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OPERATORS ON INJECTIVE BANACH LATTICES[1]

A. G. Kusraev

To Alexander Gutman
on occasion of his 50th birthday

The paper deals with some properties of bounded linear operators on injective Banach lattice using a Boolean-valued transfer principle from AL-spaces to injectives stated in author's previous work.

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1. Introduction

In this paper we consider some properties of bounded linear operators on injective Banach lattices using a *Boolean-valued transfer principle* from AL-spaces to injective Banach lattices stated in Kusraev [1]. In Section 2 we collect some Boolean valued representation results for Banach lattices and regular operators (Theorems 2.2, 2.4, and 2.5). In Section 3 we present a Daugavet type equation (Theorem 3.5 and Corollary 3.9) and a Daugavet type inequality (Theorem 3.8) for operators on injective Banach lattices. Section 4 deals with the problem when the spaces of regular (Theorem 4.4), cyclically compact (Theorem 4.7), and cone $\mathbb{B}$-summing (Theorem 4.10) operators are injective Banach lattice.

Recall some basic definitions. A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ of T_0 with $\|T_0\| = \|T\|$. A Dedekind complete AM-space with unit (Abramovich [2] and Lotz [3]) as well as an AL-space (Lotz [3]) is an injective Banach lattice, see Meyer–Nieberg [4].

We denote by $\mathbb{P}(X)$ the Boolean algebra of all band projections on a vector lattice X. A crucial role in the structure theory of injective Banach lattice plays the concept of M-projection. A band projection π in a Banach lattice X is called an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^{\perp} x\|\}$ for all $x \in X$, where $\pi^{\perp} := I_X - \pi$. The set $\mathbb{M}(X)$ of all M-projections in X forms a Boolean subalgebra of $\mathbb{P}(X)$. Haydon [5] proved that an injective Banach lattice X is an AL-space if and only if $\mathbb{M}(X) = \{0, I_X\}$.

In what follows X and Y denote Banach lattices, while $\mathscr{L}(X,Y)$ and $\mathscr{L}^r(X,Y)$ stand respectively for the spaces of bounded and regular operators from X into Y and $\|T\|_r$ stands for the regular norm of $T \in \mathscr{L}^r(X,Y)$, i. e., $\|T\|_r := \|\,|T|\,\|$. Throughout the sequel $\mathbb{B}$ is

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a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$, while $\Lambda := \Lambda(\mathbb{B})$ is a Dedekind complete AM-space with unit such that $\mathbb{B} \simeq \mathbb{P}(\Lambda)$; in this event $\mathbb{B}$ and $\mathbb{P}(\Lambda)$ are identified with $\mathbb{1}$ taken as the unit both in $\mathbb{B}$ and $\mathbb{P}(\Lambda)$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$.

For the theory of Banach lattices and positive operators we refer to the books Meyer–Nieberg [4] and Aliprantic and Burkinshaw [6]. The needed information on the theory of Boolean-valued models is briefly presented in Kusraev [7, Chapter 9] and Kusraev and Kutateladze [8, Chapter 1]; details may be found in Bell [9], Kusraev and Kutateladze [10], Takeuti and Zaring [11]. We let := denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ symbolize the naturals, the rationals, and the reals.

2. Boolean Valued Representation

In this section we present some Boolean valued representation results needed in the sequel. Assume that X is a Banach lattice and $\mathscr{B}$ is a complete subalgebra of a complete Boolean algebra $\mathbb{B}(X)$ consisting of projection bands and denote by $\mathbb{B}$ the corresponding Boolean algebra of band projections. We will identify $\mathbb{P}(\Lambda)$ and $\mathbb{B}$.

Definition 2.1. If $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$ and $(x_\xi)_{\xi\in\Xi}$ is a family in X, then there is at most one element $x \in X$ with $b_\xi x_\xi = b_\xi x$ for all $\xi \in \Xi$. This element x, if existing, is called the *mixing* of (x_ξ) by (b_ξ). Clearly, $x = o\text{-}\sum_{\xi\in\Xi} b_\xi x_\xi$. A Banach lattice X is said to be $\mathbb{B}$-*cyclic* or $\mathbb{B}$-*complete* if the mixing of every family in the unit ball $U(X)$ of X by each partition of unity in $\mathbb{B}$ (with the same index set) exists in $U(X)$.

A Banach lattice $(X, \|\cdot\|)$ is $\mathbb{B}$-cyclic with respect to a complete Boolean algebra $\mathbb{B}$ of band projections on X if and only if there exists a $\Lambda(\mathbb{B})$-valued norm $[\![\cdot]\!]$ on X such that $(X, [\![\cdot]\!])$ is a Banach–Kantorovich space, $|x| \leqslant |y|$ implies $[\![x]\!] \leqslant [\![y]\!]$ for all $x, y \in X$, and $\|x\| = \|[\![x]\!]\|_\infty$ $(x \in X)$, see Kusraev and Kutateladze [8, Theorems 5.8.11 and 5.9.1].

