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ON THE EXPANSIONS OF ANALYTIC FUNCTIONS ON CONVEX LOCALLY CLOSED SETS IN EXPONENTIAL SERIES

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In memory of G. P. Akilov

Let Q be a bounded, convex, locally closed subset of $\mathbb{C}^N$ with nonempty interior. For $N > 1$ sufficient conditions are obtained that an operator of the representation of analytic functions on Q by exponential series has a continuous linear right inverse. For $N = 1$ the criterions for the existence of a continuous linear right inverse for the representation operator are proved.

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Introduction

In the late sixties Leont'ev (see [10]) proved that each analytic function f on a convex bounded domain $Q \subset \mathbb{C}$ can be expanded in an exponential series $\sum_{j\in\mathbb{N}} c_j \exp(\lambda_j \cdot)$. This series converges absolutely to f in the Fréchet space $A(Q)$ of all functions analytic on Q, and its exponents λ_j are zeroes of an entire function on $\mathbb{C}$ which does not depend on $f \in A(Q)$. A formula for the coefficients of a some expansion in such exponential series (with the help of a system orthogonal to $(\exp(\lambda_j \cdot))_{j\in\mathbb{N}}$) was obtained only for the analytic functions on the closure of Q. Later similar results for the analytic functions on convex bounded domain $Q \subset \mathbb{C}^N$ were obtained by Leont'ev [9], Korobeinik, Le Khai Khoi [3] (if Q is a polydomain) and Sekerin [15] (if Q is a domain of which the support function is a logarithmic potential).

In [4, 5, 11] was investigated a problem of the determination of the coefficients of the expansions of all $f \in A(Q)$, where Q is a convex bounded domain in $\mathbb{C}$, in following setting. Let $K \subset \mathbb{C}$ be a convex set and suppose that L is an entire function on $\mathbb{C}$ with zero set $(\lambda_j)_{j\in\mathbb{N}}$ and with the indicator $H_Q + H_K$, where H_Q and H_K is the support function of Q resp. of K. By $\Lambda_1(Q)$ we denote a Fréchet space of all number sequence $(c_j)_{j\in\mathbb{N}}$ such that the series $\sum_{j\in\mathbb{N}} c_j \exp(\lambda_j \cdot)$ converges absolutely in $A(Q)$. In [4, 5, 11] were established the necessary and sufficient conditions under which a sequence of the coefficients $(c_j)_{j\in\mathbb{N}} \in \Lambda_1(Q)$ in a representation $f = \sum_{j\in\mathbb{N}} c_j \exp(\lambda_j \cdot)$ can be selected in such way that they depend continuously and linearly on $f \in A(Q)$. In other words, in [4, 5, 11] was solved the problem of the existence of continuous linear right inverse for the *representation operator* $R : \Lambda_1(Q) \to A(Q)$, $c \mapsto \sum_{j\in\mathbb{N}} c_j \exp(\lambda_j \cdot)$. Note that in [4, 5] a formula for continuous linear right inverse for R (if it exists) was not obtained.

In the present article we consider the following situation. Let $Q \subset \mathbb{C}^N$ be a bounded convex set with nonempty interior. We assume that Q is locally closed, i. e. Q has a fundamental sequence of compact convex subsets Q_n, $n \in \mathbb{N}$. By $A(Q)$ we denote the space of all analytic functions on Q with the topologie of $\text{proj}_{\leftarrow n} A(Q_n)$, where $A(Q_n)$ is endowed with natural (LF)-topologie. We put $e_\lambda(z) := \exp(\sum_{m=1}^{N} \lambda_m z_m)$, $\lambda, z \in \mathbb{C}^N$. For an infinite set $M \subset \mathbb{N}^N$, for a sequence $(\lambda_{(k)})_{(k)\in M} \subset \mathbb{C}^{\mathbb{N}^N}$ with $|\lambda_{(k)}| \to \infty$ as $|(k)| \to \infty$ we define a locally convex space $\Lambda_1(Q)$ of all number sequence $(c_{(k)})_{(k)\in M}$ such that the series $\sum_{(k)\in M} c_{(k)} e_{\lambda_{(k)}}$ converges absolutely in $A(Q)$. The representation operator $c \mapsto \sum_{(k)\in M} c_{(k)} e_{\lambda_{(k)}}$ maps continuously and linearly $\Lambda_1(Q)$ into $A(Q)$. We solve the problem of the existence of a continuous linear right inverse for R.

