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SOME SUBORDINATION RESULTS FOR CERTAIN CLASS WITH COMPLEX ORDER DEFINED BY SALAGEAN TYPE q-DIFFERENCE OPERATOR

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Abstract. The theory of the basic quantum calculus (that is, the basic q-calculus) plays important roles in many diverse areas of the engineering, physical and mathematical science. Making use of the basic definitions and concept details of the q-calculus, Govindaraj and Sivasubramanian [10] defined the Salagean type q-difference (q-derivative) operator. In this paper, we introduce a certain subclass of analytic functions with complex order in the open unit disk by applying the Salagean type q-derivative operator in conjunction with the familiar principle of subordination between analytic functions. Also, we derive some geometric properties such as sufficient condition and several subordination results for functions belonging to this subclass. The results presented here would provide extensions of those given in earlier works.

Key words: analytic function, subordinating factor sequence, hadamard product (or convolution), q-derivative operator, Salagean operator.

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1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We also denote by $\mathscr{K}$ the class of functions $f \in \mathscr{A}$ that are convex in $\mathbb{U}$. For two functions f and g, analytic in $\mathbb{U}$, we say that f is *subordinated* to g in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then (see [1, 2])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Given two functions $f, g \in \mathscr{A}$, where f is given by (1.1) and g is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

the *Hadamard product* (or *convolution*) $f * g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For $0 < q < 1$, the q-*derivative* of a function $f \in \mathscr{A}$ is defined by (see [3–9])

$$D_q f(z) = \begin{cases} f'(0), & z = 0, \\ \frac{f(qz)-f(z)}{(q-1)z}, & z \neq 0, \end{cases} \tag{1.3}$$

and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.3), we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \; a_k z^{k-1}, \tag{1.4}$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + \ldots + q^{k-1}, \tag{1.5}$$

and

$$\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} \frac{f(qz) - f(z)}{(q-1)z} = f'(z),$$

for a function f which is differentiable in a given subset of $\mathbb{C}$.

For $f \in \mathscr{A}$, Govindaraj and Sivasubramanian [10] defined the Salagean type q-difference operator as follows:

$$D_q^0 f(z) = f(z),$$
$$D_q^1 f(z) = z D_q f(z) = z + \sum_{k=2}^{\infty} [k]_q \; a_k z^k,$$
$$D_q^2 f(z) = z D_q \left(D_q^1 f(z) \right) = z + \sum_{k=2}^{\infty} \left([k]_q \right)^2 \; a_k z^k,$$
$$D_q^n f(z) = z D_q \left(D_q^{n-1} f(z) \right), \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$

It is easily see that

$$D_q^n f(z) = z + \sum_{k=2}^{\infty} \left([k]_q \right)^n \; a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.6}$$

We note that

$$\lim_{q \to 1^-} D_q^n f(z) = D^n f(z) = z + \sum_{k=2}^{\infty} k^n \; a_k z^k, \quad n \in \mathbb{N}_0.$$

The differential operator D^n was introduced and studied by Salagean [11] (see also Srivastava and Aouf [12]).

Let $\mathscr{G}_q^n(\lambda, b, A, B)$ denote the subclass of $\mathscr{A}$ consisting of functions $f(z)$ which satisfy

$$1 + \frac{1}{b}\left[(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right) - 1\right] \prec \frac{1+Az}{1+Bz} \tag{1.7}$$

or satisfying

$$\left|\frac{(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right) - 1}{B\left[(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right)\right] - [B + (A-B)b]}\right| < 1, \tag{1.8}$$

$$b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}; \quad 0 \leqslant \lambda \leqslant 1; \quad -1 \leqslant A < B \leqslant 1; \quad 0 < B \leqslant 1; \quad z \in \mathbb{U}.$$

We note that:

(i) $\lim_{q\to 1^-} \mathscr{G}_q^n(\lambda, b, A, B) = \mathscr{G}^n(\lambda, b, A, B)$ (see [13])

$$= \left\{f \in \mathscr{A} : 1 + \frac{1}{b}\left[(1-\lambda)\frac{D^n f(z)}{z} + \lambda\left(D^n f(z)\right)' - 1\right] \prec \frac{1+Az}{1+Bz}\right\};$$

