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ON THE ABSENCE OF SOLUTIONS TO DAMPED SYSTEM OF NONLINEAR WAVE EQUATIONS OF KIRCHHOFF-TYPE

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In higher-order function spaces, some techniques are used to give the nonexistence result to system of wave equations in the Kirchhoff type, to generalize earlier results in the literature.

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1. Introduction and Previous Work

Let us consider the problem

$$\begin{cases} (|u_1'|^{m-2}u_1')' + \left(\int\limits_\Omega |D^\kappa u_1|^2 dx\right)^\gamma (-\Delta)^\kappa u_1 + \left(a|u_1|^k + b|u_2|^l\right) u_1' = f_1(u_1,u_2); \\ (|u_2'|^{m-2}u_2')' + \left(\int\limits_\Omega |D^\kappa u_2|^2 dx\right)^\gamma (-\Delta)^\kappa u_2 + \left(c|u_2|^\theta + d|u_1|^\varrho\right) u_2' = f_2(u_1,u_2), \end{cases} \tag{1.1}$$

where all terms must be alive, we will prove that the solutions of (1.1) cannot exist for $t > 0$ with positive initial energy, where

$$u_i(x,0) = u_{i0}(x) \in H_0^\kappa(\Omega), \quad i = 1,2, \tag{1.2}$$

$$u_i'(x,0)) = u_{i1}(x) \in L^m(\Omega), \quad i = 1,2, \tag{1.3}$$

and boundary conditions

$$\frac{\partial^j u_i}{\partial \nu^j} = 0, \quad x \in \partial\Omega, \ \ i = 1,2, \ \ j = 0,1,2,\ldots,\kappa-1, \tag{1.4}$$

where ν is the outward normal to the boundary.

In the present paper, we study the system (1.1), with

$$\begin{aligned} f_1(u_1,u_2) &= (p+1)\left[a_1|u_1+u_2|^{(p-1)}(u_1+u_2) + b_1|u_1|^{\frac{(p-3)}{2}}u_1|u_2|^{\frac{(p+1)}{2}}\right], \\ f_2(u_1,u_2) &= (p+1)\left[a_1|u_1+u_2|^{(p-1)}(u_1+u_2) + b_1|u_2|^{\frac{(p-3)}{2}}u_2|u_1|^{\frac{(p+1)}{2}}\right], \end{aligned} \tag{1.5}$$

and the parametres $a_1 > 0$, $b_1 > 0$, $p > 3$, $\gamma \geqslant 0$, $m \geqslant 2$, $k, l, \theta, \varrho, \kappa \geqslant 1$ satisfying

$$p > \max\left(m-1, k+1, l+1, \theta+1, \varrho+1, 2\gamma+1\right). \tag{1.6}$$

In (1.1), $u_i = u_i(t,x)$, $i = 1,2$, where $x \in \Omega$ is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$ with a smooth boundary $\partial\Omega$, $t > 0$ and a, b, c, d are nonnegative constants.

We mention here that

$$|D^\kappa u|^2 = (\Delta^{\kappa/2} u)^2 \text{ for par value of } \kappa$$

and

$$|D^\kappa u|^2 = |\nabla(\Delta^{(\kappa-1)/2} u)|^2 \text{ for odd } \kappa,$$

where

$$|\nabla u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2, \quad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

This kind of systems appears in the models of nonlinear Kirchhoff-type. It is a generalization of a model introduced by Kirchhoff [11] in the case $n = 1$; this type of problem describes a small amplitude vibration of an elastic string. The original equation is:

$$\rho h u_{tt} + \tau u_t = \left(P_0 + \frac{Eh}{2L} \int_0^L |u_x(x,t)|^2 ds\right) u_{xx} + f, \tag{1.7}$$

where $0 \leqslant x \leqslant L$ and $t > 0$, $u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, P_0 the initial axial tension, τ the resistance modulus, E the Young modulus and f the external force (for example the action of gravity).

The blow up of the gender of our problems in the single equation has been considered in [18]; it was established a blow-up result for certain solutions with positive initial energy. In [14] local existence and blow up of the solutions, of the same equation have been studied.

A related problems with $\kappa = 1$ have attracted a great deal of attention in the last decades, and many results have been appeared on the existence and long time behavior of solutions. For the literature we quote essentially the results of [2–5], [7], [10–12], [15, 17, 19, 20, 22, 23, 30] and references therein.

The systems of nonlinear wave equations (1.1) go back to Reed [24] who proposed a similar system in three space dimensions but in the absence of the viscoelastic and damping terms. This type of system was completely analysed; for example, in [2], the authors studied the following system:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{1.8}$$

in $\Omega \times (0,T)$ with initial and boundary conditions and the nonlinear functions f_1 and f_2 satisfying appropriate conditions and in the case where $a = b = c = d = \gamma = 0$, $m = 2$, $\kappa = 1$. They proved under some restrictions on the parameters and the initial data many results on the existence of a weak solution. They also showed that any weak solution with negative initial energy blows up in finite time using the same techniques as in [8].

In the work [19], the authors considered the nonlinear viscoelastic system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\, ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)\, ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \quad x \in \Omega,\ t > 0, \tag{1.9}$$

where

$$\begin{cases} f_1(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^\rho u|v|^{(\rho+2)}, \\ f_2(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^\rho v, \end{cases} \tag{1.10}$$

and they prove a global nonexistence theorem for certain solutions with positive initial energy, the main tool of the proof is a method used in [25].

In the case of $\gamma = 0$, $\kappa = 1$, $m = 2$, problem (1.1) has been studied recently in [22] focusing on the global well-posedness of the system of nonlinear wave equations

$$\begin{cases} u_{tt} - \Delta u + \left(d|u|^k + e|v|^l\right)|u_t|^{m-1}u_t &= f_1(u,v), \\ v_{tt} - \Delta v + \left(d'|v|^\theta + e'|u|^\rho\right)|v_t|^{r-1}v_t &= f_2(u,v), \end{cases} \tag{1.11}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$, $0 < r, m < 1$, with Dirichlet boundary conditions. The nonlinearities $f_1(u,v)$ and $f_2(u,v)$ act as a strong source in the system. Under some restriction on the parameters in the system, they obtain several results on the existence and uniqueness of solutions. In addition, they prove that weak solutions blow up in finite time whenever the initial energy is negative and the exponent of the source term is more dominant than the exponents of both damping terms. This last result was extended by A. Benaissa, Ouchenane, and Zennir in [3] with positive initial energy, $r, m > 0$ and for $n > 0$.