In this paper for $N \geqslant 1$ we assume that $(\lambda_{(k)})_{(k)\in M}$ is a subset of zero set of an entire function L on $\mathbb{C}^N$ with "planar zeroes" and with indicator $H_Q + H_K$, where H_Q and H_K are the support functions of Q resp. of some convex compact set $K \subset \mathbb{C}^N$. By [15] such function L exists if and only if the support function of $\text{cl}Q + K$ is so-called *logarithmic potential* (for $N = 1$ a function L exists for each Q and each K). In contrast to [4, 5] here we do not use the structure theory of locally convex spaces. As in [11], we reduce the problem of existence a continuous linear right inverse for the representation operator to one of an extension of input function L to an entire function $\tilde{L}$ on $\mathbb{C}^{2N}$ satisfying some upper bounds. With the help of $\tilde{L}$ we construct a continuous linear left inverse for the transposed map to R. Using $\bar{\partial}$-technique, we obtain that the existence of such extension $\tilde{L}$ is equivalent to two conditions, namely, to the existence of two families of plurisubharmonic functions, first of which is associated only with Q and second is associated with K and Q. The evaluation of first condition was realized in [21]. For the evaluation of second condition we adapt as in [21] the theory of the boundary behavior of the pluricomplex Green functions of a convex domain and of a convex compact set in $\mathbb{C}^N$ which was developed in [23, 25].

For $N = 1$ we obtain more complete results. In the first place we prove the criterions for the existence of a continuous linear right inverse for R without additional suppositions on Q and K. Secondly, with the help of a function $\tilde{L}$ as above we give a formula for a continuous linear right inverse for R.

1. Preliminaries

1.1. Notations. If $B \subset \mathbb{C}^N$, by $\text{cl}B$ and $\text{int}\, B$ we will denote the closure and the interior of B, respectively. By $\text{int}_r B$, $\partial_r B$ we denote the relative interior and the relative boundary of B with respect to a certain larger set. For notations from convex analysis, we refer to Schneider [26].

1.2. Definitions and Remarks. A convex set $Q \subset \mathbb{C}^N$ admitting a countable fundamental system $(Q_n)_{n\in\mathbb{N}}$ of compact subsets of Q is called *locally closed*. Let $Q \subset \mathbb{C}^N$ be a locally closed convex set. We will write $\omega := Q \cap \partial_r Q$, where $\partial_r Q$ denotes the relative boundary of Q in its affine hull. By [21, Lemma 1.2] ω is open in $\partial_r Q$. We may assume that the sets Q_n are convex and that $Q_n \subset Q_{n+1}$ for all $n \in \mathbb{N}$. A convex set $Q \subset \mathbb{C}^N$ will be called strictly convex at $\partial_r\omega$ if the intersection of Q with each supporting hyperplane to $\text{cl}\, Q \subset \mathbb{C}^N$ is compact. If $\text{int}\, Q \neq \varnothing$, Q is strictly convex at $\partial_r\omega$ if and only if each line segment of ω is relatively compact in ω.

By [13, Lemma 3] $Q \subset \mathbb{C}^N$ is strictly convex at $\partial_r\omega$ if and only if Q has a fundamental system of convex neigborhoods.

1.3. Convention. For the sequel, we fix a bounded, convex and locally closed set $Q \subset \mathbb{C}^N$ with 0 in its nonempty interior and with a fundamental system of compact convex subsets $Q_n \subset Q_{n+1}$, $n \in \mathbb{N}$. By $(\omega_n)_{n\in\mathbb{N}}$ we shall denote some fundamental system of compact subsets of $\omega = Q \cap \partial_r Q$.

K will always denote a compact convex set in $\mathbb{C}^N$.

1.4. Notations. For each convex set $D \subset \mathbb{C}^N$ we denote by H_D the support function of D, i. e. $H_D(z) := \sup_{w\in D} \operatorname{Re}\langle z, w\rangle$, $z \in \mathbb{C}^N$. Here $\langle z, w\rangle := \sum_{j=1}^N z_j w_j$. We put $H_n := H_{Q_n}$, $n \in \mathbb{N}$.

Let $e_\lambda(z) := \exp\langle\lambda, z\rangle$, $\lambda, z \in \mathbb{C}^N$. For a locally convex space E by E'_b we denote the strong dual space of E.

1.5. Function spaces. We set $|z| := \langle z, \bar z\rangle^{1/2}$, $z \in \mathbb{C}^N$; $U(t, R) := \{z \in \mathbb{C}^N : |t-z| < R\}$, $t \in \mathbb{C}^N$, $R > 0$; $U := U(0, 1)$. For all $n, m \in \mathbb{N}$ let $E_{n,m} := A^\infty\left(Q_n + \frac{1}{m}U\right)$ denote the Banach space of all bounded holomorphic functions on $Q_n + \frac{1}{m}U$, equipped with the sup-norm. We consider the spaces $A(Q_n) = \bigcup_{m\in\mathbb{N}} E_{n,m}$ of all functions holomorphic in some neighborhood of Q_n, $n \in \mathbb{N}$, and endow them with there natural inductive limit topology. By $A(Q)$ we denote the vector space of all functions which are holomorphic on some neighborhood of Q. We have $A(Q) = \bigcap_{n\in\mathbb{N}} A(Q_n)$, and we endow this vector space with the topology of $A(Q) := \operatorname{proj}_{\leftarrow n} A(Q_n)$. This topology does not depend on the choice of the fundamental system of compact sets $(Q_n)_{n\in\mathbb{N}}$. If Q is open, $A(Q)$ is a Fréchet space of all holomorphic functions on Q.