(ii) $\lim_{q\to 1^-} \mathscr{G}_q^n(\lambda, b, 1, -1) = \mathscr{G}^n(\lambda, b)$ (see [14])

$$= \left\{f \in \mathscr{A} : \Re\left(1 + \frac{1}{b}\left[(1-\lambda)\frac{D^n f(z)}{z} + \lambda\left(D^n f(z)\right)' - 1\right]\right) > 0\right\};$$

(iii) $\mathscr{G}_q^0(\lambda, b, A, B) = \mathscr{G}_q(\lambda, b, A, B)$

$$= \left\{f \in \mathscr{A} : 1 + \frac{1}{b}\left[(1-\lambda)\frac{f(z)}{z} + \lambda D_q f(z) - 1\right] \prec \frac{1+Az}{1+Bz}\right\};$$

(iv) $\mathscr{G}_q^n(\lambda, b, -1, 1) = \mathscr{G}_q^n(\lambda, b)$

$$= \left\{f \in \mathscr{A} : \Re\left(1 + \frac{1}{b}\left[(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right) - 1\right]\right) > 0\right\};$$

(v) $\mathscr{G}_q^n(0, 1-\alpha) = \mathscr{G}_q^n(\alpha) = \left\{f \in \mathscr{A} : \Re\left[\frac{D_q^n f(z)}{z}\right] > \alpha,\ 0 \leqslant \alpha < 1\right\}$,

$\mathscr{G}_q^n(0, 1-\alpha) = \mathscr{R}_q^n(\alpha) = \left\{f \in \mathscr{A} : \Re\left[D_q\left(D_q^n f(z)\right)\right] > \alpha,\ 0 \leqslant \alpha < 1\right\}$;

(vi) $\mathscr{G}_q^n(\lambda, 1-\alpha, -1, 1) = \mathscr{G}_q^n(\lambda, \alpha)$

$$= \left\{f \in \mathscr{A} : \Re\left[(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right)\right] > \alpha\right\}, \quad 0 \leqslant \alpha < 1;$$

(vii) $\mathscr{G}_q^n\left(\lambda, e^{-i\theta}(1-\alpha)\cos\theta, -1, 1\right) = \mathscr{G}_q^n(\lambda, \alpha, \theta)$

$$= \left\{f \in \mathscr{A} : \Re\left(e^{i\theta}\left[(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right)\right]\right) > \alpha\cos\theta\right\},$$

$$|\theta| < \frac{\pi}{2}, \quad 0 \leqslant \alpha < 1.$$

2. Main Result

To prove our main result we need the following definition and lemmas.

DEFINITION 1 (Subordinating Factor Sequence [15]). A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a *subordinating factor sequence* if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z), \quad z \in \mathbb{U},\ a_1 = 1.$$

Lemma 1 [15]**.** *The sequence* $\{b_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \quad z \in \mathbb{U}.$$

Now, we prove the following lemma which gives a sufficient condition for functions to belong to the class $\mathscr{G}_q^n(\lambda, b, A, B)$.

Lemma 2. *Let the function f which is defined by* (1.1) *satisfy the following condition:*

$$\sum_{k=2}^{\infty} (1+B)\left\{1+\lambda\left([k]_q - 1\right)\right\}\left([k]_q\right)^n |a_k| \leqslant (B-A)\,|b|, \tag{2.1}$$

then $f \in \mathscr{G}_q^n(\lambda, b, A, B)$.

◁ Suppose that the inequality (2.1) holds. Then we have for $z \in \mathbb{U}$,

$$\begin{aligned}
&\left|(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right) - 1\right| \\
&\quad - \left|B\left[(1-\lambda)\frac{D_q^n f(z)}{z} + \lambda D_q\left(D_q^n f(z)\right)\right] - B - (A-B)\,b\right| \\
&= \left|\sum_{k=2}^{\infty}\left\{1+\lambda\left([k]_q-1\right)\right\}\left([k]_q\right)^n a_k z^{k-1}\right| \\
&\quad - \left|(B-A)\,b + B\sum_{k=2}^{\infty}\left\{1+\lambda\left([k]_q-1\right)\right\}\left([k]_q\right)^n a_k z^{k-1}\right| \\
&\leqslant \sum_{k=2}^{\infty}\left\{1+\lambda\left([k]_q-1\right)\right\}\left([k]_q\right)^n |a_k|\,|z|^{k-1} \\
&\quad - \left\{(B-A)\,|b| - B\sum_{k=2}^{\infty}\left\{1+\lambda\left([k]_q-1\right)\right\}\left([k]_q\right)^n |a_k|\,|z|^{k-1}\right\} \\
&\leqslant \sum_{k=2}^{\infty}(1+B)\left\{1+\lambda\left([k]_q-1\right)\right\}\left([k]_q\right)^n |a_k|\, z^{k-1} - (B-A)\,|b| \leqslant 0,
\end{aligned}$$