Our main theorem addresses to generalize earlier results in the literature. We will improve the influence of a strong sources with positive initial energy, which lead to blow up of solutions for all $t > 0$ in Theorem 3.1.

2. Notations and Preliminaries

The constants c_i, $i = 0,1,2,\dots$, used throughout this paper are positive generic constants, which may be different in various occurrences. We take $a = b = c = d = a_1 = b_1 = 1$ for convenience.

(**A1**) There exists a C^1-function $F : \mathbb{R}^2 \to \mathbb{R}$ such that

$$(p+1)F(u_1,u_2) = [u_1 f_1(u_1,u_2) + u_2 f_2(u_1,u_2)] = \left[a_1|u_1+u_2|^{p+1} + 2b_1|u_1u_2|^{\frac{(p+1)}{2}}\right], \tag{1.12}$$

where

$$\frac{\partial F}{\partial u_1} = f_1(u_1,u_2), \quad \frac{\partial F}{\partial u_2} = f_2(u_1,u_2). \tag{1.13}$$

(**A2**) There exist a positive constant $c_1 = 2^p a + b$ such that

$$F(u_1,u_2) \leqslant c_1 \sum_{i=1}^{2} |u_i|^{p+1}. \tag{1.14}$$

We introduce the following definition of weak solution to (1.1)–(1.4).

Definition 2.1. A pair of functions (u_1,u_2) is said to be a weak solution of (1.1)–(1.4) on $[0,T]$ if $u_1,u_2 \in C_w([0,T], H_0^\kappa(\Omega))$, $u_1', u_2' \in C_w([0,T], L^m(\Omega))$, $(u_{10},u_{20}) \in H_0^\kappa(\Omega) \times H_0^\kappa(\Omega)$,

$(u_{11}, u_{21}) \in L^m(\Omega) \times L^m(\Omega)$ and (u_1, u_2) satisfies,

$$\begin{aligned}
&\int_0^t \int_\Omega (|u_1'|^{m-2} u_1')' \phi \, dx ds + \int_0^t \|D^\kappa u_1\|_2^{2\gamma} \int_\Omega D^\kappa u_1 D^\kappa \phi \, dx ds \\
&+ \int_0^t \int_\Omega \left((a\,|u_1|^k + b\,|u_2|^l \right) u_1' \phi \, dx ds = \int_0^t \int_\Omega f_1(u_1, u_2) \phi \, dx ds; \\
&\int_0^t \int_\Omega (|u_2'|^{m-2} u_2')' \psi \, dx ds + \int_0^t \|D^\kappa u_2\|^{2\gamma} \int_\Omega D^\kappa u_2 D^\kappa \psi \, dx ds \\
&+ \int_0^t \int_\Omega \left(c\,|u_2|^\theta + d\,|u_1|^\varrho \right) u_2' \psi \, dx ds = \int_0^t \int_\Omega f_2(u_1, u_2) \psi \, dx ds
\end{aligned} \tag{1.15}$$

for all test functions $\phi, \psi \in H_0^\kappa(\Omega) \cap L^m(\Omega)$, for almost all $t \in [0, T]$. Where $C_w([0, T], X)$ denotes the space of weakly continuous functions from $[0, T]$ into Banach space X.

The energy functional $E(t)$ associated to our system is given by:

$$E(t) = \frac{m-1}{m} \sum_{i=1}^{2} \|u_i'\|_m^m + \frac{1}{2(\gamma+1)} \sum_{i=1}^{2} \|D^\kappa u_i\|_2^{2(\gamma+1)} - \int_\Omega F(u_1, u_2)\, dx. \tag{1.16}$$

The following Sobolev–Poincare inequality will be used frequently without mention $H_0^\kappa(\Omega) \subset L^p(\Omega)$, for

$$\begin{cases} 1 < p, & \text{if} \quad n = \kappa, 2\kappa, \\ 1 < p \leqslant \frac{4\kappa - n}{n - 2\kappa}, & \text{if} \quad n \geqslant 3\kappa. \end{cases} \tag{1.17}$$

We first state (without proof, it is similar to that in [23]) a local existence theorem for $n = 1, 2, 3$. Unfortunately, due to the strong nonlinearities on f_1, f_2 the well known techniques of constructing approximations by the Faedo–Galerkin allowed us to prove the local existence result only for $n \leqslant 3$.

Theorem 2.2. *Let $n = 1, 2, 3$. Suppose that (1.17) holds. Then, there exists a local weak solution in the sense of Definition 2.1 of problem (1.1)–(1.4) defined on $[0, T]$ for some $T > 0$, and (u_1, u_2) satisfies the energy inequality*

$$\begin{aligned}
E(t) + \int_s^t \Bigg(&\int_\Omega \left(|u_1(\tau)|^k + |u_2(\tau)|^l \right) (u_1')^2 dx \\
+ &\int_\Omega \left(|u_2(\tau)|^\theta + |u_1(\tau)|^\varrho \right) (u_2')^2 \, dx \Bigg) d\tau \leqslant E(s)
\end{aligned} \tag{1.18}$$

for all $T \geqslant t \geqslant s \geqslant 0$, where $E(t)$ is given in (1.16).

3. Results

Our main results read as follows

Theorem 3.1. *Suppose that* (1.6), (1.17) *hold. Then any solution of the problem* (1.1)–(1.4), *with initial data satisfying*

$$\sum_{i=1}^{2}\|D^{\kappa}u_{i0}\|_2^2>\alpha_1^2, \tag{1.19}$$

and

$$\sum_{i=1}^{2}\left(\|u_{i1}\|_m^m+\frac{1}{2(\gamma+1)}\|D^{\kappa}u_{i0}\|_2^{2(\gamma+1)}\right)-\int_{\Omega}F(u_{10},v_{20})\,dx<d \tag{1.20}$$

blows up for all time, where the constants α_1 *and* d *are defined in* (1.21).

We introduce the following:

$$B=\eta^{\frac{1}{p+1}},\quad \alpha_1=B^{\frac{p+1}{1-p}},\quad d=\left(\frac{1}{2(\gamma+1)}-\frac{1}{p+1}\right)\alpha_1^2, \tag{1.21}$$

where η is the constant in (1.28).