Theorem 2.2. *A restricted descent of a Banach lattice from the model $\mathbb{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-cyclic Banach lattice. Conversely, if X is a $\mathbb{B}$-cyclic Banach lattice, then in the model $\mathbb{V}^{(\mathbb{B})}$ there exists up to the isometric isomorphism a unique Banach lattice $\mathscr{X}$ whose restricted descent $\mathscr{X}{\Downarrow}$ is isometrically $\mathbb{B}$-isomorphic to X. Moreover, $\mathbb{B} = \mathbb{M}(X)$ if and only if $[\![$there is no M-projection in $\mathscr{X}$ other than 0 and $I_{\mathscr{X}}]\!] = \mathbb{1}$.*

$\lhd$ See Kusraev and Kutateladze [8, Theorem 5.9.1]. $\rhd$

Definition 2.3. The elements $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ in Theorem 2.2 and $\mathscr{T} \in \mathbb{V}^{(\mathbb{B})}$ in Theorem 2.4 below are said to be the *Boolean valued representations of* X and T, respectively.

Denote by $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ the space of all regular $\mathbb{B}$-linear operators from X to Y equipped with the *regular norm* $\|T\|_r := \inf\{\|S\| : S \in \mathscr{L}_{\mathbb{B}}(X,Y),\ \pm T \leqslant S\}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be the Boolean valued representations of $\mathbb{B}$-cyclic Banach lattices X and Y, respectively, while $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ stands for the space of all regular operators from $\mathscr{X}$ to $\mathscr{Y}$ with the regular norm within $\mathbb{V}^{(\mathbb{B})}$. The following result states that $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is the *Boolean valued representation of* $\mathscr{L}^r_{\mathbb{B}}(X,Y)$.

Theorem 2.4. *Assume that X and Y are $\mathbb{B}$-cyclic Banach lattices, while $\mathscr{X}$ and $\mathscr{Y}$ are their respective Boolean valued representation. The space $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is order $\mathbb{B}$-isometric to the bounded descent $\mathscr{L}^r(\mathscr{X},\mathscr{Y}){\Downarrow}$ of $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$. The isomorphism is set up by assigning to any $T \in \mathscr{L}^r_{\mathbb{B}}(X,Y)$ the element $\mathscr{T} := T{\uparrow}$ of $\mathbb{V}^{(\mathbb{B})}$ is uniquely determined from the formulas $[\![\, \mathscr{T} : \mathscr{X} \to \mathscr{Y} \,]\!] = \mathbb{1}$ and $[\![\, \mathscr{T}x = Tx \,]\!] = \mathbb{1}$ $(x \in X)$.*

$\lhd$ According to Theorem 2.2 we may assume without loss of generality that X and Y are the bounded descents of some Banach lattices $\mathscr{X}$ and $\mathscr{Y}$. Moreover, $\mathscr{L}_{\mathbb{B}}(X,Y)$ and

$\mathscr{L}(\mathscr{X},\mathscr{Y}){\Downarrow}$ are $\mathbb{B}$-isometric by [7, Theorem 8.3.6]. Since $T(X_+){\uparrow} = T{\uparrow}(X_+{\uparrow}) = \mathscr{T}(\mathscr{X}_+)$, it follows that $T(X_+) \subset Y_+$ if and only if $[\![\mathscr{T}(\mathscr{X}_+) \subset \mathscr{Y}_+]\!] = \mathbb{1}$. This means that the bijection $T \leftrightarrow \mathscr{T} = T{\uparrow}$ preserves positivity and hence is an order $\mathbb{B}$-isomorphism between $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ and $\mathscr{L}^r(\mathscr{X},\mathscr{Y}){\Downarrow}$. Since for $S \in \mathscr{L}^r_{\mathbb{B}}(X,Y)$ and $\mathscr{S} := S{\uparrow}$ the relations $\pm T \leqslant S$ and $[\![\pm\mathscr{T} \leqslant \mathscr{S}]\!] = \mathbb{1}$ are equivalent, we have $[\![\|\mathscr{T}\|_r = |T|_r]\!] = \mathbb{1}$, where $|T|_r = \inf\{|S| : S \in \mathscr{L}^r_{\mathbb{B}}(X,Y),\ \pm T \leqslant S\}$ and $|S| := \sup\{|Sx| : |x| \leqslant \mathbb{1}\}$. Thus, it remains to prove that $\|T\|_r = \||T|_r\|_\infty$ $(T \in \mathscr{L}^r_{\mathbb{B}}(X,Y))$.

If $\pm T \leqslant S$ then $\||T|\|_\infty \leqslant \||S|\|_\infty = \|S\|$ and hence $\|T\|_r \geqslant \||T|_r\|_\infty$. To prove the reverse inequality take an arbitrary $0 < \varepsilon \in \mathbb{R}$ and choose a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(S_\xi)_{\xi\in\Xi}$ in $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ such that $S_\xi \geqslant \pm T$ and $\pi_\xi|S_\xi| \leqslant (1+\varepsilon)|T|_r$ for all $\xi \in \Xi$. Define an operator $S \in \mathscr{L}^r_{\mathbb{B}}(X,Y)$ by $Sx := \operatorname{mix}_{\xi\in\Xi}\pi_\xi S_\xi x$ $(x \in X)$, where the mixing exists in Y, since $|S_\xi x| \leqslant (1+\varepsilon)|T|_r|x|$ and hence $(S_\xi x)$ is norm bounded in Y. Moreover, $Sx = \sum_\xi \pi_\xi S_\xi x$ in the sense of Λ-valued norm on Y. Therefore, $S \geqslant \pm T$ and $|S| \leqslant (1+\varepsilon)|T|_r$, whence $\|T\|_r \leqslant \|S\| = \||S|\|_\infty \leqslant (1+\varepsilon)\||T|_r\|_\infty$. $\rhd$

