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THE GORDON THEOREM: ORIGINS AND MEANING

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To Evgeny Gordon on occasion of his 70th birthday

Abstract. Boolean valued analysis, the term coined by Takeuti, signifies a branch of functional analysis which uses a special technique of Boolean valued models of set theory. The fundamental result of Boolean valued analysis is Gordon's Theorem stating that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the mathematical knowledge by translating information about the reals to the language of other branches of functional analysis. This is a brief overview of the mathematical events around the Gordon Theorem. The relationship between the Kantorovich's heuristic principle and Boolean valued transfer principle is also discussed.

Key words: vector lattice, Kantorovich's principle, Gordon's theorem, Boolean valued analysis.

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1. Introduction

In 1977, Evgeny Gordon, a young teacher of Lobachevsky Nizhny Novgorod State University, published the short note [1] which begins with the words:

"*This article establishes that the set whose elements are the objects representing reals in a Boolean valued model of set theory* $\mathbb{V}^{(\mathbb{B})}$ *can be endowed with the structure of a vector space and an order relation so that it becomes an extended K-space** *with base*** $\mathbb{B}$*. It is shown that in some cases this fact can be used to generalize the theorems about reals to extended K-spaces.*"

His note has become the bridge between various areas of mathematics which helps, in particular, to solve many problems of functional analysis in "semiordered vector spaces" [4] by using the techniques of Boolean valued models of set theory [5].

* A *K-space* or a *Kantorovich space* is a Dedekind complete vector lattice. An *extended K-space* is a universally complete vector lattice, cp. [2] and [3].

** The *base* of a vector lattice is the Boolean algebra of all of its bands [3].

In the same year, at the Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis (Durham, July 9–11, 1977), Gaisi Takeuti, a renowned expert in proof theory, observed that if $\mathbb{B}$ is a complete Boolean algebra of orthogonal projections in a Hilbert space H, then the set whose elements represent reals in the Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ can be identified with the vector lattice of selfadjoint operators in H whose spectral resolutions take values in $\mathbb{B}$; see [6].

These two events marked the birth of a new section of functional analysis, which Takeuti designated by the term *Boolean valued analysis*. The history and achievements of Boolean valued analysis are reflected in [7–9].

It should be mentioned that Dana Scott foresaw in 1969 [10] that the new nonstandard models must be of mathematical interest aside from the independence proof, but he was unable to give a really good evidence of this. In fact Takeuti found a narrow path whereas Gordon paved a turnpike to the core of mathematics, which justifies the vision of Scott.

Boolean valued analysis signifies the technique of studying the properties of an arbitrary mathematical object by comparison between its representations in two different Boolean valued models of set theory. As the models, we usually take the *von Neumann universe* $\mathbb{V}$ (the mundane embodiment of the classical Cantorian paradise) and the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ (a specially-trimmed universe whose construction utilizes a complete Boolean algebra $\mathbb{B}$ with a top element $\mathbb{1}$). The principal difference between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$ is the way of verification of statements: There is a natural way of assigning to each statement ϕ about $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ the *Boolean truth-value* $[\![\phi(x_1, \dots, x_n)]\!] \in \mathbb{B}$. The sentence $\phi(x_1, \dots, x_n)$ is called true in $\mathbb{V}^{(\mathbb{B})}$ if $[\![\phi(x_1, \dots, x_n)]\!] = \mathbb{1}$. All theorems of Zermelo–Fraenkel set theory with the axiom of choice are true in $\mathbb{V}^{(\mathbb{B})}$ for every complete Boolean algebra $\mathbb{B}$. There is a smooth and powerful mathematical technique for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending machinery* rests on the functors of *canonical embedding* $X \mapsto X^{\wedge}$ and *ascent* $X \mapsto X{\uparrow}$ acting from $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$ and *descent* $X \mapsto X{\downarrow}$ acting from $\mathbb{V}^{(\mathbb{B})}$ into $\mathbb{V}$; see [7, 8].

Everywhere below $\mathbb{B}$ is a complete Boolean algebra and $\mathbb{V}^{(\mathbb{B})}$ the corresponding Boolean valued model of set theory; see [5, 11, 12]. We let $:=$ denote the assignment by definition, while $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ symbolize the naturals, the reals, and the complexes.