For all $n, m \in \mathbb{N}$ let

$$A_{n,m} := \left\{ f \in A(\mathbb{C}^N) : \|f\|_{n,m} := \sup_{z\in\mathbb{C}^N} |f(z)| \exp\big(- H_n(z) - |z|/m\big) < \infty \right\}$$

and

$$A_Q := \operatorname*{ind}_{n\to} \operatorname*{proj}_{\leftarrow m} A_{n,m}.$$

1.6. Duality. *The (LF)-space $A(Q) := \operatorname{ind}_{n\to} A(Q_n)'_b$ and A_Q are isomorphic by the Laplace transformation*

$$\mathscr{F} : A(Q)' \to A_Q, \quad \mathscr{F}(\varphi)(z) := \varphi(e_z), \quad z \in \mathbb{C}^N.$$

In addition (LF)-topology of $A(Q)'$ equals the strong topology.

The assertion has been proved in [21, Lemma 1.10] (see Remark after 1.10, too)

If we identify the dual space of $A(Q)$ with A_Q by means of the bilinear form $\langle\cdot,\cdot\rangle$, then $\langle e_\lambda, f\rangle = f(\lambda)$ for all $\lambda \in \mathbb{C}^N$ and all $f \in A_Q$.

1.7. Sequence spaces. Representation operator. Let $M \subset \mathbb{N}^N$ be an infinite set and $(\lambda_{(k)})_{(k)\in M} \subset \mathbb{C}^N$ be a sequence with $|\lambda_{(k)}| \to \infty$ as $|(k)| \to \infty$. For all $n, m \in \mathbb{N}$ we introduce the Banach spaces

$$\Lambda_{n,m}(Q) := \left\{ c = (c_{(k)})_{(k)\in M} \subset \mathbb{C} : \sum_{(k)\in M} |c_{(k)}| \exp\big(H_n(\lambda_{(k)}) + |\lambda_{(k)}|/m\big) < \infty \right\},$$

$$K_{n,m}(Q) := \left\{ c = (c_{(k)})_{(k)\in M} \subset \mathbb{C} : \sup_{(k)\in M} |c_{(k)}| \exp\big(- H_n(\lambda_{(k)}) - |\lambda_{(k)}|/m\big) < \infty \right\}$$

and put

$$\Lambda_1(Q_n) := \operatorname*{ind}_{m\to} \Lambda_{n,m}(Q), \quad \Lambda_1(Q) := \operatorname*{proj}_{\leftarrow n} \Lambda_1(Q_n), \quad K_\infty(Q) := \operatorname*{ind}_{n\to} \operatorname*{proj}_{\leftarrow m} K_{n,m}(Q).$$

We note that the series $\sum_{(k)\in M} c_{(k)} e_{\lambda_{(k)}}$ converges absolutely in $A(Q)$ if and only if $c \in \Lambda_1(Q)$ (see [2, Ch. I, §§ 1, 9]).

The operator $R(c) := \sum_{(k)\in M} c_{(k)} e_{\lambda_{(k)}}$ maps continuously and linearly $\Lambda_1(Q)$ in $A(Q)$. We call R the *representation operator*. By Korobeinik [2], if $R : \Lambda_1(Q) \to A(Q)$ is surjectiv, $(e_{\lambda_{(k)}})_{(k)\in M}$ is called an *absolutely representing system* in $A(Q)$.

Let $e_{(k)} := (\delta_{(k),(m)})_{(m)\in M}$, $(k) \in M$, where $\delta_{(k),(m)}$ is the Kronecker delta.

1.8. Duality. (*i*) *The transformation* $\varphi \mapsto (\varphi(e_{(k)}))_{(k)\in M}$ *is an isomorphism of* (LF)-*space* $\Lambda_1(Q)' := \mathrm{ind}_{n\to} \Lambda_1(Q_n)'_b$ *onto* $K_\infty(Q)$. *The duality between* $\Lambda_1(Q)$ *and* $K_\infty(Q)$ *is defined by the bilinear form* $\langle c, d\rangle := \sum_{(k)\in M} c_{(k)} d_{(k)}$.
(*ii*) *A transposed map* $R' : A_Q \to K_\infty(Q)$ *to* $R : \Lambda_1(Q) \to A(Q)$ *is the restriction operator* $f \mapsto (f(\lambda_{(k)}))_{(k)\in M}$.
(*iii*) *R has a continuous linear right inverse if and only if R' has a continuous linear left inverse.*

$\lhd$ The assertions (i) and (ii) were in [13, Lemma 6] proved.

(iii): This can be proved in the same way as (*i*) $\Rightarrow$ (*ii*) in [21, Lemma 1.12]. (We note that we can not assume in advance the surjectivity of R.) $\rhd$

1.9. Notations. Let $S := \{z \in \mathbb{C}^N : |z| = 1\}$. For a convex set $D \subset \mathbb{C}^N$, $\gamma \subset D$ and $A \subset S$ we define

$$S_\gamma(D) := \{a \in S : \mathrm{Re}\langle w, a\rangle = H_D(a) \text{ for some } w \in \gamma\}$$

and

$$F_A(D) := \{w \in D : \mathrm{Re}\langle w, a\rangle = H_D(a) \text{ for some } a \in A\}.$$

We will write $S_\gamma := S_\gamma(Q)$, $\hat{A} := S_{F_A(K)}(K)$, $S_0 := S \backslash S_\omega$.