Theorem 2.5. *Let X be a $\mathbb{B}$-cyclic Banach lattice and let $\mathscr{X}$ be its Boolean valued representation in $\mathbb{V}^{(\mathbb{B})}$. Then the following hold:*

(1) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is Dedekind complete" if and only if X is Dedekind complete.*

(2) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is injective" if and only if X is injective.*

(3) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is an AM-space" if and only if X is an AM-space.*

(4) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is an AL-space" if and only if X is injective and $\mathbb{B} \simeq \mathbb{M}(X)$.*

$\lhd$ See Kusraev and Kutateladze [8, Theorems 5.9.6 (1) and 5.12.1]. $\rhd$

Remark 2.6. As was mentioned in the introduction, Boolean valued analysis approach plays a key role in the proofs below. An alternative approach relies upon Gutman's theory of bundle representation of lattice normed spaces developed in [12, 13].

3. The Daugavet Equation in Injective Banach Latices

Definition 3.1. If X is a real Banach space, a bounded linear operator $T : X \to X$ is said to satisfy the *Daugavet equation* if $\|I_X + T\| = 1 + \|T\|$.

Theorem 3.2. *If T is a bounded operator on an AL-space X then either T or $-T$ satisfies the Daugavet equation.*

$\lhd$ The proof and the history of this theorem see in Abramovich and Aliprantis [14, Theorem 11.23], see also Abramovich [15] and Schmidt [16]. $\rhd$

Definition 3.3. Fix a complete Boolean algebra $\mathbb{B}$ of band projection in X, i.e., $\mathbb{B}$ is a complete subalgebra of $\mathbb{P}(X)$. A bounded linear operator $T : X \to X$ is said to satisfiy the Daugavet equation $\mathbb{B}$-*uniformly* if $\|\pi + T\pi\| = 1 + \|T\pi\|$ for all nonzero $\pi \in \mathbb{B}$. Say that $\rho \in \mathbb{P}(X)$ is *nonzero over* $\mathbb{B}$, whenever $\pi\rho \neq 0$ for all nonzero $\pi \in \mathbb{B}$.

Lemma 3.4. *Let Λ be a normed lattice with the projection property, X be a decomposable lattice normed space over Λ and $\|x\| := \||x|\|_\infty$ $(x \in X)$. Then for $0 < p \in \mathbb{R}$ and $x, y \in X$ the inequality $|x| \geqslant (\mathbb{1} + |y|^p)^{\frac{1}{p}}$ holds if and only if $\|\pi x\| \geqslant (1 + \|\pi y\|^p)^{\frac{1}{p}}$ for all $0 \neq \pi \in \mathbb{P}(\Lambda)$.*

$\lhd$ Prove that $(\forall \pi \in \mathbb{P}(\Lambda))\ \|\pi x\| \geqslant (1 + \|\pi y\|^p)^{\frac{1}{p}}$ implies $|x| \geqslant (\mathbb{1} + |y|^p)^{\frac{1}{p}}$. If the inequality $|x| \geqslant (\mathbb{1} + |y|^p)^{\frac{1}{p}}$ is not true then there exist a nonzero $\pi_0 \in \mathbb{P}(\Lambda)$ and $0 < \varepsilon \in \mathbb{R}$ such that

$(1+\varepsilon)\pi_0|x| < \pi_0(\mathbb{1}+|y|^p)^{\frac{1}{p}}$. Notice that $|\cdot|$ is $\mathbb{P}(\Lambda)$-homogeneous, i.e., $\pi|x| = |\pi x|$ and hence $\pi(\mathbb{1} + |y|^p)^{\frac{1}{p}} = (\pi\mathbb{1} + |\pi y|^p)^{\frac{1}{p}}$ for all $x \in X$ and $\pi \in \mathbb{P}(\Lambda)$, see [8, 5.8.3]. It follows that

$$\|\pi_0 x\| < (1+\varepsilon)\|\pi_0 x\| = \|(1+\varepsilon)\pi_0|x|\,\|_\infty \leqslant \big\|\big(\pi_0\mathbb{1} + |\pi_0 y|^p\big)^{\frac{1}{p}}\big\|_\infty = \|(1+\|\pi_0 y\|^p)^{\frac{1}{p}},$$

a contradiction. The converse implication is immediate from the relations

$$|\pi x| = \pi|x| \geqslant \pi\big(\mathbb{1} + |y|^p\big)^{\frac{1}{p}} = \big(\pi\mathbb{1} + |\pi y|^p\big)^{\frac{1}{p}}, \big\|\big(\mathbb{1} + |y|_r^p\big)^{\frac{1}{p}}\big\|_\infty = \big(1 + \||y|_r\|_\infty^p\big)^{\frac{1}{p}}. \ \triangleright$$