Lemma 3.2. *Suppose that* (1.17) *holds. Let* (u_1,u_2) *be a solution of* (1.1)–(1.4). *Assume further that*

$$\sum_{i=1}^{2}\|D^{\kappa}u_{i0}\|_2^2>\alpha_1^2, \tag{1.22}$$

and

$$\sum_{i=1}^{2}\left(\frac{m-1}{m}\|u_{i1}\|_m^m+\frac{1}{2(\gamma+1)}\|D^{\kappa}u_{i0}\|_2^{2(\gamma+1)}\right)-\int_{\Omega}F(u_{10},u_{20})\,dx<d. \tag{1.23}$$

Then there exists a constant $\alpha_2>\alpha_1$ *such that*

$$\sum_{i=1}^{2}\|D^{\kappa}u_i\|_2^2>\alpha_2^2, \tag{1.24}$$

and

$$\left((p+1)\int_{\Omega}F(u_1,u_2)dx\right)^{1/(p+1)}\geqslant B\alpha_2\quad(\forall\, t\geqslant 0). \tag{1.25}$$

$\lhd$ By the definition of energy functional, we have

$$E(t)=\frac{m-1}{m}\sum_{i=1}^{2}\|u_i'\|_m^m+\frac{1}{2(\gamma+1)}\sum_{i=1}^{2}\|D^{\kappa}u_i\|_2^{2(\gamma+1)}-\int_{\Omega}F(u_1,u_2)\,dx$$

$$=\frac{m-1}{m}\sum_{i=1}^{2}\|u_i'\|_m^m+\frac{1}{2(\gamma+1)}\sum_{i=1}^{2}\|D^{\kappa}u_i\|_2^{2(\gamma+1)}-\frac{1}{p+1}\left[\|u_1+u_2\|_{p+1}^{p+1}+2\|u_1u_2\|_{\frac{p+1}{2}}^{\frac{p+1}{2}}\right].$$

By using Minkowski's inequality and embedding $H_0^{\kappa}\Omega\hookrightarrow L^{(p+1)}(\Omega)$, we get

$$\|u_1+u_2\|_{p+1}^{p+1}\leqslant 2^{\frac{p+1}{2}}\left(\sum_{i=1}^{2}\|u_i\|_{p+1}^2\right)^{\frac{p+1}{2}}\leqslant c\left(\sum_{i=1}^{2}\|D^{\kappa}u_i\|_2^{2(\gamma+1)}\right)^{\frac{p+1}{2}}. \tag{1.26}$$

Hölder's and Young's inequalities give us

$$\left\|u_1u_2\right\|_{\frac{p+1}{2}}^{\frac{p+1}{2}} \leqslant \left(\left\|u_1\right\|_{p+1}\left\|u_2\right\|_{p+1}\right)^{\frac{p+1}{2}} \leqslant c\left(\sum_{i=1}^{2}\|D^\kappa u_i\|_2^{2(\gamma+1)}\right)^{\frac{p+1}{2}}. \tag{1.27}$$

Then there exist $\eta>0$ such that

$$\|u_1+u_2\|_{p+1}^{p+1}+2\|u_1u_2\|_{\frac{p+1}{2}}^{\frac{p+1}{2}} \leqslant \eta\left(\sum_{i=1}^{2}\|D^\kappa u_i\|_2^{2(\gamma+1)}\right)^{\frac{p+1}{2}}. \tag{1.28}$$

By definition of B wo get

$$\begin{aligned} E(t) &\geqslant \frac{1}{2(\gamma+1)}\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)}-\frac{1}{p+1}\left[\left\|u_1+u_2\right\|_{p+1}^{p+1}+2\left\|u_1u_2\right\|_{\frac{p+1}{2}}^{\frac{p+1}{2}}\right] \\ &\geqslant \frac{1}{2(\gamma+1)}\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)}-\frac{\eta}{p+1}\left(\sum_{i=1}^{2}\left\|D^{2(\gamma+1)}u_i\right\|_2^{2}\right)^{\frac{p+1}{2}} \\ &\geqslant \frac{1}{2(\gamma+1)}\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)}-\frac{B^{(p+1)}}{p+1}\left(\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)}\right)^{\frac{p+1}{2}}=f(\alpha), \end{aligned} \tag{1.29}$$

where $\alpha^2=\sum_{i=1}^{2}\|D^\kappa u_i\|_2^{2(\gamma+1)}$. We can verify that the function f is increasing for $0<\alpha<\alpha_1$, decreasing for $\alpha>\alpha_1$, $f(\alpha)\to-\infty$ as $\alpha\to+\infty$, and

$$f(\alpha_1)=\frac{1}{2(\gamma+1)}\,\alpha_1^2-\frac{B^{(p+1)}}{p+1}\,\alpha_1^{p+1}=d, \tag{1.30}$$

where α_1 given in (1.21). Therefore, since $E(0)<d'$, there exists $\alpha_2>\alpha_1$ such that $f(\alpha_2)=E(0)$.

Now we set $\alpha_0^2=\sum_{i=1}^{2}\|D^\kappa u_{i0}\|_2^{2(\gamma+1)}$, then by (1.29), we have $f(\alpha_0)\leqslant E(0)$, which implies that $\alpha_0\geqslant\alpha_2$. To establish (1.24), we suppose by contradiction that $\sum_{i=1}^{2}\|D^\kappa u_i(t_0)\|_2^{2(\gamma+1)}<\alpha_2^2$ for some $t_0>0$ to choose t_0 such that $\sum_{i=1}^{2}\|D^\kappa u_i(t_0)\|_2^{2(\gamma+1)}>\alpha_1^2$.

Again using of (1.29) leads to

$$E(t_0)\geqslant f\left(\sum_{i=1}^{2}\|D^\kappa u_i(t_0)\|_2^{2(\gamma+1)}\right)>f(\alpha_2)=E(0).$$

This is impossible since $E(t)\leqslant E(0)$ $(\forall\, t\in[0,T))$.

