Vladikavkaz Mathematical Journal
2021, Volume 23, Issue 3, P. 16–26

УДК 517.927.4
DOI 10.46698/y9113-7002-9720-u

TITCHMARSH–WEYL THEORY OF THE SINGULAR HAHN–STURM–LIOUVILLE EQUATION

B. P. Allahverdiev[1] and H. Tuna[2]

[1] Department of Mathematics, Süleyman Demirel University, 32260 Isparta, Turkey;
[2] Department of Mathematics, Mehmet Akif Ersoy University, 15030 Burdur, Turkey

E-mail: `bilenderpasaoglu@sdu.edu.tr`, `hustuna@gmail.com`

Dedicated to the 80th anniversary of Stefan Grigorievich Samko

Abstract. In this work, we will consider the singular Hahn–Sturm–Liouville difference equation defined by $-q^{-1}D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x)+v(x)y(x)=\lambda y(x)$, $x\in(\omega_0,\infty)$, where λ is a complex parameter, v is a real-valued continuous function at ω_0 defined on $[\omega_0,\infty)$. These type equations are obtained when the ordinary derivative in the classical Sturm–Liouville problem is replaced by the ω,q-Hahn difference operator $D_{\omega,q}$. We develop the ω,q-analogue of the classical Titchmarsh–Weyl theory for such equations. In other words, we study the existence of square-integrable solutions of the singular Hahn–Sturm–Liouville equation. Accordingly, first we define an appropriate Hilbert space in terms of Jackson–Nörlund integral and then we study families of regular Hahn–Sturm–Liouville problems on $[\omega_0,q^{-n}]$, $n\in\mathbb{N}$. Then we define a family of circles that converge either to a point or a circle. Thus, we will define the limit-point, limit-circle cases in the Hahn calculus setting by using Titchmarsh's technique.

Key words: Hahn's Sturm–Liouville equation, limit-circle and limit-point cases, Titchmarsh–Weyl theory.

Mathematical Subject Classification (2010): 39A13, 39A70, 34B20, 39A12.

For citation: *Allahverdiev, B. P. and Tuna, H.* Titchmarsh–Weyl Theory of the Singular Hahn–Sturm–Liouville Equation, *Vladikavkaz Math. J.*, 2021, vol. 23, no. 3, pp. 16–26. DOI: 10.46698/y9113-7002-9720-u.

1. Introduction

The Titchmarsh–Weyl theory is at the heart of the area of ordinary differential operators and deals with the existence of square integrable solutions. This theory has led to important contributions over the years to our understanding of the spectral properties of differential operators. H. Weyl introduced this theory in 1910. In [1], Weyl proved that the singular Sturm–Liouville problem of the type

$$-\left(p\left(x\right)y'\left(x\right)\right)'+q\left(x\right)y\left(x\right)=\lambda y\left(x\right),\quad 0<x<\infty,$$

has a non-trivial square integrable solution. He constructed a sequence of nested circles which converges to a circle or a point and defined the limit-point, limit-circle classification. This theory has been attracting the attention of many researchers; see, for instance, [2–6].

The study of the Hahn difference operator first appeared in [7, 8] where the quantum difference operator $D_{\omega,q}$ was introduced. Such an operator is known to be a generalization of the forward difference operator and the quantum q-difference operator defined by Jackson [9]. Hahn difference operators have received considerable attention due to their applications in the construction of families of orthogonal polynomials and approximation problems, see [10–14] and the references therein.

There are some papers in the literature dealing with Hahan's difference equations. In [15, 16], Hamza and Ahmed studied the existence and uniqueness of solution for the initial value problems for Hahn's difference equations. Moreover they proved Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator and investigated the mean value theorems for this calculus. In 2016, Hamza and Makharesh [17] investigated Leibniz's rule and Fubini's theorem associated with Hahn's difference operator. Sitthiwirattham [18] consider the nonlocal boundary value problem for nonlinear Hahn's difference equation. In [19], the regular Hahn–Sturm–Liouville problem

$$-q^{-1}D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x)+v(x)y(x)=\lambda y(x),$$

$$a_1y(\omega_0)+a_2D_{-\omega q^{-1},q^{-1}}y(\omega_0)=0,$$

$$b_1y(b)+b_2D_{-\omega q^{-1},q^{-1}}y(b)=0,$$

is studied, where $\omega_0 \leqslant x < \infty$, $\alpha \in \mathbb{C}$, $a_i, b_i \in \mathbb{R} := (-\infty,\infty)$, $i = 1, 2$, and $p(\cdot)$ is a real-valued function defined on $[\omega_0, b]$ and continuous at ω_0. Annaby et al. [19] define a Hilbert space of ω, q-square summable functions. The authors discussed the formulation of the self-adjoint operator and the properties of the eigenvalues and the eigenfunctions. Furthermore, they construct the Green's function and give an eigenfunction expansion theorem.

In this paper, we attempt to study the ω, q-analogue of the classical Titchmarsh–Weyl theory.

The paper is organized as follows. In Section 2, we summarize all the necessary definitions and properties of Hahn's difference operator. In Section 3, we formulate the singular Hahn–Sturm–Liouville difference equation and develop the classical Titchmarsh–Weyl theory for such equation.