which shows that $f(z)$ belongs to the class $\mathscr{G}_q^n(\lambda, b, A, B)$. ▷

Let $\mathscr{G}_q^{n*}(\lambda, b, A, B)$ denote the class of functions $f(z) \in \mathscr{A}$ whose coefficients satisfy the condition (2.1). We note that $\mathscr{G}_q^{n*}(\lambda, b, A, B) \subseteq \mathscr{G}_q^n(\lambda, b, A, B)$. Also, let $\mathscr{G}_q^{0*}(\lambda, b, A, B) = \mathscr{G}_q^{*}(\lambda, b, A, B)$, $\mathscr{G}_q^{n*}(\lambda, b, -1, 1) = \mathscr{G}_q^{n*}(\lambda, b)$, $\mathscr{G}_q^{n*}(\lambda, 1-\alpha, -1, 1) = \mathscr{G}_q^{n*}(\lambda, \alpha)$, $\mathscr{G}_q^{n*}\left(\lambda, e^{-i\theta}(1-\alpha)\cos\theta, -1, 1\right) = \mathscr{G}_q^{n*}(\lambda, b, \theta)$ $\left(|\theta| < \frac{\pi}{2},\ 0 \leqslant \alpha < 1\right)$.

In this paper we prove several subordination relationships involving the functional class $\mathscr{G}_q^{n*}(\lambda, b, A, B)$ employing the technique used earlier by Attiya [16] and Srivastava and Attiya [17] (see also [13, 14, 18–22]).

Theorem 1. *Let the function f defined by* (1.1) *be in the class $\mathscr{G}_q^{n*}(\lambda, b, A, B)$ and $g \in \mathscr{K}$. Then*

$$\left(\frac{(1+B)(1+\lambda q)(1+q)^n}{2\left[(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|\right]}\right)(f*g)(z) \prec g(z) \tag{2.2}$$

and

$$\Re\{f(z)\} > -\frac{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}{(1+B)(1+\lambda q)(1+q)^n}, \quad z \in \mathbb{U}. \tag{2.3}$$

The constant factor $\frac{(1+B)(1+\lambda q)(1+q)^n}{2[(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|]}$ *in the subordination result* (2.2) *cannot be replaced by a larger one.*

◁ Let $f \in \mathscr{G}_q^{n*}(\lambda, b, A, B)$ and let $g(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k \in \mathscr{K}$. Then we have

$$\begin{gathered}\left(\frac{(1+B)(1+\lambda q)(1+q)^n}{2\left[(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|\right]}\right)(f*g)(z) \\ = \left(\frac{(1+B)(1+\lambda q)(1+q)^n}{2\left[(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|\right]}\right)\left(z+\sum_{k=2}^{\infty} a_k c_k z^k\right).\end{gathered} \tag{2.4}$$

Thus, by Definition 1, the subordination result (2.2) will hold true if the sequence

$$\left\{\frac{(1+B)(1+\lambda q)(1+q)^n}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}a_k\right\}_{k=1}^{\infty} \tag{2.5}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+B)(1+\lambda q)(1+q)^n}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}a_k z^k\right\} > 0, \quad z \in \mathbb{U}. \tag{2.6}$$

