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SOME ISOMORPHISM RESULTS ON COMMUTATIVE GROUP ALGEBRAS

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We prove certain results pertaining to some isomorphism properties of commutative modular group algebras and briefly review a paper by pointing out some obvious mistakes and essential incorrectness.

Key words: group, ring, group algebra, isomorphism, splitting group, p-mixed group, totally projective group.

1. Introduction

Throughout the present paper, suppose FG is the group algebra of an abelian group G written multiplicatively (and possibly mixed) over a field F of positive characteristic p. As usual, FG denotes the group algebra of G over F with unit group $U(FG)$ and normalized unit group $V(FG)$; note that the direct decomposition $U(FG) = V(FG) \times F^*$ holds where F^* is the multiplicative group of F and thus the study of $U(FG)$ reduces to the study of $V(FG)$. Moreover, let $S(FG) = V_p(FG)$ and let G_p be the p-primary components of $V(FG)$ and G, respectively. All other notions and notations are standard and follow essentially those from the reference list at the end of the article. Nevertheless, we will give below some supplementary terminology and concepts.

Two central subjects in commutative group algebras theory have been played a major role. First of all, this is the isomorphism problem for commutative modular group algebras which is still unresolved and seems insurmountable in full generality at this stage. It states as follows:

Isomorphism Conjecture. *Suppose G is a p-mixed group. If H is any group and $FG \cong FH$ as F-algebras, then $H \cong G$.*

This question was settled by many authors for various classes of abelian groups; for example the interested reader can see [1–9] together with the complete references in [10–12]. Here we shall confirm once again its truthfulness provided G is splitting whose maximal torsion subgroup G_t is a totally projective (reduced or not) p-group (compare also with the original source [1]). Note that when the torsion subgroup G_t is a torsion-complete p-group, the readers may see [3], [5] and [8].

All of this is subsumed by the second challenging topic. Specifically, we state the following yet left-open.

Direct Factor Conjecture. *Let G be a p-mixed group and let F be perfect. Then $V(FG)/G$ is simply presented and, in particular, G is a direct factor $V(FG)$ with simply presented complement.*

The structure of $V(FG)$ has a great deal of an intensively study by many authors; for instance the interested reader can see [1–9] along with the complete bibliography in [10–12].

Here we will investigate how G is situated inside $V(FG)$ in some special cases, mainly motivated by the original work [1].

2. Main Results

First and foremost, we shall recall some assertions from [1] and [2] needed for our successful presentation. A group G is said to be p-splitting if G_p is a direct factor of G.

Proposition. *Suppose G is a p-splitting group and F is perfect. Then the following hold:*

(i) *If G_p is simply presented, then $S(FG)/G_p$ is simply presented and G_p is a direct factor of $S(FG)$ with simply presented complement. In particular, $S(FG)$ is simply presented if, and only if, G_p is simply presented.*

(ii) *If H is some group such that FG and FH are F-isomorphic, then $H_p \cong G_p$, provided that G_p is simply presented.*

Remark 1. The authors of [14] just have imitated and duplicated the original proofs of (i) from [1] without any new moments but with rather more superfluous explanations which are, in fact, trivial and therefore they were omitted in [1].

Moreover, May, Mollov and Nachev reproved in ([14], Theorem 2.7) almost the same statement as that of (ii) but provided G is splitting. However, their proof is manifestly incorrect and obviously has the following serious shortcoming: Using a classical lemme of Ullery, they reduce the general case to the p-mixed case considering the p-mixed groups $G/\coprod_{q\neq p} G_q$ and $H/\coprod_{q\neq p}$ and, besides, they used that $G/\coprod_{q\neq p} G_q$ is splitting. This is, however, not correct since they need to show that if G is splitting then so is $G/\coprod_{q\neq p} G_q$.

This implication is not trivial and follows like this (see, e.g., [7]): Even more, G is p-splitting if, and only if, $G/\coprod_{q\neq p} G_q$ is splitting.