2. Notation

In this section, our aim is to present some basic concepts concerning the theory of Hahn calculus. For more details, the reader may refer [7, 8, 19, 20]. Throughout the paper, we let $q \in (0,1)$ and $\omega > 0$.

Define $\omega_0 := \omega/(1-q)$ and let I be a real interval containing ω_0.

Definition 1 [7, 8]. Let $f : I \to \mathbb{R}$ be a function. *The Hahn difference operator* is defined by

$$D_{\omega,q}f(x)=\begin{cases}\dfrac{f(\omega+qx)-f(x)}{\omega+(q-1)x}, & x\neq\omega_0,\\ f'(\omega_0), & x=\omega_0,\end{cases}\tag{1}$$

provided that f is differentiable at ω_0. In this case, we call $D_{\omega,q}f$, the ω, q-derivative of f.

Remark 1. The Hahn difference operator unifies two well known operators. When $q \to 1$, we get the forward difference operator, which is defined by

$$\Delta_\omega f(x) := \frac{f(\omega+x)-f(x)}{(\omega+x)-x}, \quad x \in \mathbb{R}.$$

When $\omega \to 0$, we get the Jackson q-difference operator, which is defined by

$$D_q f(x) := \frac{f(qx) - f(x)}{(qx) - x}, \quad x \neq 0.$$

Furthermore, under appropriate conditions, we have

$$\lim_{\substack{q \to 1, \\ \omega \to 0}} D_{\omega,q} f(x) = f'(x).$$

In what follows, we present some important properties of the ω, q-derivative.

Theorem 1 [20]**.** *Let $f, g : I \to \mathbb{R}$ be ω, q-differentiable at $x \in I$ and $h(x) := \omega + qx$. Then for all $x \in I$ we have:*

$$\text{i) } D_{\omega,q}(af + bg)(x) = aD_{\omega,q} f(x) + bD_{\omega,q} g(x), \quad a, b \in I,$$

$$\text{ii) } D_{\omega,q}(fg)(x) = D_{\omega,q}(f(x)) g(x) + f(\omega + xq) D_{\omega,q} g(x),$$

$$\text{iii) } D_{\omega,q}\left(\frac{f}{g}\right)(x) = \frac{D_{\omega,q}(f(x)) g(x) - f(x) D_{\omega,q} g(x)}{g(x) g(\omega + xq)},$$

$$\text{iv) } D_{\omega,q}\left(h^{-1}(x)\right) = D_{-\omega q^{-1}, q^{-1}} f(x).$$

The concept of the ω, q-integral of the function f can be defined as follows.

Definition 2 (Jackson–Nörlund integral [20]). Let $f : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. We define ω, q-integral of the function f from a to b by

$$\int_a^b f(x)\, d_{\omega,q}(x) := \int_{\omega_0}^b f(x)\, d_{\omega,q}(x) - \int_{\omega_0}^a f(x)\, d_{\omega,q}(x),$$

where

$$\int_{\omega_0}^x f(x)\, d_{\omega,q}(x) := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n f\left(\omega \frac{1-q^n}{1-q} + xq^n\right), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$. In this case, f is called *ω, q-integrable* on $[a, b]$.

Similarly, one can define the ω, q-integration for a function f over (ω_0, ∞) by

$$\int_{\omega_0}^{\infty} f(x)\, d_{\omega,q}(x) := ((1-q) - \omega) \sum_{n=0}^{\infty} q^n f\left(\omega \frac{1-q^n}{1-q} + q^n\right).$$

The following properties of ω, q-integration can be found in [20].

Theorem 2 [20]**.** *Let $f, g : I \to \mathbb{R}$ be ω, q-integrable on I, $a, b, c \in I$, $a < c < b$ and*

$\alpha, \beta \in \mathbb{R}$. Then the following formulas hold:

i) $\int_a^b \{\alpha f(x) + \beta g(x)\} d_{\omega,q}(x) = \alpha \int_a^b f(x) d_{\omega,q}(x) + \beta \int_a^b g(x) d_{\omega,q}(x)$,

ii) $\int_a^a f(x) d_{\omega,q}(x) = 0$,

iii) $\int_a^b f(x) d_{\omega,q}(x) = \int_a^c f(x) d_{\omega,q}(x) + \int_c^b f(x) d_{\omega,q}(x)$,

iv) $\int_a^b f(x) d_{\omega,q}(x) = -\int_b^a f(x) d_{\omega,q}(x)$.

Next, we present the ω, q-integration by parts.

Lemma 1 [20]**.** *Let $f, g : I \to \mathbb{R}$ be ω, q-integrable on I, $a, b \in I$, and $a < b$. Then the following formula holds:*

$$\int_a^b f(x) D_{\omega,q} g(x) d_{\omega,q}(x) + \int_a^b g(\omega + qx) D_{\omega,q} f(x) d_{\omega,q}(x) = f(b) g(b) - f(a) g(a).$$

The next result is the fundamental theorem of Hahn's calculus.