Theorem 3.5. *Let X be an injective Banach lattice and an operator $T \in \mathscr{L}(X)$ commutes with all M-projections. Then there exist pair-wise disjoint M-projections π_0, π_1, and π_2 in X such that $\pi_0 + \pi_1 + \pi_2 = I_X$ and the operators $\pi_1 \circ T + \pi_0 \circ T - \pi_2 \circ T$ and $\pi_1 \circ T - \pi_0 \circ T - \pi_2 \circ T$ satisfy the Daugavet equation $\mathbb{M}(X)$-uniformly. Moreover, for any nonzero M-projections $\rho_k \leqslant \pi_k$ $(k = 1, 2)$ the operators $-\rho_2 \circ T$ and $\rho_1 \circ T$ fail to satisfy the Daugavet equation.*

$\triangleleft$ Let $\mathscr{X}, \mathscr{T} \in \mathbb{V}^{(\mathbb{B})}$ be the Boolean valued representations of X and T, respectively. By Theorem 3.3 $[\![\mathscr{X}$ is an AL-space and $\mathscr{T} \in \mathscr{L}(\mathscr{X})]\!] = \mathbb{1}$. Let the formula $\psi(T)$ formalize the sentence 'T satisfies the Daugavet equation' and put $\tilde{\pi}_1 = [\![\psi(\mathscr{T})]\!]$, $\tilde{\pi}_2 = [\![\psi(-\mathscr{T})]\!]$, $\pi_0 = \tilde{\pi}_1 \circ \tilde{\pi}_2$, and $\pi_i = \tilde{\pi}_i - \pi_0$. Clearly, π_0, π_1, and π_2 are pair-wise disjoint. Boolean valued transfer principle together with Theorem 3.1 imply that $[\![$either $\mathscr{T}$ or $-\mathscr{T}$ satisfies the Daugavet equation$]\!] = \mathbb{1}$. It follows from the Transfer Principle that $\tilde{\pi}_1 \vee \tilde{\pi}_2 = [\![\psi(\mathscr{T}) \vee \psi(-\mathscr{T})]\!] = \mathbb{1}$, whence $\pi_1 + \pi_0 + \pi_2 = \mathbb{1}$. Denote by $\mathscr{S}$ the mixing of $(\mathscr{T}, \mathscr{T}, -\mathscr{T})$ by (π_1, π_0, π_2), i. e. $\pi_0 + \pi_1 \leqslant [\![\mathscr{S} = \mathscr{T}]\!]$ and $\pi_2 \leqslant [\![\mathscr{S} = -\mathscr{T}]\!]$. If $S := \mathscr{S}{\Downarrow}$ then $S := \pi_1 \circ T + \pi_0 \circ T - \pi_2 \circ T$. By applying [7, A.5 (6)] we have $\pi_0 + \pi_1 \leqslant [\![\psi(\mathscr{T})]\!] \wedge [\![\mathscr{S} = \mathscr{T}]\!] \leqslant [\![\psi(\mathscr{S})]\!]$ and $\pi_2 \leqslant [\![\psi(-\mathscr{T})]\!] \wedge [\![\mathscr{S} = -\mathscr{T}]\!] \leqslant [\![\psi(S)]\!]$ which imply $[\![\psi(\mathscr{S})]\!] = \mathbb{1}$. Since $[\![\|\mathscr{S}\| = |S|]\!] = \mathbb{1}$, we have $|I+S| = \mathbb{1}+|S|$ and taking into account Lemma 3.4 and the easy relation $\|\mathbb{1}+\lambda\|_\infty = 1+\|\lambda\|_\infty$ with $\lambda \in \Lambda$ yields $\|\pi + S\pi\| \geqslant 1 + \|S\pi\|$ and hence the required equality $\|\pi + S\| = 1 + \|S\pi\|$ for all nonzero $\pi \in \mathbb{B}$. The operator $\pi_1 \circ T - \pi_0 \circ T - \pi_2 \circ T$ is handled similarly. $\triangleright$

We now consider Daugavet type inequalities for regular operators.

For $1 \leqslant p \in \mathbb{R}$ and arbitrary $s, t \in \mathbb{R}$ we denote $t^p := \operatorname{sgn}(t)|t|^p$ and $\sigma_p(s,t) := (s^{1/p}+t^{1/p})^p$, where $1/p := p^{-1}$. In a vector lattice X, we introduce new vector operations $\oplus$ and $*$, while the original ordering $\leqslant$ remain unchanged:

$$x \oplus y := \sigma_p(x,y) := (x^{1/p} + y^{1/p})^p, \quad t * x := t^p x \quad (x, y \in X;\ t \in \mathbb{R}).$$

Then $X^{(p)} := (X, \oplus, *, \leqslant)$ is again a vector lattice. Moreover, $(X^{(p)}, \|\cdot\|_p)$ with $\|x\|_p := \|x\|^{1/p}$ is a Banach lattice called the *p-convexification* of X, see Lindenstrauss and Tzafriri [17, pp. 53, 54]. Observe that $\mathbb{P}(X^{(p)}) = \mathbb{P}(X)$ and $\mathbb{M}(X^{(p)}) = \mathbb{M}(X)$. Given a Banach lattice $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ and $1 \leqslant p \in \mathbb{R}$, we denote by $\mathscr{X}^{(p)} := \mathscr{X}^{(p^\wedge)}$ the $p^\wedge$-convexification of $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$. Moreover, if $|\cdot|$ and $|\cdot|_p$ are the respective descents (see [8, 1.5.6]) of $\|\cdot\|$ and $\|\cdot\|_p$ then $|x|_p = |x|^{1/p}$ for all $x \in X$.