To prove (1.25), we exploit the definition of E, to get

$$\frac{1}{2(\gamma+1)}\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)}\leqslant E(0)+\frac{1}{p+1}\left[\left\|u_1+u_2\right\|_{p+1}^{p+1}+2\left\|u_1u_2\right\|_{\frac{p+1}{2}}^{\frac{p+1}{2}}\right].$$

Consequently, (1.24) gives

$$\begin{aligned} &\frac{1}{p+1}\left[\left\|u_1+u_2\right\|_{p+1}^{p+1}+2\left\|u_1u_2\right\|_{\frac{p+1}{2}}^{\frac{p+1}{2}}\right] \\ &\geqslant \frac{1}{2(\gamma+1)}\sum_{i=1}^{2}\left\|D^\kappa u_i(t_0)\right\|_2^{2(\gamma+1)}-E(0)\geqslant\frac{\eta}{p+1}\alpha_2^{(p+1)}\quad(\forall\, t\geqslant 0). \ \triangleright \end{aligned}$$

◁ Proof of Theorem 3.1. We set

$$H(t) = d - E(t). \tag{1.31}$$

By using (1.16), (1.31) we get

$$\begin{aligned} H'(t) = &\int\limits_{\Omega} \Big(|u_1(t)|^k + |u_2(t)|^l\Big)\, |u_1'(t)|^2\, dx \\ &+ \int\limits_{\Omega} \Big(|u_2(t)|^\theta + |u_1(t)|^\varrho\Big)\, |u_2'(t)|^2\, dx \geqslant 0 \quad (\forall\, t \geqslant 0). \end{aligned} \tag{1.32}$$

Therefore,

$$\begin{aligned} 0 < H(0) \leqslant H(t) = d - \frac{m-1}{m} \sum_{i=1}^{2} \|u_i'\|_m^m \\ - \frac{1}{2(\gamma+1)} \sum_{i=1}^{2} \big\|D^\kappa u_i\big\|_2^{2(\gamma+1)} + \frac{1}{p+1}\left[\big\|u_1+u_2\big\|_{p+1}^{p+1} + 2\big\|u_1u_2\big\|_{\frac{(p+1)}{2}}^{\frac{(p+1)}{2}}\right]. \end{aligned}$$

From (1.24), we obtain that for all $t \geqslant 0$ the estimates hold

$$\begin{aligned} d - \frac{1}{2(\gamma+1)} \sum_{i=1}^{2} \big\|D^\kappa u_i(t_0)\big\|_2^{2(\gamma+1)} + \frac{1}{p+1}\left[\big\|u_1+u_2\big\|_{p+1}^{p+1} + 2\big\|u_1u_2\big\|_{\frac{(p+1)}{2}}^{\frac{(p+1)}{2}}\right] \\ < d - \frac{1}{2(\gamma+1)}\, \alpha_1^2 + \frac{1}{p+1}\left[\big\|u_1+u_2\big\|_{p+1}^{p+1} + 2\big\|u_1u_2\big\|_{\frac{(p+1)}{2}}^{\frac{(p+1)}{2}}\right] \\ < -\frac{1}{p+1}\, \alpha_1^2 + \frac{1}{p+1}\left[\big\|u_1+u_2\big\|_{p+1}^{p+1} + 2\big\|u_1u_2\big\|_{\frac{(p+1)}{2}}^{\frac{(p+1)}{2}}\right] \\ < \frac{c_0}{p+1}\left[\big\|u_1+u_2\big\|_{p+1}^{p+1} + 2\big\|u_1u_2\big\|_{\frac{(p+1)}{2}}^{\frac{(p+1)}{2}}\right]. \end{aligned}$$

Hence by (**A2**), we have

$$0 < H(0) \leqslant H(t) \leqslant \frac{c_1}{p+1} \sum_{i=1}^{2} \big\|u_i\big\|_{p+1}^{p+1}.$$

Then we introduce

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int\limits_{\Omega} \sum_{i=1}^{2} u_i |u_i'|^{m-2} u_i' dx, \tag{1.33}$$

for ε small to be chosen later and

$$0 < \sigma \leqslant \min\left\{\frac{p-(k+1)}{p+1}, \frac{p-(l+1)}{p+1},\ \frac{p-(\varrho+1)}{p+1}, \frac{p-(\theta+1)}{p+1}, \frac{(p-(m-1))}{m(p+1)}\right\}. \tag{1.34}$$

We will show that $L(t)$ satisfies

$$L'(t) \geqslant \xi L^{1+\nu}(t), \quad \text{for all} \ \ t \geqslant 0,\ \nu > 0,\ \xi > 0, \tag{1.35}$$

defined in $[0,\infty)$. By taking a derivative of (1.33) and using (1.1), we obtain

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\sum_{i=1}^{2}\|u_i'\|_m^m + \varepsilon\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)}$$
$$-\varepsilon\int_\Omega u_1\left(|u_1(t)|^k + |u_2(t)|^l\right)u_1'\,dx - \varepsilon\int_\Omega u_2\left(\left|u_2(t)\right|^\theta + \left|u_1(t)\right|^\varrho\right)u_2'dx$$
$$+\varepsilon\int_\Omega \left(u_1f_1(u_1,u_2) + u_2f_2(u_1,u_2)\right)\,dx.$$

Then

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\sum_{i=1}^{2}\|u_i'\|_m^m + \varepsilon\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)}$$
$$-\varepsilon\int_\Omega u_1\left(|u_1(t)|^k + |u_2(t)|^l\right)u_1'dx - \varepsilon\int_\Omega u_2\left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right)u_2'dx$$
$$+\varepsilon\left(\left\|u_1+u_2\right\|_{p+1}^{p+1} + 2\left\|u_1u_2\right\|_{\frac{(p+1)}{2}}^{\frac{(p+1)}{2}}\right).$$

By exploiting (1.16) and (1.21), equation (1.36) takes the form

$$L'(t) \geqslant (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\frac{m+2(\gamma+1)(m-1)}{m}\sum_{i=1}^{2}\|u_i'\|_m^m$$
$$+\varepsilon 2(\gamma+1)H(t) - \varepsilon 2(\gamma+1)d + \varepsilon 2\sum_{i=1}^{2}\|D^\kappa u_i\|_2^{2(\gamma+1)}$$
$$-\varepsilon\int_\Omega u_1\left(|u_1(t)|^k + |u_2(t)|^l\right)u_1'dx - \varepsilon\int_\Omega u_2\left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right)u_2'dx$$
$$+\varepsilon(1-\frac{2(\gamma+1)}{p+1})\left(\left\|u_1+u_2\right\|_{p+1}^{p+1} + 2\left\|u_1u_2\right\|_{\frac{(p+1)}{2}}^{\frac{(p+1)}{2}}\right).$$