Theorem 3 [20]**.** *Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t) d_{\omega,q}(t), \quad x \in I.$$

Then F is continuous at ω_0. Moreover, $D_{\omega,q}F(x)$ exists for every $x \in I$ and $D_{\omega,q}F(x) = f(x)$. Conversely,

$$\int_a^b D_{\omega,q} F(x) d_{\omega,q}(x) = f(b) - f(a).$$

Let $L^2_{\omega,q}(\omega_0, \infty)$ be the space of all complex-valued functions defined on $[\omega_0, \infty)$ such that

$$\|f\| := \left(\int_{\omega_0}^{\infty} |f(x)|^2 d_{\omega,q} x \right)^{\frac{1}{2}} < \infty.$$

The space $L^2_{\omega,q}(\omega_0, \infty)$ is a separable Hilbert space with the inner product

$$(f, g) := \int_{\omega_0}^{\infty} f(x) \overline{g(x)} d_{\omega,q} x, \quad f, g \in L^2_{\omega,q}(\omega_0, \infty)$$

(see [20]).

The ω, q-Wronskian of functions $y(\cdot)$, $z(\cdot)$ is defined as

$$W_{\omega,q}(y, z)(x) := y(x) D_{\omega,q} z(x) - z(x) D_{\omega,q} y(x), \quad x \in [\omega_0, \infty). \tag{2}$$

3. The Singular Hahn–Sturm–Liouville Equation and Titchmarsh–Weyl Theory

In this section, we introduce the Titchmarsh–Weyl theory of the singular Hahn–Sturm–Liouville difference equation.

Consider the singular Hahn–Sturm–Liouville difference equation

$$\Gamma (y) := -q^{-1}D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x) + v(x)\,y(x) = \lambda y(x), \quad x \in (\omega_0, \infty), \tag{3}$$

where λ is a complex parameter, v is a real-valued continuous function at ω_0 defined on $[\omega_0, \infty)$.

We note that there exists a unique solution of (4) satisfying the conditions [19]

$$y(\omega_0) = d_1, \quad D_{-\omega q^{-1},q^{-1}}y(\omega_0) = d_2,$$

where $d_1, d_2 \in \mathbb{C}$.

Lemma 2 (see [19]). *For any y and z in $L^2_{\omega,q}(\omega_0,\infty)$, the following relation holds*

$$\int_{\omega_0}^{x} \Gamma(y)\,\overline{z}\,d_{\omega,q}t - \int_{\omega_0}^{x} y\,\overline{\Gamma(z)}\,d_{\omega,q}t = [y,z](x) - [y,z](\omega_0), \tag{4}$$

where

$$[y,z] = y\,\overline{(D_{-\omega q^{-1},\,q^{-1}}z)} - (D_{-\omega q^{-1},\,q^{-1}}y)\,\overline{z}.$$

Theorem 4. *For each $\lambda \in \mathbb{C}$ and $x \in [\omega_0,\infty)$, the ω,q-Wronskian of any two solution of equation (4) is independent of x.*

$\lhd$ Let y and z be two solutions of equation (4). It follows from (4) that

$$\int_{\omega_0}^{x} \Gamma(y)\overline{z}\,d_{\omega,q}t - \int_{\omega_0}^{x} y\overline{\Gamma(z)}\,d_{\omega,q}t = [y,z](x) - [y,z](\omega_0)$$
$$= W_{\omega,q}(y,\overline{z})\left(h^{-1}(x),\lambda\right) - W_{\omega,q}(y,\overline{z})\left(h^{-1}(\omega_0),\lambda\right).$$

Since $\Gamma(y) = \lambda y$ and $\Gamma(z) = \lambda z$, we have

$$\begin{aligned}\int_{\omega_0}^{x} \Gamma(y)z\,d_{\omega,q}t - \int_{\omega_0}^{x} y\Gamma(z)\,d_{\omega,q}t = (\lambda-\lambda)\int_{\omega_0}^{x} y(t,\lambda)z(t,\lambda)d_{\omega,q}t = 0\\ = W_{\omega,q}(y,z)\left(h^{-1}(x),\lambda\right) - W_{\omega,q}(y,z)\left(h^{-1}(\omega_0),\lambda\right).\end{aligned} \tag{5}$$

Then we have $W_{\omega,q}(y,z)(h^{-1}(x),\lambda) = W_{\omega,q}(y,z)(h^{-1}(\omega_0),\lambda)$, i. e., the Wronskian is independent of x ($x \in [\omega_0,\infty)$). $\rhd$

Now we impose a boundary condition for the solution y of equation (4) as

$$D_{\omega,q}y\left(q^{-n}\right)\sin\alpha + y\left(q^{-n}\right)\cos\alpha = 0, \quad \alpha \in \mathbb{R},\ n \in \mathbb{N} := \{1,2,\dots\}. \tag{6}$$

Let $\chi(x,\lambda)$ and $\varphi(x,\lambda)$ be the solutions of the equation (4) satisfying

$$\begin{aligned}\chi(\omega_0,\lambda) = \sin\zeta, \quad \varphi(\omega_0,\lambda) = \cos\zeta,\\ D_{-\omega q^{-1},q^{-1}}\chi(\omega_0,\lambda) = -\cos\zeta, \quad D_{-\omega q^{-1},q^{-1}}\varphi(\omega_0,\lambda) = \sin\zeta,\end{aligned} \tag{7}$$

where $0 \leqslant \zeta < \pi$. Since $W_{\omega,q}(\chi,\varphi) = 1$, the solutions χ and φ are linearly independent.