Theorem 3.6. *Let $\mathscr{X}$ be an AL-space with a weak order unit and $\mathscr{T}$ be a regular linear operator on $\mathscr{X}^{(p)}$, $1 \leqslant p \in \mathbb{R}$. Then $\mathscr{T} \perp I_{\mathscr{X}^{(p)}}$ if and only if $\|\rho \pm \mathscr{T}\rho\|_r \geqslant (1 + \|\mathscr{T}\rho\|_r^p)^{\frac{1}{p}}$ for all nonzero band projections ρ in $\mathscr{X}^{(p)}$.*

$\triangleleft$ This is a reformulation of the main result (Theorem 9) in Schep [18], since in the case of a function space $\mathscr{X}$ we have $\mathscr{X}^{(p)} = \{f :\ |f|^p \in \mathscr{X}\}$. $\triangleright$

To perform the Boolean valued interpretation of Theorem 3.6 we need an auxiliary fact.

Lemma 3.7. *Let $\mathscr{X}$ be a Banach lattice within $\mathbb{V}^{(\mathbb{B})}$. Then for each $1 \leqslant p \in \mathbb{R}$ we have*

$$(\mathscr{X}^{(p)})\Downarrow = (\mathscr{X}\Downarrow)^{(p)}.$$

$\lhd$ See Kusraev [19, Lemma 4]. $\rhd$

Theorem 3.8. *Let X be an injective Banach lattice with a weak order unit and an operator $T \in \mathscr{L}^r(X^{(p)})$ commutes with M-projections on $X^{(p)}$. The following are equivalent:*

(1) $T \perp I_{X^{(p)}}$.

(2) $\|\pi\rho \pm \pi T\rho\|_r \geqslant (1 + \|\pi T\rho\|_r^p)^{\frac{1}{p}}$ *with $0 \neq \pi \in \mathbb{M}(X)$ and $\rho \in \mathbb{P}(X)$ nonzero over $\mathbb{M}(X)$.*

(3) $\|\rho \pm T\rho\|_r \geqslant (1 + \|T\rho\|_r^p)^{\frac{1}{p}}$ *for all nonzero $\rho \in \mathbb{P}(X)$.*

$\lhd$ Let again $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ stand for the Boolean valued representation of X. Then $\mathscr{X}$ is an AL-space with a weak order unit within $\mathbb{V}^{(\mathbb{B})}$ by Theorem 2.5, while $X^{(p)} = (\mathscr{X}^{(p)})\Downarrow$ by Lemma 3.7. According to Theorem 2.4 there exists a regular operator $\mathscr{T} \in \mathscr{L}^r(\mathscr{X}^{(p)})$ such that $T = \mathscr{T}\Downarrow$. By Boolean valued transfer principle, Schep's result (Theorem 3.6) is valid within $\mathbb{V}^{(\mathbb{B})}$, i.e., $[\![\mathscr{T} \perp I_{\mathscr{X}^{(p)}}]\!] = \mathbb{1}$ if and only if $[\![\|\rho \pm \mathscr{T}\rho\|_r \geqslant (1 + \|\mathscr{T}\rho\|_r^{p^\wedge})^{1/p^\wedge}$ for all nonzero band projections ρ on $\mathscr{X}]\!] = \mathbb{1}$. Clearly, $[\![\rho \neq 0]\!] = \mathbb{1}$ if and only if $\pi\rho \neq 0$ for all nonzero $\pi \in \mathbb{M}(\mathbb{X})$. Observe also that $|T|_r = |\|T\||$ and thus $\|T\|_r = \||T|_r\|_\infty$, since an injective Banach lattice is order complete, see [8, Corollary 5.10.7]. It follows that $T \perp I_{X^{(p)}}$ if and only if $|\rho \pm T\rho|_r \geqslant \big(\mathbb{1} + |T\rho|_r^p\big)^{\frac{1}{p}}$ for each $\rho \in \mathbb{P}(X)$ nonzero over $\mathbb{M}(X)$. By Lemma 3.4 the last inequality is equivalent to $\|\pi\rho \pm \pi T\rho\|_r \geqslant (1 + \|\pi T\rho\|_r^p)^{\frac{1}{p}}$ for all nonzero $\pi \in \mathbb{M}(X)$ and $\rho \in \mathbb{P}(X)$ nonzero over $\mathbb{M}(X)$. Thus, (1) $\Longleftrightarrow$ (2), while (3) $\Longrightarrow$ (2) is trivial. To prove (2) $\Longrightarrow$ (3), take arbitrary nonzero $\rho \in \mathbb{P}(X)$ and put $\pi_0 := \sup\{\pi \in \mathbb{M}(X) : \pi\rho = 0\}$, $\bar{\rho} := \rho + \pi_0$, and $\bar{\pi} := \pi_0^\perp$. Then $\bar{\rho}$ is nonzero over $\mathbb{M}(X)$ and $\bar{\pi}\bar{\rho} = \rho$. Now, making use of (2), we deduce $\|\rho \pm T\rho\|_r = \|\bar{\pi}\bar{\rho} \pm \bar{\pi}T\bar{\rho}\|_r \geqslant (1 + \|\bar{\pi}T\bar{\rho}\|_r^p)^{\frac{1}{p}} = (1 + \|T\rho\|_r^p)^{\frac{1}{p}}$. $\rhd$