Definition 1.10. (a) Given an open subset $B \subset S$ and a compact convex set $K \subset \mathbb{C}^N$. K is called *smooth* in the directions of the boundary of B if for each compact set $\kappa \subset B$ the compact set $\hat{\kappa} := S_{F_\kappa(K)}(K)$ is still contained in B.

Note that the condition is fulfilled if ∂K is of class C^1.

(b) A convex compact set $K \subset \mathbb{C}^N$ is called *not degenerate* in the directions of $B \subset S$, if K is not contained all in the supporting hyperplane $\{z \in \mathbb{C}^N \ \mathrm{Re}\langle z, a\rangle = H_K(a)\}$ of K for each $a \in B$.

Note that the condition is fulfilled if $\mathrm{int}\, K \neq \varnothing$.

Remark 1.11. (a) Under the hypotheses of the Definition 1.10 (a) the following holds: Let $S_1 \subset S$ be an open neighborhood of $S\backslash B$ (with respect to S). For $\kappa := S\backslash S_1$, the set $\hat{\kappa}$ is a compact subset of B. Hence if $S_2 \subset S\backslash\hat{\kappa}$ is compact, we have $\hat{S}_2 \cap \kappa = \varnothing$ and thus $\hat{S}_2 \subset S_1$. (Otherwise it would follow that $\hat{\kappa} \cap S_2 \neq \varnothing$.)

(b) Let K have 0 as an interior point. K is smooth in the directions of the boundary of B if and only if the convex set $\mathrm{int}K^0 \cup \omega'$ is strictly convex at $\partial_r\omega'$, where $\omega' := \partial K^0 \cap \Gamma(B)$, $K^0 := \{z \in \mathbb{C}^N \,|\, H_K(z) \leqslant 1\}$ and $\Gamma(B) := \{tb \,|\, t > 0\}$.

2. Conditions of existence of a continuous linear right inverse for the representation operator

2.1. Notations, Definitions and Remarks. (a) Let f be an entire functions of exponential type on $\mathbb{C}^N$. By h^*_f we denote the (radial) indicator of f, i. e.

$$h^*_f(z) := \limsup_{z'\to z}(\limsup_{r\to+\infty} \log|f(rz')|/r) \text{ for all } z \in \mathbb{C}^N.$$

(b) An entire function f of exponential type on $\mathbb{C}$ is called *function of completelly regular growth* (by Levin–Pflüger), if there is a set of circles $U(\mu_j, r_j)$, $j \in \mathbb{N}$, with $|\mu_j| \to \infty$ as $j \to \infty$, such that $\lim_{R\to\infty} \frac{1}{R}\sum_{|\mu_j|<R} r_j = 0$ and outside of $\cup_{j\in\mathbb{N}} U(\mu_j, r_j)$ the following asymptotic equality holds:

$$\log|f(z)| = h_f^*(z) + \overline{o}(|z|) \text{ as } |z| \to \infty.$$

By Krasichkov–Ternovskii [6], in Definition (b) we can choose the exclusive circles $U(\mu_j, r_j)$ so that they are mutually disjoint.

(c) By Gruman [18] an entire function f of exponential type on $\mathbb{C}^N$ is called function of completelly regular growth, if for almost all $a \in S$ the function $f(az)$ of one complex variable has completelly regular growth on $\mathbb{C}$.

(d) There are other definitions of the functions of completelly regular growth of Azarin [1] and of Lelon, Gruman [8, Ch. IV, 4.1]. By Papush [14], if f is an entire function on $\mathbb{C}^N$ with "planar" zeroes, i. e. the zero set $\{z \in \mathbb{C}^N : f(z) = 0\}$ of f is the union of the hyperplanes $\{z \in \mathbb{C}^N : \langle z, a_k\rangle = c_k\}$, $a_k \in S$, $c_k \in \mathbb{C}$, $k \in \mathbb{N}$, all these definitions (for f) are equivalent. From this and from [7, 22] it follows that an entire function f on $\mathbb{C}^N$ with "planar" zeroes has completely regular growth on $\mathbb{C}^N$ if and only if f is *slowly decreasing* on $\mathbb{C}^N$.

We recall the some definitions and results from Sekerin [15].