We will estimate, for some constance $\lambda_1, \lambda_2 > 0$, two terms as

$$\begin{aligned}&\int_\Omega\left(|u_1(t)|^k + |u_2(t)|^l\right)|u_1u_1'|dx\\ &\leqslant \lambda_1\int_\Omega\left(|u_1(t)|^k + |u_2(t)|^l\right)|u_1|^2\,dx + \frac{1}{4\lambda_1}\int_\Omega\left(|u_1(t)|^k + |u_2(t)|^l\right)|u_1'|^2dx,\end{aligned}\tag{1.36}$$

and

$$\begin{aligned}&\int_\Omega\left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right)|u_2u_2'|dx\\ &\leqslant \lambda_2\int_\Omega\left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right)|u_2|^2\,dx + \frac{1}{4\lambda_2}\int_\Omega\left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right)|u_2'|^2\,dx.\end{aligned}\tag{1.37}$$

Then,

$$\begin{aligned}L'(t) \geqslant (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\frac{m+2(\gamma+1)(m-1)}{m}\sum_{i=1}^{2}\|u_i'\|_m^m \\ +\varepsilon 2\sum_{i=1}^{2}\left\|D^\kappa u_i\right\|_2^{2(\gamma+1)} + 2(\gamma+1)\varepsilon H(t) \\ +\varepsilon\left(1-\frac{2(\gamma+1)}{p+1}\right)\left(\|u_1+u_2\|_{p+1}^{p+1} + 2\|u_1u_2\|_{\frac{p+1}{2}}^{\frac{p+1}{2}}\right) \\ -\varepsilon\lambda_1\int_\Omega\left(|u_1(t)|^k + |u_2(t)|^l\right)|u_1|^2dx - \varepsilon\frac{1}{4\lambda_1}\int_\Omega\left(|u_1(t)|^k+|u_2(t)|^l\right)|u_1'|^2\,dx \\ -\varepsilon\lambda_2\int_\Omega\left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right)|u_2|^2dx - \varepsilon\frac{1}{4\lambda_2}\int_\Omega\left(|u_2(t)|^\theta+|u_1(t)|^\varrho\right)|u_2'|^2\,dx.\end{aligned} \tag{1.38}$$

Consequently, by using Young's inequality for some $\delta, \delta_1 > 0$, we have

$$\begin{aligned}\int_\Omega\left(|u_1(t)|^k + |u_2(t)|^l\right)|u_1|^2dx = \|u_1\|_{k+2}^{k+2} + \int_\Omega |u_2|^l|u_1|^2dx \\ \leqslant \|u_1\|_{k+2}^{k+2} + \frac{l}{l+2}\delta^{(l+2)/l}\|u_2\|_{l+2}^{l+2} + \frac{2}{l+2}\delta^{-(l+2)/(2)}\|u_1\|_{l+2}^{l+2},\end{aligned}$$

and

$$\begin{aligned}\int_\Omega\left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right)|u_2|^2dx = \|u_2\|_{\theta+2}^{\theta+2} + \int_\Omega |u_1|^\varrho|u_2|^2dx \\ \leqslant \|u_2\|_{\theta+2}^{\theta+2} + \frac{\varrho}{\varrho+2}\delta_1^{(\varrho+2)/\varrho}\|u_1\|_{\varrho+2}^{\varrho+2} + \frac{2}{\varrho+2}\delta_1^{-(\varrho+2)/(2)}\|u_2\|_{\varrho+2}^{\varrho+2}.\end{aligned}$$

Then,

$$\begin{aligned}L'(t) \geqslant (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\frac{m+2(\gamma+1)(m-1)}{m}\sum_{i=1}^{2}\|u_i'\|_m^m \\ +\varepsilon 2\sum_{i=1}^{2}\|D^\kappa u_i\|_2^{2(\gamma+1)} + 2(\gamma+1)\varepsilon H(t) \\ +\varepsilon\left(1-\frac{2(\gamma+1)}{p+1}\right)\left(\|u_1+u_2\|_{p+1}^{p+1} + 2\|u_1u_2\|_{\frac{p+1}{2}}^{\frac{p+1}{2}}\right) \\ -\varepsilon\frac{1}{4\lambda_1}\int_\Omega\left(|u_2(t)|^\theta+|u_1(t)|^\varrho\right)|u_2'|^2dx - \varepsilon\frac{1}{4\lambda_2}\int_\Omega\left(|u_1(t)|^k+|u_2(t)|^l\right)|u_1'|^2dx \\ -\varepsilon\lambda_1\left(\|u_2\|_{\theta+2}^{\theta+2} + \frac{\varrho}{\varrho+2}\delta_1^{(\varrho+2)/\varrho}\|u_1\|_{\varrho+2}^{\varrho+2} + \frac{2}{\varrho+2}\delta_1^{-(\varrho+2)/(2)}\|u_2\|_{\varrho+2}^{\varrho+2}\right) \\ -\varepsilon\lambda_2\left(\|u_1\|_{k+2}^{k+2} + \frac{l}{l+2}\delta^{(l+2)/l}\|u_2\|_{l+2}^{l+2} + \frac{2}{l+2}\delta^{-(l+2)/(2)}\|u_1\|_{l+2}^{l+2}\right).\end{aligned}$$

Choosing λ_1, λ_2 such that

$$\frac{1}{4\lambda_1} = m_1H^{-\sigma}(t), \quad \frac{1}{4\lambda_2} = m_2H^{-\sigma}(t), \quad m_1, m_2 > 0. \tag{1.39}$$