Since

$$\Psi(k) = \left\{1+\lambda\left([k]_q-1\right)\right\}\left([k]_q\right)^n, \quad k \geqslant 2,\ 0 \leqslant \lambda \leqslant 1,\ 0 < q < 1,\ n \in \mathbb{N}_0,$$

is an increasing function of k ($k \geqslant 2$), when $|z| = r < 1$, we have

$$\begin{aligned}&\Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+B)(1+\lambda q)(1+q)^n}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}a_k z^k\right\} \\ &= \Re\Bigg\{1+\frac{(1+B)(1+\lambda q)(1+q)^n}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}z \\ &\quad + \frac{(1+B)\sum\limits_{k=2}^{\infty}(1+\lambda q)(1+q)^n}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}a_k z^k\Bigg\} \\ &\geqslant 1-\frac{(1+B)(1+\lambda q)(1+q)^n}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}r-\frac{\sum\limits_{k=2}^{\infty}(1+B)\left\{1+\lambda\left([k]_q-1\right)\right\}\left([k]_q\right)^n|a_k|}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}r^k \\ &> 1-\frac{(1+B)(1+\lambda q)(1+q)^n}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}r-\frac{(B-A)|b|}{(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|}r \\ &= 1-r > 0, \quad |z| = r < 1,\end{aligned}$$

where we have also made use of assertion (2.1) of Lemma 2. Thus (2.6) holds true in $\mathbb{U}$ and also the subordination result (2.2) asserted by Theorem 1. The inequality (2.3) follows from (2.2) by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$. To prove the sharpness of the constant $\frac{(1+B)(1+\lambda q)(1+q)^n}{2[(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|]}$, we consider the function $f_0(z) \in \mathscr{G}_q^{n*}(\lambda, b, A, B)$ given by

$$f_0(z) = z - \frac{(B-A)|b|}{(1+B)(1+\lambda q)(1+q)^n} z^2. \tag{2.7}$$

Thus from (2.2), we have

$$\frac{(1+B)(1+\lambda q)(1+q)^n}{2[(1+B)(1+\lambda q)(1+q)^n + (B-A)|b|]} f_0(z) \prec \frac{z}{1-z}, \quad z \in \mathbb{U}. \tag{2.8}$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.7) that

$$\min_{|z|\leqslant r} \left\{ \Re \left(\frac{(1+B)(1+\lambda q)(1+q)^n}{2[(1+B)(1+\lambda q)(1+q)^n + (B-A)|b|]} f_0(z) \right) \right\} = -\frac{1}{2}. \tag{2.9}$$

This shows that the constant $\frac{(1+B)(1+\lambda q)(1+q)^n}{2[(1+B)(1+\lambda q)(1+q)^n+(B-A)|b|]}$ is the best possible, this completes the proof of Theorem 1. ▷

Putting $n = 0$ in Theorem 1, we have

Corollary 1. *Let the function f defined by (1.1) be in the class $\mathscr{G}_q^{*}(\lambda, b, A, B)$ and $g \in \mathscr{K}$. Then*

$$\left(\frac{(1+B)(1+\lambda q)}{2[(1+B)(1+\lambda q) + (B-A)|b|]} \right) (f * g)(z) \prec g(z), \quad z \in \mathbb{U}, \tag{2.10}$$

and

$$\Re\{f(z)\} > -\frac{(1+B)(1+\lambda q) + (B-A)|b|}{(1+B)(1+\lambda q)}. \tag{2.11}$$

The constant factor $\frac{(1+B)(1+\lambda q)}{2[(1+B)(1+\lambda q)+(B-A)|b|]}$ in the subordination result (2.10) cannot be replaced by a larger one.

Putting $A = -1$ and $B = 1$ in Theorem 1, we have

Corollary 2. *Let the function f defined by (1.1) be in the class $\mathscr{G}_q^{n*}(\lambda, b)$ and $g \in \mathscr{K}$. Then*

$$\left(\frac{(1+\lambda q)(1+q)^n}{2[(1+\lambda q)(1+q)^n + |b|]} \right) (f * g)(z) \prec g(z), \quad z \in \mathbb{U}, \tag{2.12}$$

and

$$\Re\{f(z)\} > -\frac{(1+\lambda q)(1+q)^n + |b|}{(1+\lambda q)(1+q)^n}, \quad z \in \mathbb{U}. \tag{2.13}$$

The constant factor $\frac{(1+\lambda q)(1+q)^n}{2[(1+\lambda q)(1+q)^n+|b|]}$ in the subordination result (2.12) cannot be replaced by a larger one.