Lemma 3. *For $x \in [\omega_0, \infty)$ and $\lambda \in \mathbb{C}$, we have*

$$\overline{\chi(x,\lambda)} = \chi\left(x, \overline{\lambda}\right), \quad \overline{\varphi(x,\lambda)} = \varphi\left(x, \overline{\lambda}\right).$$

$\lhd$ If $\chi(x,\lambda)$ is a solution of the equation (4), then we have

$$-q^{-1} D_{-\omega q^{-1}, q^{-1}} D_{\omega,q} \chi(x,\lambda) + v\left(x\right) \chi(x,\lambda) = \lambda \chi(x,\lambda), \quad x \in (\omega_0, \infty).$$

Taking the complex conjugate, we obtain

$$-q^{-1} D_{-\omega q^{-1}, q^{-1}} D_{\omega,q} \overline{\chi(x,\lambda)} + v\left(x\right) \overline{\chi(x,\lambda)} = \overline{\lambda}\, \overline{\chi(x,\lambda)}, \quad x \in (\omega_0, \infty).$$

By (7), $\overline{\chi(x,\lambda)}$ is a solution of

$$-q^{-1} D_{-\omega q^{-1}, q^{-1}} D_{\omega,q} u\left(x\right) + v\left(x\right) u\left(x\right) = \overline{\lambda} u\left(x\right), \quad x \in (\omega_0, \infty).$$

But $u = \chi(x, \overline{\lambda})$ is also a solution of the equation (4) with the same conditions (7). By the uniqueness of solutions we get the desired result. $\rhd$

Lemma 4. *If $y(x,\lambda)$ is a solution of equation (4), then we have*

$$2i\sigma \int_{\omega_0}^{b} \left|y(x,\lambda)\right|^2 d_{\omega,q} x = W_{\omega,q}\left(y, \overline{y}\right)\left(h^{-1}\left(b\right), \lambda\right) - W_{\omega,q}\left(y, \overline{y}\right)\left(h^{-1}\left(\omega_0\right), \lambda\right), \qquad (8)$$

$$\sigma = \operatorname{Im} \lambda, \quad b > 0, \quad \lambda \in \mathbb{C}.$$

$\lhd$ Substituting $z(x,\lambda) = y(x,\lambda)$ in (4), we have (5). Thus, we get (8). $\rhd$

Now using (7) we shall construct a solution ϑ of (4) as

$$\vartheta(x,\lambda) := \chi(x,\lambda) + \eta \varphi(x,\lambda), \quad x \in [\omega_0, \infty),$$

where η is a constant. If we substitute ϑ for y in (6) we obtain

$$(\Omega + \eta F) \sin\alpha + (\Upsilon + \eta \Phi) \cos\alpha = 0, \qquad (9)$$

where

$$\Omega = D_{\omega,q} \chi\left(q^{-n}, \lambda\right), \quad F = D_{\omega,q} \varphi\left(q^{-n}, \lambda\right),$$
$$\Upsilon = \chi\left(q^{-n}, \lambda\right), \quad \Phi = \varphi\left(q^{-n}, \lambda\right) \quad (n \in \mathbb{N}).$$

Then, we get

$$\eta = -\frac{\Upsilon \cos\alpha + \Omega \sin\alpha}{\Phi \cos\alpha + F \sin\alpha}, \qquad (10)$$

η is a meromorphic function of λ because $\chi(x,\lambda)$ and $\varphi(x,\lambda)$ are entire functions of λ. Furthermore, since the eigenvalues of the regular problem are real, all poles of η are real and simple. If $\cos\alpha$ is replaced by a complex variable z, then we have

$$\eta = -\frac{\Upsilon z + \Omega}{\Phi z + F}. \qquad (11)$$

It follows from the theory of Möbius transformations [21] that the equation (11) is a one-to-one conformal mapping in z for every λ. Hence η describes a circle $C_{q^{-n}}$ in the complex plane.

The task is now to find the center and the radius of $C_{q^{-n}}$ $(n\in\mathbb{N})$. Let us denote by Θ_n, r_n the center of the circles $C_{q^{-n}}$ $(n\in\mathbb{N})$ and its radius, respectively.