Using (1.39) and the fact that

$$H'(t) = \int\limits_{\Omega} \left(|u_1(t)|^k + |u_2(t)|^l\right) |u_1'(t)|^2 dx + \int\limits_{\Omega} \left(|u_2(t)|^\theta + |u_1(t)|^\varrho\right) |u_2'(t)|^2 dx \quad (\forall\, t \geqslant 0),$$

to obtain for $M = m_1 + m_2$ and assumption (**A2**),

$$\begin{aligned} L'(t) \geqslant & ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) + \varepsilon \frac{m + 2(\gamma+1)(m-1)}{m} \sum_{i=1}^{2} \|u_i'\|_m^m \\ & + \varepsilon 2 \sum_{i=1}^{2} \left\|D^\kappa u_i\right\|_2^{2(\gamma+1)} + 2(\gamma+1)\varepsilon H(t) + \varepsilon c_2 \sum_{i=1}^{2} \|u_i\|_{p+1}^{p+1} \\ & - \varepsilon \frac{1}{4m_1} H^\sigma(t) \left(\|u_2\|_{\theta+2}^{\theta+2} + \frac{\varrho}{\varrho+2} \delta_1^{(\varrho+2)/\varrho} \|u_1\|_{\varrho+2}^{\varrho+2} + \frac{2}{\varrho+2} \delta_1^{-(\varrho+2)/(2)} \|u_2\|_{\varrho+2}^{\varrho+2} \right) \\ & - \varepsilon \frac{1}{4m_2} H^\sigma(t) \left(\|u_1\|_{k+2}^{k+2} + \frac{l}{l+2} \delta^{(l+2)/l} \|u_2\|_{l+2}^{l+2} + \frac{2}{l+2} \delta^{-(l+2)/(2)} \|u_1\|_{l+2}^{l+2} \right). \end{aligned}$$

Since (1.6) holds, we obtain by using condition (1.34)

$$\begin{cases} H^\sigma(t) \|u_1\|_{i+2}^{i+2} \leqslant c_3 \left(\|u_1\|_{(p+1)}^{\sigma(p+1)+(i+2)} + \|u_2\|_{(p+1)}^{\sigma(p+1)} \|u_1\|_{i+2}^{i+2} \right); \\ H^\sigma(t) \|u_2\|_{j+2}^{j+2} \leqslant c_4 \left(\|u_2\|_{(p+1)}^{\sigma(p+1)+(j+2)} + \|u_1\|_{(p+1)}^{\sigma(p+1)} \|u_2\|_{j+2}^{j+2} \right), \end{cases} \tag{1.40}$$

where $i = k, l, \varrho$ and $j = \theta, \varrho, l$. Then

$$\begin{aligned} L'(t) \geqslant & ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) + \varepsilon \frac{m + 2(\gamma+1)(m-1)}{m} \sum_{i=1}^{2} \|u_i'\|_m^m \\ & + \varepsilon 2 \sum_{i=1}^{2} \|D^\kappa u_i\|_2^{2(\gamma+1)} + 2(\gamma+1)\varepsilon H(t) + \varepsilon c_2 \sum_{i=1}^{2} \|u_i\|_{p+1}^{p+1} \\ & - \varepsilon \frac{1}{4m_1} c_4 \left(\|u_2\|_{(p+1)}^{\sigma(p+1)+(\theta+2)} + \|u_1\|_{(p+1)}^{\sigma(p+1)} \|u_2\|_{\theta+2}^{\theta+2} \right) \\ & - \varepsilon \frac{1}{4m_1} \frac{\varrho}{\varrho+2} \delta_1^{(\varrho+2)/\varrho} c_3 \left(\|u_1\|_{(p+1)}^{\sigma(p+1)+(\varrho+2)} + \|u_2\|_{(p+1)}^{\sigma(p+1)} \|u_1\|_{\varrho+2}^{\varrho+2} \right) \\ & - \varepsilon \frac{1}{4m_1} \frac{2}{\varrho+2} \delta_1^{-(\varrho+2)/(2)} c_4 \left(\|u_2\|_{(p+1)}^{\sigma(p+1)+(\varrho+2)} + \|u_1\|_{(p+1)}^{\sigma(p+1)} \|u_2\|_{\varrho+2}^{\varrho+2} \right) \\ & - \varepsilon \frac{1}{4m_2} c_3 \left(\|u_1\|_{(p+1)}^{\sigma(p+1)+(k+2)} + \|u_2\|_{(p+1)}^{\sigma(p+1)} \|u_1\|_{k+2}^{k+2} \right) \\ & - \varepsilon \frac{1}{4m_2} \frac{l}{l+2} \delta^{(l+2)/l} c_4 \left(\|u_2\|_{(p+1)}^{\sigma(p+1)+(l+2)} + \|u_1\|_{(p+1)}^{\sigma(p+1)} \|u_2\|_{l+2}^{l+2} \right) \\ & - \varepsilon \frac{1}{4m_2} \frac{2}{l+2} \delta^{-(l+2)/(2)} c_3 \left(\|u_1\|_{(p+1)}^{\sigma(p+1)+(l+2)} + \|u_2\|_{(p+1)}^{\sigma(p+1)} \|u_1\|_{l+2}^{l+2} \right). \end{aligned}$$

By using (1.34) and the algebraic inequality

$$z^\nu \leqslant (z+1) \leqslant \left(1 + \frac{1}{a}\right)(z+a) \quad (\forall\, z \geqslant 0,\ 0 < \nu \leqslant 1,\ a \geqslant 0), \tag{1.41}$$

we have, for all $t \geqslant 0$,

$$\|u_i\|_{(p+1)}^{\sigma(p+1)+j+2} \leqslant b\left(\|u_i\|_{(p+1)}^{(p+1)} + H(0)\right) \leqslant b\left(\|u_i\|_{(p+1)}^{(p+1)} + H(t)\right), \tag{1.42}$$

where $b = 1 + 1/H(0)$, $j = k, \theta, l, \varrho$ and $i = 1, 2$, so that we obtain