The following corollary generalizes Theorem 1 from Shvidkoy [21].

Corollary 3.9. *Assume that X is an injective Banach lattice with a weak order unit and an operator $T \in \mathscr{L}^r(X)$ commutes with all M-projections on X. Then $T \perp I_X$ if and only if T satisfy the Daugavet equation $\mathbb{P}(X)$-uniformly.*

$\lhd$ This is immediate from Theorem 3.8 and Proposition 2 in Shvidkoy [21]. $\rhd$

4. Injective Banach Latices of Operators

We consider now under which conditions the space of regular operators between Banach lattices is an injective Banach lattice. First, we state results obtained by Wickstead in [22].

Theorem 4.1. *If $\mathscr{X}$ and $\mathscr{Y}$ are Banach lattices, neither of which is the zero space, with $\mathscr{Y}$ Dedekind complete then $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AL-space under the regular norm if and only if $\mathscr{X}$ is an AM-space and $\mathscr{Y}$ is an AL-space.*

$\lhd$ See Wickstead [22, Theorem 2.1]. $\rhd$

Theorem 4.2. *If $\mathscr{Y}$ is a nonzero Dedekind complete Banach lattices then $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AM-space under the regular norm for every AL-space $\mathscr{X}$ if and only if $\mathscr{Y}$ is an AM-space with a Fatou norm.*

$\lhd$ See Wickstead [22, Theorem 2.3]. $\rhd$

Denote by $\mathscr{K}^r(\mathscr{X},\mathscr{Y})$ the linear span of positive compact operators from $\mathscr{X}$ to $\mathscr{Y}$ endowed with the k-norm defined as $\|T\|_k := \inf\{\|S\| : \pm T \leqslant S \in \mathscr{K}(\mathscr{X},\mathscr{Y})\}$, see [22].

Theorem 4.3. *If $\mathscr{X}$ and $\mathscr{Y}$ are nonzero Banach lattices, then $\mathscr{K}^r(\mathscr{X},\mathscr{Y})$ is an AL-space under the k-norm if and only if $\mathscr{X}$ is an AM-space and $\mathscr{Y}$ is an AL-space.*

$\lhd$ See Wickstead [22, Theorem 2.5 (i)]. $\rhd$

By Boolean valued transfer principle the above three theorems are true within each Boolean valued model. The proofs below are carried out by externalization of these internal facts with $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{T}$ standing for Boolean valued representations of X, Y and T, respectively.

Theorem 4.4. *Let X and Y be $\mathbb{B}$-cyclic Banach lattices with Y Dedekind complete. Then $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an injective Banach lattice under the regular norm with $\mathbb{B}\simeq\mathbb{M}(\mathscr{L}^r_{\mathbb{B}}(X,Y))$ if and only if X is an AM-space and Y is an injective Banach lattice with $\mathbb{B}\simeq\mathbb{M}(Y)$.*

$\lhd$ This a Boolean valued interpretation of Theorem 4.1. According to Theorems 2.4 and 2.5 (4) $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an injective Banach lattice under the regular norm with $\mathbb{B}(\mathscr{L}^r_{\mathbb{B}}(X,Y))$ isomorphic to $\mathbb{B}$ if and only if $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AL-space under the regular norm within $\mathbb{V}^{(\mathbb{B})}$. Theorem 4.1 (applicable by Theorem 2.5 (1)) tells us that the latter is equivalent to saying that $\mathscr{X}$ is an AM-space and $\mathscr{Y}$ is an AL-space. It remains to refer again to Theorem 2.5 (3, 4). $\rhd$

Theorem 4.5. *Let Y be a nonzero $\mathbb{B}$-cyclic Dedekind complete Banach lattices. Then $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an AM-space under the regular norm with $\mathbb{M}(\mathscr{L}^r_{\mathbb{B}}(X,Y))\simeq\mathbb{B}$ for every injective Banach lattice X with $\mathbb{B}=\mathbb{M}(X)$ if and only if Y is an AM-space with a Fatou norm.*