Sketch of proof. Indeed, if $G = G_p \times M$, then $G = G_t M$ and hence $G/\coprod_{q\neq p} G_q = \big(G_t/\coprod_{q\neq p} G_q\big)\big(M(\coprod_{q\neq p} G_q)/\coprod_{q\neq p} G_q\big)$. But $G_t/\coprod_{q\neq p} G_q = \big(G/\coprod_{q\neq p} G_q\big)_t$ and by the modular law we have that

$$\Big[M\Big(\coprod_{q\neq p} G_q\Big)\Big] \cap G_t = \Big(\coprod_{q\neq p} G_q\Big)\Big[M \cap G_t\Big] = \Big(\coprod_{q\neq p} G_q\Big)\Big[M \cap (G_p \times M_t)\Big]$$
$$= M_t\Big(\coprod_{q\neq p} G_q\Big)\Big[M \cap G_p\Big] = \coprod_{q\neq p} G_q$$

since $M_t = \coprod_{q\neq p} M_q \subseteq \coprod_{q\neq p} G_q$.

Concerning the converse, it has an identical verification as that demonstrated above. So, we are done.

Remark 2. In [14] was claimed also that in reference [4] from [14], i. e. in [Da], is used the fact, which as stated by them is certainly wrong, that if a subgroup B is isomorphic to a direct factor of an abelian group A, then B is a direct factor of A. However, in [1] this was utilized not for an arbitrary subgroup; reciprocally it was used for a special subgroup of $V(FG)$ for which this is really true. No counterexample in [14] is given, only redundant comments that are false.

We next will reprove an assertion of ours first established in [1]. The idea is given in [9] but the proof here is a little more conceptual.

Theorem. *Suppose G is a splitting p-mixed abelian group such that G_p is simply presented and $KH \cong KG$ are K-isomorphic for some group H and some field K of prime characteristic p. Then $H \cong G$.*

Without loss of generality, and as it is well-known, we may assume that $KH = KG$ since $KG \cong KH$ implies that $KG = KH'$ for some $H \cong H' \leqslant V(KG)$. Moreover, it is well known that K may be chosen to be algebraically closed and hence perfect. Therefore, one may write that $A = V(KG) = V(KH)$. Suppose L is a perfect field of characteristic p, containing K and having the same identity, such that LA exists. Thus $I(KG; G) = I(KH; H)$ and consequently $LA \cdot I(KG; G) = LA \cdot I(KH; H)$, i. e., $I(LA; G) = I(LA; H)$ (see cf. [10]). Furthermore, $L(A/G) \cong LA/I(LA; G) = LA/I(LA; H) \cong L(A/H)$. But, according to Proposition 2.1 (i), $A/G = V(KG)/G \cong S(KG)/G_p$ is simply presented p-torsion. Consequently, by the classical isomorphism result from [13], we deduce that $A/H = V(KH)/H$ is simply presented p-torsion as well. Therefore, H is a direct factor of $V(KH)$ and hence H_p is a direct factor $S(KH)$. But $G = G_p \times M$ implies that $V(KG) = S(KG) \times T$. Thus $V(KH) = S(KH) \times T$ and by what we have shown H_p is a direct factor of $V(KH)$, whence of $H \subseteq V(KH)$. Writing $G \cong G_p \times (G/G_p)$ and $H \cong H_p \times (H/H_p)$, it follows that $G \cong H$ because again with [10] or [11] at hand we derive $G_p \cong H_p$ and $G/G_p \cong H/H_p$. $\triangleright$

Remark 3. Owing to our isomorphism theorem alluded to above, Corollary 4.3 and Theorem 4.6 both from [14] are unnecessary since they are its elementary consequences.

Moreover, it is not clear to the (non-expert) reader why in the formulation of Theorem 4.6, the subgroup G_p must be reduced. The result (perhaps) follows for not necessarily reduced G_p.

And finally, in [14] have criticized that some statements from papers of Danchev (see, e.g., the bibliography in [14]) are not completely proved in the sense that not all details are given in an explicit form. In this way, this is really almost true, but the reason is that these assertions and the missing parts and points are trivial the readers can verify that by seeing Lemmas 2.1–2.5 from [14].

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ОБ ИЗОМОРФИЗМАХ КОММУТАТИВНЫХ ГРУППОВЫХ АЛГЕБР

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Установлены некоторые результаты об изоморфизмах коммутативных групповых алгебр и приводится краткий реферат статьи [14] с указанием очевидных ошибок и существенных неточностей.

Ключевые слова: группа, кольцо, групповая алгебра, изоморфизм, расщепляющая группа, p-смешанная группа, проективная группа.