$$\begin{aligned}
L'(t) \geqslant{}& ((1-\sigma) - M\varepsilon)H^{-\sigma}(t)H'(t) + \varepsilon\frac{m + 2(\gamma+1)(m-1)}{m}\sum_{i=1}^{2}\|u_i'\|_m^m \\
&+\varepsilon 2\sum_{i=1}^{2}\|D^{\kappa}u_i\|_2^{2(\gamma+1)} + 2(\gamma+1)\varepsilon H(t) + \varepsilon c_2\sum_{i=1}^{2}\|u_i\|_{p+1}^{p+1} \\
&-\varepsilon\frac{1}{4m_1}c_4\left(b\left(\|u_2\|_{(p+1)}^{(p+1)} + H(t)\right) + \|u_1\|_{(p+1)}^{\sigma(p+1)}\|u_2\|_{\theta+2}^{\theta+2}\right) \\
&-\varepsilon\frac{1}{4m_1}\frac{\varrho}{\varrho+2}\delta_1^{(\varrho+2)/\varrho}c_3\left(b\left(\|u_1\|_{(p+1)}^{(p+1)} + H(t)\right) + \|u_2\|_{(p+1)}^{\sigma(p+1)}\|u_1\|_{\varrho+2}^{\varrho+2}\right) \\
&-\varepsilon\frac{1}{4m_1}\frac{2}{\varrho+2}\delta_1^{-(\varrho+2)/(2)}c_4\left(b\left(\|u_2\|_{(p+1)}^{(p+1)} + H(t)\right) + \|u_1\|_{(p+1)}^{\sigma(p+1)}\|u_2\|_{\varrho+2}^{\varrho+2}\right) \\
&-\varepsilon\frac{1}{4m_2}c_3\left(b\left(\|u_1\|_{(p+1)}^{(p+1)} + H(t)\right) + \|u_2\|_{(p+1)}^{\sigma(p+1)}\|u_1\|_{k+2}^{k+2}\right) \\
&-\varepsilon\frac{1}{4m_2}\frac{l}{l+2}\delta^{(l+2)/l}c_4\left(b\left(\|u_2\|_{(p+1)}^{(p+1)} + H(t)\right) + \|u_1\|_{(p+1)}^{\sigma(p+1)}\|u_2\|_{l+2}^{l+2}\right) \\
&-\varepsilon\frac{1}{4m_2}\frac{2}{l+2}\delta^{-(l+2)/(2)}c_3\left(b\left(\|u_1\|_{(p+1)}^{(p+1)} + H(t)\right) + \|u_2\|_{(p+1)}^{\sigma(p+1)}\|u_1\|_{l+2}^{l+2}\right).
\end{aligned}$$

Also, since $(X+Y)^s \leqslant C(X^s + Y^s)$, $X, Y \geqslant 0$, making use of (1.34) we conclude

$$\begin{aligned}
\|u_2\|_{(p+1)}^{\sigma(p+1)}\|u_1\|_{j+2}^{j+2} &\leqslant |\Omega|^{\frac{(p+1)-(j+2)}{(p+1)}}\left(\|u_2\|_{(p+1)}^{\sigma(p+1)}\|u_1\|_{p+1}^{j+2}\right) \\
&\leqslant |\Omega|^{\frac{(p+1)-(j+2)}{(p+1)}}\left(\|u_2\|_{(p+1)}^{\sigma}\|u_1\|_{p+1}^{\frac{j+2}{p+1}}\right)^{p+1} \\
\leqslant |\Omega|^{\frac{(p+1)-(j+2)}{(p+1)}}&\left(c'\|u_2\|_{(p+1)}^{\frac{\sigma(p+1)+(j+2)}{(p+1)}} + c''\|u_1\|_{p+1}^{\frac{\sigma(p+1)+(j+2)}{p+1}}\right)^{p+1} \leqslant c_5\sum_{i=1}^{2}\|u_i\|_{(p+1)}^{(p+1)},
\end{aligned} \tag{1.43}$$

where $c' = \frac{\sigma(p+1)}{\sigma(p+1)+(i+2)}$, $c'' = \frac{i+2}{\sigma(p+1)+(i+2)}$, for $j = k, l, \varrho$.

Similarly,

$$\|u_1\|_{(p+1)}^{\sigma(p+1)}\|u_2\|_{j+2}^{j+2} \leqslant c_6\sum_{i=1}^{2}\|u_i\|_{(p+1)}^{(p+1)}, \tag{1.44}$$

for $j = \theta$, ϱ, l.

Taking into account (1.43), (1.44), we deduce

$$
\begin{aligned}
L'(t) \geqslant{} & ((1-\sigma)-M\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon\frac{m+2(\gamma+1)(m-1)}{m}\sum_{i=1}^{2}\left\|u_i'\right\|_m^m \\
& +\varepsilon 2\sum_{i=1}^{2}\left\|D^{\kappa}u_i\right\|_2^{2(\gamma+1)}+2(\gamma+1)\varepsilon H(t)+\varepsilon c_2\sum_{i=1}^{2}\left\|u_i\right\|_{p+1}^{p+1} \\
& -\varepsilon\frac{1}{4m_1}c_4\left(b\left(\left\|u_2\right\|_{(p+1)}^{(p+1)}+H(t)\right)+c_6\sum_{i=1}^{2}\left\|u_i\right\|_{(p+1)}^{(p+1)}\right) \\
& -\varepsilon\frac{1}{4m_1}\frac{\varrho}{\varrho+2}\delta_1^{(\varrho+2)/\varrho}c_3\left(b\left(\left\|u_1\right\|_{(p+1)}^{(p+1)}+H(t)\right)+c_5\sum_{i=1}^{2}\left\|u_i\right\|_{(p+1)}^{(p+1)}\right) \\
& -\varepsilon\frac{1}{4m_1}\frac{2}{\varrho+2}\delta_1^{-(\varrho+2)/(2)}c_4\left(b\left(\left\|u_2\right\|_{(p+1)}^{(p+1)}+H(t)\right)+c_6\sum_{i=1}^{2}\left\|u_i\right\|_{(p+1)}^{(p+1)}\right) \\
& -\varepsilon\frac{1}{4m_2}c_3\left(b\left(\left\|u_1\right\|_{(p+1)}^{(p+1)}+H(t)\right)+c_5\sum_{i=1}^{2}\left\|u_i\right\|_{(p+1)}^{(p+1)}\right) \\
& -\varepsilon\frac{1}{4m_2}\frac{l}{l+2}\delta^{(l+2)/l}c_4\left(b\left(\left\|u_2\right\|_{(p+1)}^{(p+1)}+H(t)\right)+c_6\sum_{i=1}^{2}\left\|u_i\right\|_{(p+1)}^{(p+1)}\right) \\
& -\varepsilon\frac{1}{4m_2}\frac{2}{l+2}\delta^{-(l+2)/(2)}c_3\left(b\left(\left\|u_1\right\|_{(p+1)}^{(p+1)}+H(t)\right)+c_5\sum_{i=1}^{2}\left\|u_i\right\|_{(p+1)}^{(p+1)}\right).
\end{aligned}
\tag{1.45}
$$