Putting $b = 1 - \alpha (0 \leqslant \alpha < 1)$, $A = -1$ and $B = 1$ in Theorem 1, we have

Corollary 3. *Let the function f defined by (1.1) be in the class $\mathscr{G}_q^{n*}(\lambda, \alpha)$ $(0 \leqslant \alpha < 1)$ and $g \in \mathscr{K}$. Then*

$$\left(\frac{(1+\lambda q)(1+q)^n}{2[(1+\lambda q)(1+q)^n + 1 - \alpha]} \right) (f * g)(z) \prec g(z), \quad z \in \mathbb{U}, \tag{2.14}$$

and

$$\Re\{f(z)\} > -\frac{(1+\lambda q)(1+q)^n + 1 - \alpha}{(1+\lambda q)(1+q)^n}, \quad z \in \mathbb{U}. \tag{2.15}$$

The constant factor $\frac{(1+\lambda q)(1+q)^n}{2[(1+\lambda q)(1+q)^n+1-\alpha]}$ *in the subordination result* (2.14) *cannot be replaced by a larger one.*

Putting $b = e^{-i\theta}(1-\alpha)\cos\theta$ $\left(|\theta| < \frac{\pi}{2};\ 0 \leqslant \alpha < 1\right)$, $A = -1$ and $B = 1$ in Theorem 1, we have

Corollary 4. *Let the function* f *defined by* (1.1) *be in the class* $\mathscr{G}_q^{n*}(\lambda, \alpha, \theta)$ $\left(|\theta| < \frac{\pi}{2};\ 0 \leqslant \alpha < 1\right)$ *and* $g \in \mathscr{K}$. *Then*

$$\left(\frac{(1+\lambda q)(1+q)^n}{2\left[(1+\lambda q)(1+q)^n + (1-\alpha)\cos\theta\right]}\right)(f * g)(z) \prec g(z), \quad z \in \mathbb{U}, \tag{2.16}$$

and

$$\Re\{f(z)\} > -\frac{(1+\lambda q)(1+q)^n + (1-\alpha)\cos\theta}{(1+\lambda q)(1+q)^n}, \quad z \in \mathbb{U}. \tag{2.17}$$

The constant factor $\frac{(1+\lambda q)(1+q)^n}{2[(1+\lambda q)(1+q)^n+(1-\alpha)\cos\theta]}$ *in the subordination result* (2.14) *cannot be replaced by a larger one.*

Remark 1. Taking $A = -1$, $B = 1$ and letting $q \to 1^-$ in Theorem 1, we get the result obtained by Aouf [14, Theorem 1].

Remark 2. Replacing A by $-A$, B by $-B$ and letting $q \to 1^-$ in Theorem 1, we obtain the result get by Sivasubramanian et al. [13, Theorem 2.2].

Remark 3. Putting $n = \lambda = 0$, $b = 1 - \alpha$ $(0 \leqslant \alpha < 1)$ and letting $q \to 1^-$ in Corollary 2, we get the result obtained by Aouf [14, Corollary 3].

Remark 4. Putting $n = 0$, $\lambda = 1$, $b = 1-\alpha$ $(0 \leqslant \alpha < 1)$ and letting $q \to 1^-$ in Corollary 2, we get the result obtained by Aouf [14, Corollary 4].

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НЕКОТОРЫЕ РЕЗУЛЬТАТЫ О ПОДЧИНЕНИИ ДЛЯ ОДНОГО ФУНКЦИОНАЛЬНОГО КЛАССА, ОПРЕДЕЛЯЕМОГО q-РАЗНОСТНЫМ ОПЕРАТОРОМ ТИПА САЛАГИНА

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Аннотация. Теория базового квантового исчисления (то есть базового q-исчисления) играет важную роль в различных областях знания, инженерии, физико-математических науках. Используя основные определения и некоторые детали q-исчисления, Говиндарадж и Сивасубраманиан [10] определили q-разностный (q-производный) оператор типа Салагина. В этой статье мы вводим определенный подкласс аналитических функций со сложным порядком в открытом единичном круге, применяя q-производный оператор типа Салагина в сочетании с известным принципом подчинения между аналитическими функциями. Кроме того, мы выводим некоторые геометрические свойства и несколько результатов о подчинении для функций, принадлежащих этому подклассу. Представленные здесь результаты расширяют результаты, представленные в более ранних работах.

Ключевые слова: аналитическая функция, подчиняющая фактор последовательность, произведение Адамара (или конволюция), q-производный оператор, оператор Салагина.

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