2. Kantorovich's Heuristic Principle

The unexplained terms of vector lattice theory can be found in [2, 3, 13, 14]. All vector lattices below are assumed to be Archimedean.

Definition 1. A *vector lattice* or a *Riesz space* is a real vector space X equipped with a partial order $\leqslant$ for which the *join* $x \vee y$ and the *meet* $x \wedge y$ exist for all $x, y \in X$, and such that the *positive cone* $X_+ := \{x \in X : 0 \leqslant x\}$ is closed under addition and multiplication by positive reals and for any $x, y \in X$ the relations $x \leqslant y$ and $0 \leqslant y - x$ are equivalent. A *band* in a vector lattice X is the *disjoint complement* $Y^{\perp}$ of any subset $Y \subset X$ where $Y^{\perp} := \{x \in X : \ (\forall\, y \in Y)\, |x| \wedge |y| = 0\}$. Let $\mathbb{B}(X)$ and $\mathbb{P}(X)$ stand for the inclusion ordered sets of all bands and all band projections in X, respectively.

Definition 2. A subset $U \subset X$ is *order bounded* if U lies in an *order interval* $[a, b] := \{x \in X : \ a \leqslant x \leqslant b\}$ for some $a, b \in X$. A vector lattice X is *Dedekind complete* (respectively, *laterally complete*) if each nonempty order bounded set (respectively, each nonempty set of pairwise disjoint positive vectors) U in X has a least upper bound $\sup(U) \in X$. Note that $\mathbb{B}(X)$ and $\mathbb{P}(X)$ are isomorphic Boolean algebras for such X. The vector lattice that is laterally complete and Dedekind complete simultaneously is referred to as *universally complete*.

Definition 3. An *f-algebra* is a vector lattice X equipped with a distributive multiplication such that if $x, y \in X_+$ then $xy \in X_+$, and if $x \wedge y = 0$ then $(ax) \wedge y = (xa) \wedge y = 0$ for all $a \in X_+$. An f-algebra is *semiprime* provided that $xy = 0$ implies $x \perp y$ for all x and y. A *complex vector lattice* $X_{\mathbb{C}}$ is the complexification $X_{\mathbb{C}} := X \oplus iX$ (with i standing for the imaginary unity) of a real vector lattice X.

Leonid Kantorovich was among the first who studied operators in ordered vector spaces. He distinguished an important instance of ordered vector spaces, a Dedekind complete vector lattice, often called a *Kantorovich space* or a *K-space*. This notion appeared in Kantorovich's first fundamental article [15] on this topic where he wrote:

"In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (*those with values in such a space*) *as linear functionals."*

Here Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces, claiming that the elements of a K-space can be considered as generalized reals. Essentially, this principle turned out to be one of those profound ideas that, playing an active and leading role in the formation of a new branch of analysis, led eventually to a deep and elegant theory of K-space rich in various applications. At the very beginning of the development of the theory, attempts were made at formalizing the above heuristic argument. In this direction, there appeared the so-called *identity preservation theorems* which claimed that if some proposition involving finitely many relations is proven for the reals then an analogous fact remains valid automatically for the elements of every K-space (see [3, 4, 14]). The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis. See more about the Kantorovich's universal heuristics and innate integrity of his methodology in [16]. The contemporary forms of above mentioned relation preservation theorems, basing on Boolean valued models, may be found in Gordon [17–19] and Jech [20, 21].

3. Boolean Valued Reals

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the mathematical knowledge by translating information about the reals to the language of other branches of functional analysis.

According to the principles of Boolean valued set theory there exists an internal field of reals $\mathscr{R}$ in $\mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. In other words, there exists $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals $]\!] = \mathbb{1}$. Moreover, if $[\![\, \mathscr{R}'$ is a field of reals $]\!] = \mathbb{1}$ for some $\mathscr{R}' \in \mathbb{V}^{(\mathbb{B})}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = \mathbb{1}$.