Then

$$
\begin{aligned}
L'(t) \geqslant{} & ((1-\sigma)-M\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon\frac{m+2(\gamma+1)(m-1)}{m}\sum_{i=1}^{2}\|u_i'\|_m^m \\
& +\varepsilon 2\sum_{i=1}^{2}\|D^{\kappa}u_i\|_2^{2(\gamma+1)}+\varepsilon\left(2(\gamma+1)+\frac{1}{4m_1}c_7+\frac{1}{4m_2}c_8\right)H(t) \\
& +\varepsilon\left(c_2+\frac{1}{4m_1}c_7+\frac{1}{4m_2}c_9\right)\sum_{i=1}^{2}\|u_i\|_{p+1}^{p+1}.
\end{aligned}
\tag{1.46}
$$

For large values of m_1 and m_2 we can find positive constants A and B such that

$$
\begin{aligned}
L'(t) \geqslant{} & ((1-\sigma)-M\varepsilon)H^{-\sigma}(t)H'(t) \\
& +\varepsilon\frac{m+2(\gamma+1)(m-1)}{m}\sum_{i=1}^{2}\left\|u_i'\right\|_m^m+\varepsilon AH(t)+\varepsilon B\sum_{i=1}^{2}\left\|u_i\right\|_{p+1}^{p+1}.
\end{aligned}
\tag{1.47}
$$

We pick ε small enough so that $((1-\sigma)-M\varepsilon)\geqslant 0$ and $L(0)>0$.
Consequently, there exists $\Gamma>0$ such that

$$
L'(t)\geqslant\varepsilon\Gamma\left(H(t)+\sum_{i=1}^{2}\|u_i'\|_m^m+\sum_{i=1}^{2}\|u_i\|_{(p+1)}^{(p+1)}\right). \tag{1.48}
$$

Thus, we have $L(t) \geqslant L(0) > 0$, for all $t \geqslant 0$.

On the other hand, we have

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \Big(H^{1-\sigma}(t) + \varepsilon \int\limits_\Omega \sum_{i=1}^{2} u_i |u_i'|^{m-2} u_i'(x,t)dx\Big)^{\frac{1}{1-\sigma}} \\ &\leqslant c_{10}\Big(H(t) + \Big|\int\limits_\Omega \sum_{i=1}^{2} u_i |u_i'|^{m-1}dx\Big|^{\frac{1}{1-\sigma}}\Big). \end{aligned} \tag{1.49}$$

By Hölder's and Young's inegualities, taking (1.6) into accaunt, we estimate

$$\Big|\int\limits_\Omega u_i|u_i'|^{m-1}dx\Big|^{\frac{1}{1-\sigma}} \leqslant \|u_i\|_m^{\frac{1}{1-\sigma}} \|u_i'\|_m^{\frac{m-1}{1-\sigma}} \leqslant C|\Omega|^{\frac{1}{m}-\frac{1}{p+1}} \Big(\|u_i\|_{p+1}^{\frac{m-m\sigma}{(1-\sigma)(1-m\sigma)}} + \|u_i'\|_m^m\Big), \quad i=1,2,$$

and also using (1.34), we have

$$\begin{aligned} \Big|\int\limits_\Omega \sum_{i=1}^{2} u_i|u_i'|^{m-1}\Big|^{\frac{1}{1-\sigma}} &\leqslant \Big|\int\limits_\Omega u_1|u_1'|^{m-1}\Big|^{\frac{1}{1-\sigma}} + \Big|\int\limits_\Omega u_2|u_2'|^{m-1}\Big|^{\frac{1}{1-\sigma}} \\ &\leqslant C\Big(\sum_{i=1}^{2} \|u_i\|_{p+1}^{\frac{m}{(1-m\sigma)}} + \sum_{i=1}^{2} \|u_i'\|_m^m\Big). \end{aligned} \tag{1.50}$$

By using again (1.34) and (1.41) we get

$$\|u_i\|_{(p+1)}^{\frac{m}{(1-m\sigma)}} \leqslant b\Big(\|u_i\|_{(p+1)}^{(p+1)} + H(t)\Big) \quad (i=1,2,\ \forall\, t \geqslant 0). \tag{1.51}$$

Therefore,

$$L^{\frac{1}{1-\sigma}}(t) \leqslant c_{11}\Big[H(t) + \sum_{i=1}^{2} \|u_i\|_{(p+1)}^{(p+1)} + \sum_{i=1}^{2} \|u_i'\|_m^m\Big] \quad (\forall\, t \geqslant 0). \tag{1.52}$$

With (1.52) and (1.48), we arrive at

$$L'(t) \geqslant a_0 L^{\frac{1}{1-\sigma}}(t) \quad (\forall\, t \geqslant 0). \tag{1.53}$$

Finally, a simple integration of (1.53) gives the desired result. ▷

5. Comments and Question

Remark. Let us mention that our main contributions in this article is the study of the influence of strong source terms on the existence of solutions with positive initial energy and in the higher-order function spaces, where f_1, f_2 drive the solution of our system to blow up for all t if they dominate the damping terms, for large values of p.

Noting that one need carefully following the proofs of results in this paper to prove the nonexistence of solutions of the viscoelactic cases, using some well known assumptions on the memory terms, but it will be interresting to see the energy decay rate which will be according with that of the relaxation functions.

Question: One can consider the problem

$$\begin{cases} u_1'' - \phi\left(\|\nabla u_1\|_2\right)\Delta u_1 + \psi\left(\|\nabla u_1\|_2\right)\int\limits_0^t g_1(t-s)\Delta u_1(s)\,ds = 0, \\ u_2'' - \phi\left(\|\nabla u_2\|_2\right)\Delta u_2 + \psi\left(\|\nabla u_2\|_2\right)\int\limits_0^t g_2(t-s)\Delta u_2(s)\,ds = 0, \end{cases} \tag{1.54}$$

and may ask questions on asymptotic behavior of the solutions (If it existes): as time goes to infinity, what is the asymptotic behavior of solutions? More generally, what is the long time behavior of solutions when initial data vary in any bounded set in a Sobolev space associated with the problem (1.54).

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НЕСУЩЕСТВОВАНИЕ РЕШЕНИЯ ЗАТУХАЮЩЕЙ СИСТЕМЫ НЕЛИНЕЙНЫХ ВОЛНОВЫХ УРАВНЕНИЙ ТИПА КИРХГОФА

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Изучается влияние сильного источника на существование решений в пространстве с высоким порядком суммируемости в затухающей системе нелинейных волновых уравнений типа Кирхгофа.

Ключевые слова: взрыв, уравнение типа Кирхгофа, вырождающиеся затухающие системы, сильно нелинейный источник, положительная начальная энергия.