2.2. A special entire function. A structure of the exponents $\lambda_{(k)}$. (a) Below we shall exploit an entire function L on $\mathbb{C}^N$ of order 1, which satisfies the following conditions:
(i) The zero set $V(L)$ of L is a sequence of pairwise distinct hyperplanes $P_k := \{z \in \mathbb{C}^N : \langle a_k, z\rangle = c_k\}$, $k \in \mathbb{N}$, where $|a_k| = 1$ and $c_k \neq 0$. If for $k_1 < k_2 < \ldots < k_N$ the intersection $P_{k_1} \cap P_{k_2} \cap \ldots \cap P_{k_N}$ is not empty, then it consits of a single point $\lambda_{(k)}$, where (k) denotes multiindex $(k_1, k_2, \ldots, k_N)$. Further M is the set of the such multiindexes (k). Moreover, $L_{(k)}(\lambda_{(k)}) \neq 0$, where $L_{(k)}(z) := L(z)/l_{(k)}(z)$ and $l_{(k)}(z) := \prod_{j=1}^N(\langle a_{k_j}, z\rangle - c_{k_j})$, $(k) \in M$.
(ii) L is a function of completely regular growth with indicator $H_Q + H_K$.
(iii) $|L_{(k)}(\lambda_{(k)})| = \exp(H_Q(\lambda_{(k)}) + H_K(\lambda_{(k)}) + \bar{o}(|\lambda_k|))$ as $|(k)| \to \infty$.

We write $l_k(z) := \langle a_k, z\rangle - c_k$, $z \in \mathbb{C}^N$, $k \in \mathbb{N}$.

(b) (i) By [15, Theorem 1], for each $f \in A_{\operatorname{int} Q+K}$ the Lagrange interpolation formula holds:

$$f(\lambda) = \sum_{(k)\in M} \frac{L_{(k)}(\lambda)}{L_{(k)}(\lambda_{(k)})} f(\lambda_{(k)}), \quad \lambda \in \mathbb{C}^N, \tag{1}$$

where the series converges uniformly on compact sets of $\mathbb{C}^N$. From (1) it follows that $(\lambda_{(k)})_{(k)\in M}$ is the uniqueness set for $A_{\operatorname{int} Q+K}$, i. e. from $f \in A(\mathbb{C}^N)$, $h_f^*(z) < H_Q(z) + H_K(z)$ for all $z \in \mathbb{C}^N\backslash\{0\}$ it follows that $f \equiv 0$.

(ii) There is a function $\alpha(z) = \bar{o}(|z|)$ as $|z| \to \infty$ such that $|L_{(k)}(z)| \leqslant \exp(H_Q(z) + H_K(z) + \alpha(z))$ for all $z \in \mathbb{C}^N$ and all $(k) \in M$.

(c) A plurisubharmonic function u on $\mathbb{C}^N$ will be called a logarithmic potential if there exists a Borel measure $\mu \geqslant 0$ on $[0, \infty) \times S^N$ such that for every $R \in (0, \infty)$ there is a pluriharmonic function u_R on $U(0, R)$ with

$$u(z) = \int\limits_{[0,R]\times S^N} \log|t - \langle z, w\rangle|\, d\mu(t) + u_R(z) \text{ for all } z \in U(0, R).$$

By [15] for a bounded convex domain D with $0 \in D$ the support function H_D is a logarithmic potential for example if D is a polydomain, a ball, an ellipsoid, a polyhedra with symmetric faces, and in the case of $\mathbb{C}^2$, if $D = D_1 + iD_2$, where D_1 and D_2 are any centrally symmetric convex domains in $\mathbb{R}^2$; if D is symmetric with respect to 0 and $\operatorname{cl} D$ is a Steiner compact set (see Matheron [19, § 4.5]).

For each bounded convex domain $D \subset \mathbb{C}$ with $0 \in \operatorname{int} D$ the function H_D is a logarithmic potential.

(d) By [15, Theorem 5], there exists a function L satisfying the conditions (i)–(iii) in 2.2 (a) if and only if $H_Q + H_K$ is a logarithmic potential. $H_Q + H_K$ is a logarithmic potential if H_Q and H_K are the logarithmic potentials.

(e) Let $H_{Q+K} = H_Q + H_K$ be a logarithmic potential. By [15] the representation operator $R : \Lambda_1(\operatorname{int} Q + K) \to A(\operatorname{int} Q + K)$ is surjective. By [13, Theorem 14] $R : \Lambda_1(Q) \to A(Q)$ is surjective, if Q is strictly convex at $\partial_r \omega$, K is smooth in the directions of $\partial_r S_\omega$ and not degenerate in the directions of S_ω.

Theorem 2.3. *Let Q be strictly convex at $\partial_r \omega$ and L be an entire function on $\mathbb{C}^N$ satisfying the conditions* 2.2 (a). *Then* (II) $\Leftrightarrow$ (III) $\Rightarrow$ (I)*:*

(I) *The representation operator $R : \Lambda_1(A) \to A(Q)$ has a continuous linear right inverse.*

(II) *There is a positively homogeneous of order 1 plurisubharmonic function P on $\mathbb{C}^{2N}$ such that $P(z,z) \geqslant H_Q(z) + H_K(z)$ and $(\forall n)$ $(\exists n')$ $(\forall s)$ $(\exists s')$ with*

$$P(z,\mu) \leqslant H_{n'}(z) + |z|/s + H_K(\mu) + H_Q(\mu) - H_n(\mu) - |\mu|/s' \quad (\forall z, \mu \in \mathbb{C}^N).$$

