{R(y_k, b_2, \ldots, b_n)\}\big) = R, \ \ R \in Y_G^*$$

Now,

$$\begin{gathered}(S_X \oplus S_Y)\big(\{(T \oplus R)(x_k \oplus y_k, a_2 \oplus b_2, \ldots, a_n \oplus b_n)\}\big)\\ = (S_X \oplus S_Y)\big(\{T(x_k, a_2, \ldots, a_n)\} \oplus \{R(y_k, b_2, \ldots, b_n)\}\big)\\ = S_X\big(\{T(x_k, a_2, \ldots, a_n)\}\big) \oplus S_Y\big(\{R(y_k, b_2, \ldots, b_n)\}\big) = T \oplus R \quad (\forall\, T \oplus R \in Z_{F\oplus G}^*).\end{gathered}$$

Hence, the family $(\{x_k \oplus y_k\}, S_X \oplus S_Y)$ is a retro Banach frame associated to $(a_2 \oplus b_2, \ldots, a_n \oplus b_n)$ for $Z_{F\oplus G}^*$ with respect to Z_d^*. $\rhd$

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Received November 9, 2021

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Владикавказский математический журнал
2023, Том 25, Выпуск 1, С. 48–63

ОБ УСТОЙЧИВОСТИ РЕТРО БАНАХОВА ФРЕЙМА ОТНОСИТЕЛЬНО b-ЛИНЕЙНОГО ФУНКЦИОНАЛА В n-БАНАХОВОМ ПРОСТРАНСТВЕ

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Аннотация. Вводится понятие ретро банахова фрейма относительно ограниченного b-линейного функционала в n-банаховом пространстве и устанавливается, что сумма двух ретро банаховых фреймов в n-банаховом пространстве с разными операторами реконструкции также является ретро банаховым фреймом в n-банаховом пространстве. Также определяется ретро банахова последовательность Бесселя относительно ограниченного b-линейного функционала в n-банаховом пространстве. Получено необходимое и достаточное условие устойчивости ретро банахова фрейма относительно ограниченного b-линейного функционала в n-банаховом пространстве. Далее, доказано, что ретро банахов фрейм относительно ограниченного b-линейного функционала в n-банаховом пространстве устойчив по отношению к возмущению элементов фрейма положительно ограниченной последовательностью скаляров. Изучены некоторые свойства возмущении ретро банахова фрейма в n-банаховом пространстве с помощью ограниченного b-линейного функционала. Наконец, даются достаточное условие того, чтобы конечная сумма ретро банаховых фреймов была ретро банаховым фреймом в n-банаховом пространстве. В заключении рассматривается ретро банахов фрейм относительно ограниченного b-линейного функционала в декартовом произведении двух n-банаховых пространств.

Ключевые слова: фрейм, банахов фрейм, ретро банахов фрейм, устойчивость, n-банахово пространство, b-линейный функционал.

AMS Subject Classification: 42C15, 46C07, 46M05, 47A80.

Образец цитирования: *Ghosh P. and Samanta T. K.* On Stability of Retro Banach Frame with Respect to b-Linear Functional in n-Banach Space // Владикавк. мат. журн.—2023.—Т. 25, № 1.—С. 48–63 (in English). DOI: 10.46698/o3961-3328-9819-i.