Theorem 5. *Let $\lambda\in\mathbb{C}$, $\sigma=\operatorname{Im}\lambda\neq 0$. Then, we have*

$$\Theta_n(\lambda)=-\frac{W_{\omega,q}(\chi,\overline{\varphi})(q^{-n},\lambda)}{W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda)}, \tag{12}$$

$$r_n(\lambda)=\left(2\sigma\int_{\omega_0}^{q^{-n}}|\varphi(x,\lambda)|^2\,d_{\omega,q}\,x\right)^{-1}\quad (n\in\mathbb{N}). \tag{13}$$

$\lhd$ $\Theta_n(\lambda)$ $(n\in\mathbb{N})$ is the symmetric point at ∞. Let z' and z'' are in the z-plane such that

$$\eta(\lambda,z')=\infty,\quad \eta(\lambda,z'')=\Theta_n(\lambda).$$

Then z' and z'' must be symmetric with respect to the real axis of the $z-$plane, i. e., $z'=\overline{z''}$. But $\eta(\lambda,z')=\infty$ if and only if

$$z'=-\frac{D_{\omega,q}\varphi(q^{-n},\lambda)}{\varphi(q^{-n},\lambda)}.$$

Hence, we have

$$\Theta_n(\lambda)=\eta\left(\lambda,-\frac{D_{\omega,q}\varphi(q^{-n},\lambda)}{\varphi(q^{-n},\lambda)}\right)=-\frac{\chi(q^{-n},\lambda)\left(-\overline{\frac{D_{\omega,q}\varphi(q^{-n},\lambda)}{\varphi(q^{-n},\lambda)}}\right)+D_{\omega,q}\chi(q^{-n},\lambda)}{\varphi(q^{-n},\lambda)\left(-\overline{\frac{D_{\omega,q}\varphi(q^{-n},\lambda)}{\varphi(q^{-n},\lambda)}}\right)+D_{\omega,q}\varphi(q^{-n},\lambda)}$$

$$=-\frac{\chi(q^{-n},\lambda)D_{\omega,q}\varphi(q^{-n},\overline{\lambda})-\varphi(q^{-n},\overline{\lambda})D_{\omega,q}\chi(q^{-n},\lambda)}{\varphi(q^{-n},\lambda)D_{\omega,q}\varphi(q^{-n},\overline{\lambda})-\varphi(q^{-n},\overline{\lambda})D_{\omega,q}\varphi(q^{-n},\lambda)}$$

$$=-\frac{W_{\omega,q}(\chi,\overline{\varphi})(q^{-n},\lambda)}{W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda))}\quad (n\in\mathbb{N}).$$

It is evident that $r_n(\lambda)$ is the distance between the center of $C_{q^{-n}}$ and the point $\eta(\lambda,0)$ on $C_{q^{-n}}(\lambda)$. It follows from $W_{\omega,q}(\chi,\varphi)(q^{-n})=1$ $(n\in\mathbb{N})$ that

$$r_n(\lambda)=\left|\frac{D_{\omega,q}\chi(q^{-n},\lambda)}{D_{\omega,q}\varphi(q^{-n},\lambda)}-\frac{W_{\omega,q}(\chi,\overline{\varphi})(q^{-n},\lambda)}{W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda)}\right|=\left|\frac{D_{\omega,q}\varphi(q^{-n},\overline{\lambda})W_{\omega,q}(\chi,\varphi)(q^{-n},\lambda)}{D_{\omega,q}\varphi(q^{-n},\lambda)W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda)}\right|$$

$$=\left|\frac{W_{\omega,q}(\chi,\varphi)(q^{-n},\lambda)}{W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda)}\right|=\frac{1}{|W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda)|}\quad (n\in\mathbb{N}).$$

By virtue of Lemma 4, we conclude that

$$W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda)=2i\sigma\int_{\omega_0}^{q^{-n}}|\varphi(x,\lambda)|^2\,d_{\omega,q}\,x.$$

Thus, we get

$$\left|W_{\omega,q}(\varphi,\overline{\varphi})(q^{-n},\lambda)\right|=2|\sigma|\int_{\omega_0}^{q^{-n}}|\varphi(x,\lambda)|^2\,d_{\omega,q}\,x,$$

which proves the theorem. $\rhd$

Now, our next concern will be the behavior of the family of circles $\{C_{q^{-n}}(\lambda)\}$ $(n \in \mathbb{N})$.

Theorem 6. *Let $\lambda = \rho + i\sigma$. If $\operatorname{Im}\lambda = \sigma > 0$, then the upper half-plane is associated with the exterior of the circle $C_{q^{-n}}$ $(n \in \mathbb{N})$.*

◁ It follows from (2) that

$$\operatorname{Im}\left\{\frac{D_{\omega,q}\varphi\left(q^{-n},\lambda\right)}{\varphi\left(q^{-n},\lambda\right)}\right\} = \frac{1}{2}\,i\left\{-\frac{D_{\omega,q}\varphi\left(q^{-n},\lambda\right)}{\varphi\left(q^{-n},\lambda\right)} + \frac{D_{\omega,q}\varphi\left(q^{-n},\overline{\lambda}\right)}{\varphi\left(q^{-n},\overline{\lambda}\right)}\right\}$$
$$= \frac{1}{2}\,i\,\frac{W_{\omega,q}\left(\overline{\varphi},\varphi\right)\left(q^{-n},\lambda\right)}{\left|\varphi\left(q^{-n},\lambda\right)\right|^2} = -\frac{1}{2}\,i\,\frac{W_{\omega,q}\left(\varphi,\overline{\varphi}\right)\left(q^{-n},\lambda\right)}{\left|\varphi\left(q^{-n},\lambda\right)\right|^2}$$
$$= \frac{\sigma}{\left|\varphi\left(q^{-n},\lambda\right)\right|^2}\int_{\omega_0}^{q^{-n}} |\varphi(x,\lambda)|^2\, d_{\omega,q}\,x > 0 \quad (n \in \mathbb{N}),$$

i. e., if $\sigma > 0$, the exterior of $C_{q^{-n}}(\lambda)$ is mapped onto the upper half-plane of the z-plane. ▷

Now, we can prove the following result.