$\lhd$ The proof is similar to that of Theorem 4.4: Theorem 4.2 is true within $\mathbb{V}^{(\mathbb{B})}$ and hence $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AM-space under the regular norm for every AL-space $\mathscr{X}$ if and only if $\mathscr{Y}$ is an AM-space with a Fatou norm. Moreover, Y has the Fatou norm if and only if $[\![\mathscr{Y}$ has the Fatou norm$]\!] = \mathbb{1}$, see [8, Theorem 5.9.6 (2)]. Now, combining Theorems 2.4 and 2.5 completes the proof. $\rhd$

Definition 4.6. Denote by $\mathrm{Prt}(\mathbb{B})$ (respectively, $\mathrm{Prt}_\sigma(\mathbb{B})$) the set of all partitions (respectively, countable partitions) of unity in $\mathbb{B}$. A set U in X is said to be mix-*complete* if, for all $(\pi_\xi)_{\xi\in\Xi}\in\mathrm{Prt}(\mathbb{B})$ and $(u_\xi)_{\xi\in\Xi}\subset U$, there is $u\in U$ such that $u=\mathrm{mix}_{\xi\in\Xi}\,\pi_\xi u_\xi$. Suppose that X is a $\mathbb{B}$-cyclic Banach lattice, $(x_n)_{n\in\mathbb{N}}\subset X$, and $x\in X$. Say that a *sequence* $(x_n)_{n\in\mathbb{N}}$ $\mathbb{B}$-*approximates* x if, for each $k\in\mathbb{N}$, we have $\inf\{\sup_{n\geqslant k}\|\pi_n(x_n-x)\| :\ (\pi_n)_{n\geqslant k}\in\mathrm{Prt}_\sigma(\mathbb{B})\}=0$. Call a set $K\subset X$ mix-*compact* if K is mix-complete and for every sequence $(x_n)_{n\in\mathbb{N}}\subset K$ there is $x\in K$ such that $(x_n)_{n\in\mathbb{N}}$ $\mathbb{B}$-approximates x. Observe that if $\|x\|=\|\,|x|\,\|_\infty$ $(x\in X)$ with a $\Lambda(\mathbb{B})$-valued norm $|\cdot|$, then a sequence $(x_n)_{n\in\mathbb{N}}$ in X $\mathbb{B}$-approximates x if and only if $\inf_{n\geqslant k}|x_n-x|$ for all $k\in\mathbb{N}$. An operator with values in a $\mathbb{B}$-cyclic Banach lattice is called *cyclically compact* if (or mix-*compact*) the image of any bounded subset is contained in a cyclically compact set.

It is clear that in case $E=\mathbb{R}$ mix-compactness is equivalent to compactness in the norm topology. Note also that the concept of mix-compactness in Gutman and Lisovskaya [20] coincides with that of cyclically compactness introduced by Kusraev [7], see [20, Theorem 3.4] and [8, Proposition 2.12.C.5].

Given $\mathbb{B}$-cyclic Banach lattices X and Y, denote by $\mathscr{K}^r_{\mathbb{B}}(X,Y)$ the linear span of positive $\mathbb{B}$-linear cyclically compact operators from X to Y, see [7, 8.5.5]. This is a Banach lattice under the k-norm defined as

$$\|T\|_k := \inf\{\|S\| :\ \pm T\leqslant S\in\mathscr{K}^r_{\mathbb{B}}(X,Y)\}.$$

Note that $\mathscr{K}^r(X,Y) := \mathscr{K}^r_{\mathbb{B}}(X,Y)$, whenever $\mathbb{B}=\{0,\mathbb{1}\}$, cp. [22].

Theorem 4.7. *Let X and Y be $\mathbb{B}$-cyclic Banach lattices. Then $\mathscr{K}^r_{\mathbb{B}}(X,Y)$ is an injective Banach lattice under the k-norm with $\mathbb{M}(\mathscr{L}^r_{\mathbb{B}}(X,Y)) \simeq \mathbb{B}$ if and only if X is an AM-space and Y is an injective Banach lattice with $\mathbb{M}(Y) \simeq \mathbb{B}$.*

$\lhd$ The proof runs along the lines of the proof of Theorem 4.4. We have only to observe that an operator $T \in \mathscr{L}^r_{\mathbb{B}}(X,Y)$ is mix-compact if and only if $[\![\, \mathscr{T} = T\!\uparrow$ is a compact linear operator from $\mathscr{X}$ into $\mathscr{Y}\,]\!] = \mathbb{1}$, see [7, Proposition 8.5.5 (1)]. Thus the $\mathbb{B}$-isometry between $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ and $\mathscr{L}^r(\mathscr{X},\mathscr{Y})\!\Downarrow$ induces a $\mathbb{B}$-isometry between $\mathscr{K}^r_{\mathbb{B}}(X,Y)$ and $\mathscr{K}^r(\mathscr{X},\mathscr{Y})\!\Downarrow$. $\rhd$

Definition 4.8. Let X be a Banach lattice and Y be a $\mathbb{B}$-cyclic Banach space. Denote by $\mathscr{P}_{\text{fin}}(X)$ the collection of all finite subsets of X. For every $T \in \mathscr{L}(X,Y)$ define

$$\sigma(T) := \sup\left\{ \inf_{(\pi_k)\in \mathrm{Prt}_\sigma(\mathbb{B})} \sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k T x_i\| : \ \{x_1,\dots,x_n\} \in \mathscr{P}_{\text{fin}}(X),\ \left\| \sum_{i=1}^{n} |x_i| \right\| \leqslant 1 \right\}.$$