By the same reasons there exists an internal field of complexes $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R}$. We call $\mathscr{R}$ and $\mathscr{C}$ the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

The fundamental result of Boolean valued analysis is *Gordon's Theorem* [1] which reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model.* Formally:

Gordon Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra, $\mathscr{R}$ be a field of reals within $\mathbb{V}^{(\mathbb{B})}$. Endow $\mathbf{R} := \mathscr{R}{\downarrow}$ with the descended operations and order. Then*

(1) *The algebraic structure $\mathbf{R}$ is an universally complete vector lattice.*

(2) *The field $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ can be chosen so that $[\![\, \mathbb{R}^{\wedge}$ is a dense subfield of $\mathscr{R} \,]\!] = \mathbb{1}$.*

(3) *There is a Boolean isomorphism χ from $\mathbb{B}$ onto $\mathbb{P}(\mathbf{R})$ such that*

$$\chi(b)x = \chi(b)y \Longleftrightarrow b \leqslant [\![x = y]\!],$$
$$\chi(b)x \leqslant \chi(b)y \Longleftrightarrow b \leqslant [\![x \leqslant y]\!]$$
$$(x, y \in \mathbf{R};\ b \in \mathbb{B}).$$

The converse is also true: *Each Archimedean vector lattice embeds into an appropriated Boolean valued model, becoming a vector sublattice of the reals* (*viewed as such over some dense subfield of the reals*). More details on the Boolean valued theory of vector lattices and positive operators can be found in [7–9, 22].

Gutman [23] characterized those complete Boolean algebras $\mathbb{B}$ for which the internal fields $\mathbb{R}^{\wedge}$ and $\mathscr{R}$ coincide: $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^{\wedge}$ *if and only if* $\mathbb{B}$ *is the vector lattice σ-distributive** *if and only if $\mathscr{R}{\downarrow}$ is locally one-dimensional***. He also proved that there exist nondiscrete locally one-dimensional Dedekind complete vector lattice. Observe also some additional properties of Boolean valued reals, multiplicative structure and complexification:

Corollary 1. *The universally complete vector lattice $\mathscr{R}{\downarrow}$ with the descended multiplication is a semiprime f-algebra with the ring unity $\mathbb{1} := 1^{\wedge}$. Moreover, for every $b \in \mathbb{B}$ the band projection $\chi(b) \in \mathbb{P}(\mathbf{R})$ acts as multiplication by $\chi(b)\mathbb{1}$.*

Corollary 2. *Let $\mathscr{C}$ be the complexes within $\mathbb{V}^{(\mathbb{B})}$. Then the algebraic system $\mathscr{C}{\downarrow}$ is a universally complete complex f-algebra. Moreover, $\mathscr{C}{\downarrow}$ is the complexification of the universally complete real f-algebra $\mathscr{R}{\downarrow}$; i. e., $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$.*

Example 1. Assume that a measure space (Ω, Σ, μ) is semi-finite; i. e., if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(B) < \infty$. The vector lattice $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$ (of cosets) of μ-measurable functions on Ω is universally complete if and only if (Ω, Σ, μ) is *localizable* ($\equiv$ *Maharam*). In this event $L^p(\Omega, \Sigma, \mu)$ is Dedekind complete; see [24, 241G]. Note that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$. In [25], Scott observed that in the algebra of random variables of a probability space (as in a Boolean structure) all Boolean truth values of the axioms of the field of reals are $\mathbb{1}$.

Example 2. Given a complete Boolean algebra $\mathbb{B}$ of orthogonal projections in a Hilbert space H, denote by $\langle\mathbb{B}\rangle$ the space of all selfadjoint operators on H whose spectral resolutions are in $\mathbb{B}$; i. e., $A \in \langle\mathbb{B}\rangle$ if and only if $A = \int_{\mathbb{R}} \lambda\, dE_\lambda$ and $E_\lambda \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$. Define the partial order in $\langle\mathbb{B}\rangle$ by putting $A \geqslant B$ whenever $\langle Ax, x\rangle \geqslant \langle Bx, x\rangle$ for all $x \in \mathscr{D}(A) \cap \mathscr{D}(B)$, where $\mathscr{D}(A) \subset H$ stands for the domain of A. Then $\langle\mathbb{B}\rangle$ is a universally complete vector lattice and the Boolean algebras $\mathbb{P}(\langle\mathbb{B}\rangle)$ and $\mathbb{B}$ are isomorphic.