Theorem 7. *Let $\operatorname{Im}\lambda = \sigma > 0$. Then η lies on the circles $C_{q^{-n}}$ $(n \in \mathbb{N})$ if and only if*

$$\int_{\omega_0}^{q^{-n}} \left|\chi(x,\lambda) + \eta\varphi(x,\lambda)\right|^2 d_{\omega,q}x = \frac{\operatorname{Im}\eta}{\sigma} \quad (n \in \mathbb{N}), \tag{14}$$

and η belongs to the interior of $C_{q^{-n}}$ if and only if

$$\int_{\omega_0}^{q^{-n}} \left|\chi(x,\lambda) + \eta\varphi(x,\lambda)\right|^2 d_{\omega,q}x < \frac{\operatorname{Im}\eta}{\sigma} \quad (n \in \mathbb{N}).$$

◁ Let $\eta \in \mathbb{C}$. Then, we have

$$\begin{aligned} W_{\omega,q}\big(\chi + \eta\varphi, \overline{\chi + \eta\varphi}\big)(\omega_0,\lambda) &= W_{\omega,q}\left(\chi,\overline{\chi}\right)(\omega_0,\lambda) + \eta W_{\omega,q}\left(\varphi,\overline{\chi}\right)(\omega_0,\lambda) \\ &+ \overline{\eta}\, W_{\omega,q}\left(\chi,\overline{\varphi}\right)(\omega_0,\lambda) + |\eta|^2\, W_{\omega,q}\left(\varphi,\overline{\varphi}\right)(\omega_0,\lambda) = -\eta + \overline{\eta} = -2i\operatorname{Im}\eta. \end{aligned} \tag{15}$$

It follows from Lemma 4 that

$$2\sigma\int_{\omega_0}^{q^{-n}} \left|\chi(x,\lambda) + \eta\varphi(x,\lambda)\right|^2 d_{\omega,q}x = \frac{1}{i}\left(W_{\omega,q}(\chi+\eta\varphi, \chi+\eta\varphi)(q^{-n},\lambda) + 2i\operatorname{Im}\eta\right) \quad (n \in \mathbb{N}). \tag{16}$$

From Theorem 6, if $\operatorname{Im} z < 0$, then η is inside $C_{q^{-n}}$ $(n \in \mathbb{N})$ for $v > 0$. By (11), we obtain

$$z = -\frac{\Omega + \eta F}{\Upsilon + \eta\Phi}, \quad \eta = -\frac{\Upsilon z + \Omega}{\Phi z + F},$$
$$\Omega = D_{\omega,q}\chi\left(q^{-n},\lambda\right), \quad F = D_{\omega,q}\varphi\left(q^{-n},\lambda\right),$$
$$\Upsilon = \chi\left(q^{-n},\lambda\right), \quad \Phi = \varphi\left(q^{-n},\lambda\right) \quad (n \in \mathbb{N}).$$

Hence,

$$i\left(z - \overline{z}\right) = i\left\{-\frac{\Omega + \eta F}{\Upsilon + \eta\Phi} + \frac{\overline{\Omega} + \overline{\eta F}}{\overline{\Upsilon} + \overline{\eta\Phi}}\right\} = i\,\frac{W_{\omega,q}\big(\chi + \eta\varphi, \overline{\chi + \eta\varphi}\,\big)\left(q^{-n},\lambda\right)}{|\Upsilon + \eta\Phi|^2} \quad (n \in \mathbb{N}).$$

Then, $\operatorname{Im} z < 0$ if and only if

$$i\,W_q\big(\chi + \eta\varphi, \overline{\chi + \eta\varphi}\,\big)\,\big(q^{-n}, \lambda\big) > 0 \quad (n \in \mathbb{N}). \tag{17}$$

By virtue of (16) and (17), we conclude that

$$\int\limits_{\omega_0}^{q^{-n}} \big|\chi(x,\lambda) + \eta\varphi(x,\lambda)\big|^2\, d_{\omega,q}x < \frac{\operatorname{Im}\eta}{\sigma} \quad (n \in \mathbb{N}). \tag{18}$$

On the other hand, η is on the circle $C_{q^{-n}}$ if and only if $\operatorname{Im} z = 0$. Therefore, we have

$$W_{\omega,q}\big(\chi + \eta\varphi, \overline{\chi + \eta\varphi}\,\big)\,\big(q^{-n}, \lambda\big) = 0 \quad (n \in \mathbb{N}). \tag{19}$$

Substituting (19) in (16), we obtain the equality (14). ▷

Theorem 8. *Let* $\operatorname{Im}\lambda = \sigma > 0$. *The circles* $C_{q^{-n}}$ *are nested as* $n \to \infty$.