An operator $T \in \mathscr{L}(X,Y)$ is said to be cone $\mathbb{B}$-*summing* if $\sigma(T) < \infty$. Thus, T is cone $\mathbb{B}$-*summing* if and only if there exists a positive constant C such that for any finite collection $x_1,\dots,x_n \in X$ there is a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{B}$ with

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k T x_i\| \leqslant C \left\| \sum_{i=1}^{n} |x_i| \right\|;$$

moreover, in this event $\sigma(T) = \inf\{C\}$. Denote by $\mathscr{S}_{\mathbb{B}}(X,Y)$ the set of all cone $\mathbb{B}$-summing operators. The class $\mathscr{S}_{\mathbb{B}}(X,Y)$ was introduced in Kusraev [23, Definition 7.1], see also Kusraev and Kutateladze [8, 5.13.1]. Observe that if $\mathbb{B} = \{0, I_Y\}$ then $\mathscr{S}(X,Y) := \mathscr{S}_{\mathbb{B}}(X,Y)$ is the space of cone absolutely summing operators, see Schaefer [24, Ch. 4, § 3, Proposition 3.3 (d)] or (which is the same) 1-concave operators, see Diestel, Jarchow, and Tonge [25, p. 330]. Cone absolutely summing operators were introduced by Levin [26] and later independently by Schlotterbeck, see [24, Ch. 4].

Theorem 4.9. *Let $\mathscr{X}$ and $\mathscr{Y}$ be nonzero Banach lattices. The following are equivalent:*

(1) *$\mathscr{S}(\mathscr{X},\mathscr{Y})$ is an AL-space.*

(2) *$\mathscr{X}$ is an AM-space and $\mathscr{Y}$ is an AL-space.*

$\lhd$ This result was obtained by Schlotterbeck, see Schaefer [24, Ch. 4, Proposition 4.5]. $\rhd$

Theorem 4.10. *Let X be a nonzero Banach lattice and Y be a $\mathbb{B}$-cyclic Banach lattice. The following are equivalent:*

(1) *$\mathscr{S}_{\mathbb{B}}(X,Y)$ is an injective Banach lattice with $\mathbb{M}(\mathscr{S}_{\mathbb{B}}(X,Y))$ isomorphic to $\mathbb{B}$.*

(2) *X is an AM-space and Y is an injective Banach lattice with $\mathbb{M}(Y)$ isomorphic to $\mathbb{B}$.*

$\lhd$ Suppose that X is a Banach lattice, $\mathscr{X}$ is the completion of the metric space $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$, and $\mathscr{Y}$ is the Boolean valued representation of a $\mathbb{B}$-cyclic Banach space Y. Then $[\![\, \mathscr{X}$ is a Banach lattice$\,]\!] = \mathbb{1}$ and the map $x \mapsto x^\wedge$ is a lattice isometry from X to $\mathscr{X}\!\Downarrow$. Moreover, for every $T \in \mathscr{S}_{\mathbb{B}}(X,Y)$ there exists a unique $\mathscr{T} := T\!\uparrow \in \mathbb{V}^{(\mathbb{B})}$ determined from the formulas

$$[\![\, \mathscr{T} \in \mathscr{S}(\mathscr{X},\mathscr{Y}) \,]\!] = \mathbb{1}, \quad [\![\, \mathscr{T} x^\wedge = Tx \,]\!] = \mathbb{1} \quad (x \in X).$$

The map $T \mapsto \mathscr{T}$ is an order preserving $\mathbb{B}$-isometry from $\mathscr{S}_{\mathbb{B}}(X,Y)$ onto the restricted descent $\mathscr{S}(\mathscr{X},\mathscr{Y})\!\Downarrow$, see Kusraev and Kutateladze [7, 8.3.4] and [8, Theorem 5.13.6]. Note also that X is an AM-space if and only if $[\![\, \mathscr{X}$ is an AM-space$\,]\!] = \mathbb{1}$ by Theorem 2.5 (3). Now, the proof can be carried out in similar lines by Boolean valued interpretation of Theorem 4.9. $\rhd$

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Kusraev Anatoly Georgievich
Vladikavkaz Science Center of the RAS, *Chaiman*
22 Markus Street, Vladikavkaz, 362027, Russia;
North Ossetian State University
44–46 Vatutin Street, Vladikavkaz, 362025, Russia
E-mail: `kusraev@smath.ru`

ОПЕРАТОРЫ В ИНЪЕКТИВНЫХ БАНАХОВЫХ РЕШЕТКАХ

Кусраев А. Г.

Изучаются некоторые свойства ограниченных линейных операторов в инъективных банаховых решетках, используя булевозначный принцип переноса с AL-пространств на инъективные банаховы решетки, полученный в работе автора [1].

Ключевые слова: AM-пространство, AL-пространство, инъективная банахова решетка, булевозначная модель, булевозначный принцип переноса, уравнение Даугавета, циклически компактный оператор, $\mathbb{B}$-суммирующий оператор.