(III) *There are the plurisubharmonic functions u_t, v_t, $t \in S$, on $\mathbb{C}^N$ such that $u_t(t) \geqslant 0$, $v_t(t) \geqslant 0$ and $(\forall n)$ $(\exists n')$ $(\forall s)$ $(\exists s')$ with*
(a) $u_t(z) \leqslant H_{n'}(z) - H_n(t) + |z|/s - 1/s'$ *and*
(b) $v_t(\mu) \leqslant H_K(\mu) + H_Q(\mu) - H_n(\mu) - H_K(t) - H_Q(t) + H_{n'}(t) - |\mu|/s' + 1/s$ *for all $z, \mu \in \mathbb{C}^N$ and all $t \in S$.*

$\lhd$ (II) $\Rightarrow$ (III). We may choose

$$u_t(z) := P(z,t) - H_Q(t) - H_K(t), \quad v_t(\mu) := P(t,\mu) - H_Q(t) - H_K(t)$$

for all $z, \mu \in \mathbb{C}^N$ and $t \in S$.

(III) $\Rightarrow$ (II). We put

$$P_0(z,\mu) := \Big(\sup_{t \in S} \big(u_t(z) + v_t(\mu) + H_Q(z) + H_K(\mu) \big) \Big)^*, \quad z, \mu \in \mathbb{C}^N,$$

where f^* denotes the regularization of a function f. P_0 is the plurisubharmonic function on $\mathbb{C}^{2N}$ with

$$P(z,z) \geqslant H_Q(z) + H_K(z) \quad (\forall z \in S).$$

By (III) we have: $(\forall m)$ $(\exists n')$ $(\forall s)$ $(\exists r)$ with

$$u_t(z) \leqslant H_{n'}(z) - H_m(t) + |z|/s - 1/r \text{ for all } z \in \mathbb{C}^N \text{ and all } t \in S$$

and $(\forall n)$ $(\exists m)$ $(\forall r)$ $(\exists s')$ with

$$v_t(\mu) + H_Q(t) + H_K(t) \leqslant H_K(\mu) + H_Q(\mu) - H_n(\mu) + H_m(t) - |\mu|/s + 1/r$$

for all $\mu \in \mathbb{C}^N$ and all $t \in S$. By adding the last inequalities, we obtain that $(\forall n)$ $(\exists n')$ $(\forall s)$ $(\exists s')$ with

$$u_t(z) + v_t(\mu) + H_Q(t) + H_K(t) \leqslant H_{n'}(z) + H_K(\mu) + H_Q(\mu) - H_n(\mu) + |z|/s - |\mu|/s'$$

for all $z, \mu \in \mathbb{C}^N$ and $t \in S$. From this it follows that P_0 satisfies the upper bounds in (II). As P we may choose $P(z,\mu) := (\limsup_{t\to+\infty} P(tz,t\mu)/t)^*$, $z, \mu \in \mathbb{C}^N$.

(III) $\Rightarrow$ (I). By (the proof of) [16, Theorem 4.4.3] (see [8, Theorem 7.1], too) there is a $\tilde{L} \in A(\mathbb{C}^{2N})$ with $\tilde{L}(z,z) = L(z)$ and $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ $(\exists\, C)$: $(\forall\, z, \mu \in \mathbb{C}^N)$

$$|\tilde{L}(z,\mu)| \leqslant C \exp\big(H_{n'}(z) + |z|/s + H_K(\mu) + H_Q(\mu) - H_n(\mu) - |\mu|/s'\big). \tag{2}$$

We define

$$\kappa_1(c)(z) := \sum_{(k)\in M} \frac{L_{(k)}(z)\tilde{L}(z,\lambda_{(k)})}{L_{(k)}(\lambda_{(k)})} c_{(k)}, \quad c \in K_\infty(Q),\ z \in \mathbb{C}^N. \tag{3}$$

From (2) it follows that the series in (3) converges absolutely in A_{2Q+K}. (By [21, Remark 1.5] $2Q + K$ is locally closed and $(2Q_n + K)_{n\in\mathbb{N}}$ is a fundamental system of compact subsets of $2Q + K$.) Hence κ_1 maps $K_\infty(Q)$ in A_{2Q+K} continuously (and linearly). Since, by (2), for all $f \in A_Q$ and $z \in \mathbb{C}^N$ the function $\tilde{L}(z,\cdot)f$ belongs to $A_{\operatorname{int} Q+K}$, by 2.2 (b) for all $z \in \mathbb{C}^N$

$$\kappa_1(R'(f))(z) = \sum_{(k)\in M} \frac{L_{(k)}(z)\tilde{L}(z,\lambda_{(k)})}{L_{(k)}(\lambda_{(k)})} f(\lambda_{(k)}) = \tilde{L}(z,z)f(z) = L(z)f(z).$$

From here it follows that $\kappa_1 \circ R'$ is the operator of multiplication by L. By [21, Proposition 2.7] there is a continuous linear left inverse $\kappa_2 : A_{2Q+K} \to A_Q$ for $\kappa_1 \circ R'$. The operator $\kappa := \kappa_1 \circ \kappa_2$ is a continuous linear left inverse for R'.