◁ Let us consider another point k such that $q^{-k} < q^{-n}$. Using (16) we may write

$$\int\limits_{\omega_0}^{q^{-k}} \big|\chi(x,\lambda) + \eta\varphi(x,\lambda)\big|^2\, d_{\omega,q}x < \int\limits_{\omega_0}^{q^{-n}} \big|\chi(x,\lambda) + \eta\varphi(x,\lambda)\big|^2\, d_{\omega,q}x < \frac{\operatorname{Im}\eta}{\sigma}.$$

This implies that the point η must be inside the circle $C_{q^{-k}}$. ▷

Corollary 1. *The circles* $C_{q^{-n}}$ *may converge either to a circle or a point as* $n \to \infty$.

Definition 3. If C_∞ is a point, then the equation (4) is said to be *in the limit-point case*. Similarly, if C_∞ is a circle, then the equation (4) is said to be in the limit-circle case.

Theorem 9. *Let* η *be a point lying on or inside the limiting circle* C_∞, *and* $\Psi(x,\lambda) := \chi(x,\lambda) + \eta\varphi(x,\lambda)$, $\operatorname{Im}\lambda = \sigma > 0$, *be the solution of* (4). *Then* $\Psi(\cdot,\lambda) \in L^2_{\omega,q}(\omega_0,\infty)$, *i. e.,*

$$\int\limits_{\omega_0}^{\infty} \big|\chi(x,\lambda) + \eta\varphi(x,\lambda)\big|^2\, d_{\omega,q}x < \infty.$$

We note that η is called a *Titchmarsh–Weyl function*, and $\Psi(x,\lambda)$ is called a *Weyl solution* of the equation (4).

◁ Let η be a point lying on or inside the limiting circle C_∞. Then we have

$$\int\limits_{\omega_0}^{q^{-n}} \big|\chi(x,\lambda) + \eta\varphi(x,\lambda)\big|^2 d_{\omega,q}x < \frac{\operatorname{Im}\eta}{\sigma} \quad (n \in \mathbb{N}). \tag{20}$$

Since the right-hand side of (20) is independent of the point q^{-n}, we may pass to the limit as $n \to \infty$. Thus, we get

$$\int\limits_{\omega_0}^{\infty} |\Psi(x,\lambda)|^2\, d_{\omega,q}x < \frac{\operatorname{Im}\eta}{\sigma},$$

which completes the proof. ▷

References

1. *Weyl, H.* Über gewöhnlicke Differentialgleichungen mit Singuritaten und die zugehörigen Entwicklungen willkürlicher Funktionen, *Mathematische Annalen*, 1910, vol. 68, no. 2, pp. 220–269. DOI: 10.1007/BF01474161.
2. *Titchmarsh, E. C. Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I*, 2nd Edition, Oxford, Clarendon Press, 1962.

3. *Levitan, B. M. and Sargsjan, I. S. Sturm–Liouville and Dirac Operators, Mathematics and its Applications* (*Soviet Series*), Dordrecht, Kluwer Academic Publishers Group, 1991. DOI: 10.1007/978-94-011-3748-5.
4. *Yosida, K.* On Titchmarsh–Kodaira's Formula Concerning Weyl–Stone's Eigenfunction Expansion, *Nagoya Mathematical Journal*, 1950, vol. 1, pp. 49–58. DOI: 10.1017/S0027763000022820.
5. *Yosida, K. Lectures on Differential and Integral Equations*, New York, Springer, 1960.
6. *Levinson, N.* A Simplified Proof of the Expansion Theorem for Singular Second Order Linear Differential Equations, *Duke Mathematical Journal*, 1951, vol. 18, no. 1, pp. 57–71. DOI: 10.1215/S0012-7094-51-01806-6.
7. *Hahn, W.* Über Orthogonalpolynome, die q-Differenzengleichungen genügen, *Mathematische Nachrichten*, 1949, vol. 2, no. 1–2, pp. 4–34. DOI: 10.1002/mana.19490020103.
8. *Hahn, W.* Ein Beitrag zur Theorie der Orthogonalpolynome, *Monatshefte für Mathematik*, 1983, vol. 95, no. 1, pp. 19–24. DOI: 10.1007/BF01301144.
9. *Jackson, F. H.* q-Difference Equations, *American Journal of Mathematics*, 1910, no. 32, pp. 305–314.
10. *Álvarez-Nodarse, R.* On Characterizations of Classical Polynomials, *Journal of Computational and Applied Mathematics*, 2006, vol. 196, no. 1, pp. 320–337. DOI: 10.1016/j.cam.2005.06.046.
11. *Dobrogowska, A. and Odzijewicz, A.* Second Order q-Difference Equations Solvable by Factorization Method, *Journal of Computational and Applied Mathematics*, 2006, vol. 193, no. 1, pp. 319–346. DOI: 10.1016/j.cam.2005.06.009.
12. *Kwon, K. H., Lee, D. W., Park, S. B. and Yoo, B. H.* Hahn Class Orthogonal Polynomials, *Kyungpook Mathematical Journal*, 1998, vol. 38, pp. 259–281.
13. *Lesky, P. A. Eine Charakterisierung der klassischen kontinuierlichen-, diskreten- und q-Orthogonalpolynome*, Aachen, Shaker, 2005.
14. *Petronilho, J.* Generic Formulas for the Values at the Singular Points of Some Special Monic Classical $H_{q,\omega}$-Orthogonal Polynomials, *Journal of Computational and Applied Mathematics*, 2007, vol. 205, no. 1, pp. 314–324. DOI: 10.1016/j.cam.2006.05.005.
15. *Hamza, A. E. and Ahmed, S. A.* Existence and Uniqueness of Solutions of Hahn Difference Equations, *Advances in Difference Equations*, 2013, vol. 316, no. 1, pp. 1–15. DOI: 10.1186/1687-1847-2013-316.
16. *Hamza, A. E. and Ahmed, S. A.* Theory of Linear Hahn Difference Equations, *Journal of Advances in Mathematics*, 2013, vol. 4, no. 2, pp. 440–460.
17. *Hamza, A. E. and Makharesh, S. D.* Leibnizs Rule and Fubinis Theorem Associated with Hahn Difference Operator, *Journal of Advanced Mathematical*, 2016, vol. 12, no. 6, pp. 6335–6345. DOI: 10.24297/jam.v12i6.3836.
18. *Sitthiwirattham, T.* On a Nonlocal Bundary Value Problem for Nonlinear Second-Order Hahn Difference Equation with two Different q,ω-Derivatives, *Advances in Difference Equations*, 2016, vol. 2016, no. 1, article no. 116. DOI: 10.1186/s13662-016-0842-2.
19. *Annaby, M. H., Hamza, A. E. and Makharesh, S. D.*, A Sturm–Liouville Theory for Hahn Difference Operator, *Frontiers of Orthogonal Polynomials and q-Series*, Singapore, World Scientific, 2018, pp. 35–84. DOI: 10.1142/9789813228887_0004.
20. *Annaby, M. H., Hamza, A. E. and Aldwoah, K. A.* Hahn Difference Operator and Associated Jackson–Nörlund Integrals, *Journal of Optimization Theory and Applications*, 2012, vol. 154, no. 1, pp. 133–153. DOI: 10.1007/s10957-012-9987-7.
21. *Knopp, K. Elements of the Theory of Functions*, New York, Dover, 1952.