If μ is a Maharam measure and $\mathbb{B}$ in the Gordon Theorem is the algebra of all μ-measurable sets modulo μ-negligible sets, then $\mathscr{R}{\downarrow}$ is lattice isomorphic to $L^0(\mu)$; see Example 1. If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space H then $\mathscr{R}{\downarrow}$ is isomorphic to $\langle\mathbb{B}\rangle$; see Example 2. The two indicated particular cases of Gordon's Theorem were intensively and fruitfully exploited by Takeuti [6, 26]. The object $\mathscr{R}{\downarrow}$ for the general Boolean algebras was also studied by Jech [20, 21], who in fact rediscovered Gordon's Theorem. The difference is that in [20] a (complex) universally complete vector lattice with unit is defined by another system of axioms and is referred to as a complete *Stone algebra*.

* A Boolean algebra $\mathbb{B}$ is called *σ-distributive* if for every double sequence $(b_{n,m})_{n,m\in\mathbb{N}}$ in $\mathbb{B}$ the following equation holds: $\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b_{n,m} = \bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \bigvee_{n\in\mathbb{N}} b_{n,\varphi(n)}$.

** A universally complete vector lattice G is called *locally one-dimensional* if all positive elements of G are *locally constants* with respect to and arbitrary order unit $\mathbb{1}$, that is, every $x \in G_+$ is reprentable as $e = \sup_{\xi\in\Xi} \lambda_\xi \pi_\xi \mathbb{1}$ for some numeric family $(\lambda_\xi)_{\xi\in\Xi}$ and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections.

4. Concluding Remarks

1. In 1963 Cohen discovered his *method of forcing* and also proved the independence of the Continuum Hypothesis. A comprehensive presentation of the Cohen forcing method gave rise to the *Boolean valued models of set theory*, which were first introduced by Scott and Solovay (see Scott [10]*) and Vopěnka [27]. In an extremely interesting and illuminating foreword to Bell's book [5] written by Scott, the development following Cohen's discovery is characterized as follows:

"*It was in 1963 that we were hit by a real bomb, however, when Paul J. Cohen discovered his method of 'forcing', which started a long chain reaction of independence results stemming from his initial proof of the independence of the Continuum Hypothesis. Set theory could never be the same after Cohen, and there is simply no comparison whatsoever in the sophistication of our knowledge about models for set theory today as contrasted to the pre-Cohen era.*"

Many delicate properties of the objects within $\mathbb{V}^{(\mathbb{B})}$ depend essentially on the structure of the initial complete Boolean algebra $\mathbb{B}$. The diversity of opportunities together with a great stock of information on particular Boolean algebras ranks Boolean valued models among the most powerful tools of foundational studies. A systematic account of the theory of Boolean valued models and its applications to independence proofs can be found in [5, 11, 12, 28].

2. Recal that ZF is *Zermelo–Fraenkel set theory*, AC and DC stand for the *Axiom of Choice* and the *Principle of Dependent Choice*, respectively, ZFC=ZF+AC, and LM denotes the sentence "Every set of reals is Lebesgue measurable." Solovay, in his celebrated work [29], proved the following result by constructing a model for ZF+DC+LM.

Theorem 1. *If the existence of an inaccessible cardinal is consistent with* ZFC, *then*

(1) *The statement "Every subset of* $\mathbb{R}$ *definable by a countable sequence of ordinals is Lebesgue measurable" is consistent with* ZF.

(2) LM *is consistent with* ZF+DC.

Solovay then posed the famous problem: *Does Theorem 1 remains true without assumption of consistency of the existence of an inaccessible cardinal?*

Solovay's model have many interesting properties. For example, the Hahn-Banach theorem fails in the model of Theorem 1, while it follows readily from DC for separable Banach spaces [29, p. 3]. Moreover, in Solovay's model each linear operator on a Hilbert space is a bounded linear operator; see [30, Theorem 6].

3. Gordon came to his theorem, while trying to attack Solovay's problem. He failed to solve the problem but proved the following weaker statement; see [1, Theorem 7] and [31].

Theorem 2. *The statement "The Lebesgue measure on* $\mathbb{R}$ *can be extended to a* σ*-additive invariant measure on the* σ*-algebra of sets definable by a countable sequence of ordinals" is consistent with* ZFC.