Now we shall evaluate the abstract condition (III) (b) of Theorem 2.3. The condition (III) (a) was evalueted in [21, Proposition 3.6]. $\rhd$

We recall some definitions from [23] and [25].

Definition 2.4. If $D \subset \mathbb{C}^N$ is bounded, convex and $c > 0$, let $v^0_{H_D,c}$ be the largest plurisubharmonic function on $\mathbb{C}^N$ bounded by H_D and with $v^0_{H_D,c}(z) \leqslant c \log|z| + O(1)$ as $|z| \to 0$. A function $C^0_{H_D} : S \to [0,\infty]$ is defined by

$$\big\{z \in \mathbb{C}^N : v^0_{H_D,c}(z) = H_D(z)\big\} = \big\{\lambda a : a \in S,\ 1/C^0_{H_D}(a) \leqslant \lambda < \infty\big\}.$$

If $0 \in \operatorname{int} D$ and if $C > 0$, let $v^\infty_{H_D,C}$ be the largest plurisubharmonic function on $\mathbb{C}^N$ bounded by H_D and with $v^\infty_{H_D,C}(z) \leqslant C \log|z| + O(1)$ as $|z| \to \infty$. A function $C^\infty_{H_0} : S \to [0,\infty]$ is defined by

$$\big\{z \in \mathbb{C}^N : v^\infty_{H_D,C}(z) = H_D(z)\big\} = \big\{\lambda a : a \in S,\ 0 \leqslant \lambda \leqslant 1/C^\infty_{H_D}(a)\big\}.$$

Instead $C^0_{H_D}$ and $C^\infty_{H_D}$ we shall write briefly C^0_D resp. C^∞_D.

Proposition 2.5. *Let Q be strictly convex at the $\partial_r\omega$ and suppose that $0 \in \operatorname{int} K$. For $N > 1$ assume that K is smooth in the directions of $\partial_r S_\omega$. The following are equivalent:*

(i) *There are plurisubharmonic functions v_t $(t \in S)$ on $\mathbb{C}^N$ with $v_t(t) \geqslant 0$ such that: $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ with*

$$v_t \leqslant H_K + H_Q - H_n - |\cdot|/s' - H_K(t) - H_Q(t) + H_{n'}(t) + 1/s \quad (\forall\, t \in S).$$

(ii) *$1/C^0_K$ is bounded on some neighborhood of S_0 and C^∞_K is bounded on each compact subset of S_ω.*

$\lhd$ (i) $\Rightarrow$ (ii). Choose n' according to (i) for $n=1$. On S_o we have $H_{n'} < H_Q$. Thus there are a neigborhood $\tilde{S}$ of S_o and some $\varepsilon > 0$ with $H_{n'} + \varepsilon \leqslant H_Q$ on $\tilde{S}$. We put

$$v := \big(\sup_{t\in\tilde{S}}(v_t + H_K(t))\big)^*.$$

This function is plurisubharmonic on $\mathbb{C}^N$ with $v \geqslant H_K$ on $\tilde{S}$ and satisfies: $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ such that

$$v \leqslant H_K + |\cdot|/n + \max_{t\in\tilde{S}}\{-H_Q(t) + H_{n'}\} + 1/s.$$

Since $H_{n'} \leqslant H_Q$, this gives $v \leqslant H_K$ on $\mathbb{C}^N$. The bounds for $n=1$ give $v(0) \leqslant -\varepsilon$.

From [25, 2.14] it follows that $1/C^0_K$ is bounded on $\tilde{S}$.

Let $\kappa \subset S_\omega$. We define

$$v := \big(\sup_{t\in\kappa}(v_t + H_K(t))\big)^*.$$

This function is plurisubharmonic on $\mathbb{C}^N$ with $v \geqslant H_K$ on κ and satisfies: $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ such that $v \leqslant H_K + H_Q - H_n - |\cdot|/s' + 1/s \leqslant H_K + H_Q - H_n + 1/s$. This shows that $v \leqslant H_K$.

Now choose n with $\kappa \subset S_{\omega_n}$, i. e. with $H_Q = H_n$ on $\hat{\kappa}$. Choose $n' \geqslant n$ according to (i). Choose s' for $s=1$. Then there is a neighborhood $\tilde{\kappa}$ of κ in S such that

$$H_Q - H_n - |\cdot|/s' \leqslant -|\cdot|/(2s') \quad \text{on} \quad \Gamma(\tilde{\kappa})$$

and thus

$$v \leqslant H_K - |\cdot|/(2s') + 1 \quad \text{on} \quad \Gamma(\tilde{\kappa}).$$

In order to reach our claim that C^∞_K is bounded on κ, we need an estimate like the previous one on all $\mathbb{C}^N$ (not only on the particular cone). For this purpose we are going to modify v. First note that, if $N=1$, it follows from what we have already proved that ∂K has to be of class C^1 (see [20, 2.10, 2.14]). For $N>1$ we use our special hypothesis. For this reason we may assume that we have constructed v for the set $\hat{\kappa}$ instead of κ.