Received December 30, 2020

Bilender P. Allahverdiev
Department of Mathematics, Süleyman Demirel University,
32260 Isparta, Turkey,
Professor
E-mail: `bilenderpasaoglu@sdu.edu.tr`
https://orcid.org/0000-0002-9315-4652

Hüseyin Tuna
Department of Mathematics, Mehmet Akif Ersoy University,
15030 Burdur, Turkey,
Professor
E-mail: `hustuna@gmail.com`
https://orcid.org/0000-0001-7240-8687

Владикавказский математический журнал
2021, Том 23, Выпуск 3, С. 16–26

ТЕОРИЯ ТИТЧМАРША — ВЕЙЛЯ СИНГУЛЯРНОГО УРАВНЕНИЯ ХАНА — ШТУРМА — ЛИУВИЛЛЯ

Аллахвердиев Б. П.[1], Туна Х.[2]

[1] Университет имени Сулеймана Демиреля, Турция, 32260, Испарта;
[2] Университет Мехмета Акифа Эрсоя, Турция, 15030, Бурдур

E-mail: bilenderpasaoglu@sdu.edu.tr, hustuna@gmail.com

Посвящается 80-летию профессора
Стефана Григорьевича Самко

Аннотация. В этой работе рассматривается сингулярное разностное уравнение Хана — Штурма — Лиувилля, определяемый уравнением $-q^{-1}D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x)+v(x)y(x)=\lambda y(x)$, $x\in(\omega_0,\infty)$, где λ — комплексный параметр, v — вещественнозначная функция, определенная на $[\omega_0,\infty)$ и непрерывная в точке ω_0. Такого вида уравнения возникают, когда обычную производную в классической задаче Штурма — Лиувилля заменяется на (ω,q)-Хан разностным оператором $D_{\omega,q}$. Развивается (ω,q)-аналог классической теории Титчмарша — Вейля для таких уравнений. Другими словами, изучается существование квадратично интегрируемое решение сингулярного уравнения Хана — Штурма — Лиувилля. Сначала определяется подходящее гильбертово пространство в терминах интеграла Джексона — Нёрлунда. Затем изучаются семейства регулярных задач Хана — Штурма — Лиувилля на $[\omega_0,q^{-n}]$, $n\in\mathbb{N}$. Далее, определяется семейство окружностей, сходящейся либо к точке, либо к кругу. Тем самым, в исчислении Хана возникают случаи предельной точки или предельной окружности, используя технику Титчмарша.

Ключевые слова: уравнение Хана — Штурма — Лиувилля, предельная окружность и предельная точка, теория Титчмарша — Вейля.

Mathematical Subject Classification (2010): 39A13, 39A70, 34B20, 39A12.

Образец цитирования: *Allahverdiev, B. P. and Tuna, H.* Titchmarsh–Weyl Theory of the Singular Hahn–Sturm–Liouville Equation // Владикавк. мат. журн.—2021.—Т. 23, № 3.—С. 16–26 (in English). DOI: 10.46698/y9113-7002-9720-u.