In order to prove Theorem 2, he needed to consider the elements $\mathscr{B}, \mu \in \mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B}$ is a complete Boolean algebra with measure and

$$[\![(\mathscr{B}, \mu) \text{ is a complete Boolean algebra with measure}]\!] = \mathbb{1},$$

and identify in $\mathbb{V}$ the descent $\mu{\downarrow} : \mathscr{B}{\downarrow} \to \mathscr{R}{\downarrow}$ of μ as a vector measure on the complete Boolean algebra $\mathscr{B}{\downarrow}$ with values in $\mathscr{R}{\downarrow}$. The fact that $\mathscr{B}{\downarrow}$ is a complete Boolean algebra that contains $\mathbb{B}$ as a complete subalgebra was known from the paper [28] about the iterated forcing.

* There are many references in the literature to the Scott–Solovay unpublished paper *"Boolean valued models of set theory."* The reasons for this are discussed in the preface to the book [5].

So, he considered the algebraic structure $\mathscr{R}{\downarrow}$ and proved that it is the extended K-space with the base isomorphic to $\mathbb{B}$. He learned about K-spaces from the book [32]. Now, since $\mathbb{B}$ is an algebra with measure, a real-valued measure on $\mathscr{B}{\downarrow}$ can be produced by integration of elements $\mu{\downarrow}(b) \in \mathscr{R}{\downarrow}$ for all $b \in \mathscr{B}{\downarrow}$.

4. The Solovay problem was settled by Shelah [33], who showed that the assumption about inaccessible cardinal cannot be removed from Theorem 1. More precisely, he proved that ZF+DC+LM implies that ω_1 is inaccessible in L, the universe of Gödel constructible sets. It is also worth mentioning that Sacks [34] obtained the following result without assuming the existence of an inaccessible cardinal.

Theorem 3. *The statement "The Lebesgue measure on $\mathbb{R}$ can be extended to the σ-additive invariant measure defined on all subsets of $\mathbb{R}$" is consistent with* ZF+DC.

In particular, Shelah's result brings to light the importance of Theorems 2 and 3.

5. Two more remarkable independence results are worth mentioning here. We first recall the following abbreviations:

SH (*Souslin's Hypothesis*): Every order complete order dense linearly ordered set having neither bottom nor top element is order isomorphic to the ordered set of the reals $\mathbb{R}$, provided that every collection of mutually disjoint nonempty open intervals in it is countable.

NDH (*No Discontinuous Homomorphisms*): For each compact space X, each homomorphism from $C(X, \mathbb{C})$, the Banach algebra of all continuous complex-valued functions on X, into arbitrary complex Banach algebra is continuous. NDH is equivalent to saying that every algebra norm on $C(X, \mathbb{C})$ is equivalent to the uniform norm.

The problem whether or not SH is true was posed by Souslin in 1920. The corresponding problem for NDH dates back to the Kaplansky article of 1948.

Theorem 4. *Both statements SH and NDH are independent of ZFC.*

Tennenbaum [35] and Jech [36] both gave models in which SH is false. Solovay and Tennenbaum [28] extended Cohen's method to define models in which SH holds. The consistency of $\neg$NDH is due to Dales and Esterly, while the consistency of NDH was proved by Solovay and Woodin; see [37] for details. Thus, like the Continuum Hypothesis, SH and NDH are undecidable on using the contemporary axioms of set theory.

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ТЕОРЕМА ГОРДОНА: ИСТОКИ И СМЫСЛ

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Аннотация. Термин булевозначный анализ, введенный Такеути, обозначает раздел функционального анализа, в котором используется специальная техника булевозначных моделей теории множеств. Фундаментальным результатом булевозначного анализа является теорема Гордона о том, что каждое внутреннее поле вещественных чисел булевозначной модели спускается в универсально полную векторную решетку. Таким образом, открывается замечательная возможность расширить и обогатить математические знания, переводя информацию о вещественных числах на язык других разделов функционального анализа. Настоящая работа — краткий обзор математических событий вокруг теоремы Гордона. Обсуждается также связь между эвристическим принципом Канторовича и принципом булевозначного переноса.

Ключевые слова: векторная решетка, принцип Канторовича, теорема Гордона, булевозначный анализ.

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