Define

$$L(z) := \sup_{w\in F_\kappa} \operatorname{Re}\langle w, z\rangle, \quad z \in \mathbb{C}^N.$$

The positively homogeneous function L satisfies $L \leqslant H_K$ on $\mathbb{C}^N$, and $L = H_K$ on κ. If $L(a) = H_K(a)$, there is $w \in F_\kappa$ with $\operatorname{Re}\langle w, a\rangle = H_K(a)$, hence $a \in S_{F_\kappa}$. Thus $L < H$ on S outside the compact set $\hat{\kappa}$. We replace v by $\tilde{v} := v/2 + L/2$ and obtain $\tilde{v} \leqslant H_K$ on $\mathbb{C}^N$, $\tilde{v} = H_K$ on κ and $\tilde{v} < H_K$ outside a neighborhood of the origin. By [23, 2.1] this shows that C^∞_K is bounded on κ.

(ii) $\Rightarrow$ (i). By the hypothesis, $1/C^0_K$ is bounded on some neighborhood $\tilde{S}$ of S_0. Hence there is $c > 0$ such that the plurisubharmonic function $v^0_{H_K,c}$ equals H_K on $\tilde{S}$. Let $n \in \mathbb{N}$. Since $H_n < H_Q$ on S_0, there is a compact neighborhood S_n of S_0 with $H_n < H_Q$ on S_n. We may assume $S_n \subset S_{n-1} \subset \ldots \subset S_1 \subset \tilde{S}$. Since C^∞_K is bounded on $S\backslash S_n$, there is $C_n > 0$ with $v^\infty_n := v^\infty_{H_K,C_n} = H_K$ on $S\backslash S_{n+2}$.

Again for $N=1$ it follows from (ii) that ∂K is of class C^1. For $N>1$ we apply the extra hypothesis to obtain (as in the first part of the proof) a positively homogeneous function L_n bounded by H on $\mathbb{C}^N$, which equals H on $\kappa = S_{n+1}$, and such that $L_n < H$ outside the compact set $\hat{S}_{n+1} \subset S_n$ (see Remark 1.11 (a)). Then the plurisubharmonic function $v^0_n := v^0_{H_K,c}/2 + L_n/2$ satisfy $v_n \leqslant H_K$ on $\mathbb{C}^N$, $v_n = H_K$ on S_{n+1}, $v_n \leqslant (H_K + L_n)/2 < H_K$ on $S\backslash S_n$.

Fix $n \in \mathbb{N}$. Since $v_n^0 \leqslant (H_K + L_n)/2 < H_K + H_Q - H_n$ on S, and since $v_n^0(0) < 0$, there is $\tilde{n}$ with

$$v_n^0 \leqslant H_K + H_Q - H_n - D/2 - 1/\tilde{n} \ \text{ on } \ \mathbb{C}^N,$$

where

$$D := H_K + H_Q - H_n - (H_K + L_n)/2 = (H_K - L_n)/2 + H_Q - H_n.$$

Choose n' with $H_Q - H_{n'} \leqslant 1/\tilde{n}$ on S_{n+1}. Then for each s there is s' with $D/2 \geqslant |\cdot|/s'$ on $\mathbb{C}^N$ such that

$$v_n^0 \leqslant H_K + H_Q - H_n - |\cdot|/s' - H_Q(t) + H_{n'}(t) + 1/s \quad (\forall\, t \in S_{n+1}).$$

For the functions v_n^∞ we get: Choose n' (in addition) so large that $H_Q = H_{n'}$ on $S \backslash S_{n+2}$. For each s we choose s' (in addition) so large that $v_n^\infty \leqslant H_K - |\cdot|/s' + 1/s$ (see Definition 2.4). This gives

$$v_n^\infty \leqslant H_K + H_Q - H_n - |\cdot|/s' - H_Q(t) + H_{n'}(t) + 1/s \quad (\forall\, t \in S \backslash S_{n+2}).$$

Note that $v_1^0 \geqslant \ldots \geqslant v_n^0 \geqslant v_{n+1}^0$ and that $v_1^\infty \leqslant \ldots \leqslant v_n^\infty \leqslant v_{n+1}^\infty$. That is why for each $l \in \mathbb{N}$ the following holds: $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ with

$$v_l^0 \leqslant H_K + H_Q - H_n - |\cdot|/s' - H_Q(t) + H_{n'}(t) + 1/s \quad (\forall\, t \in S_{n+1}),$$

and $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ with

$$v_l^\infty \leqslant H_K + H_Q - H_n - |\cdot|/s' - H_Q(t) + H_{n'}(t) + 1/s \quad (\forall\, t \in S \backslash S_{n+2})$$

By the construction, $\lim_{l\to\infty} v_l^0 =: v_\infty^0$ exists and defines a plurisubharmonic function with $v_\infty^0 = H_K$ on S_0.

For $t \in S \backslash S_2$ define $\tilde{v}_t := v_1^\infty$. For $t \in S_{l+1} \backslash S_{l+2}$ we put $\tilde{v}_t := v_l^0/2 + v_l^\infty/2$. For $t \in S_0$ we define $\tilde{v}_t := v_\infty^0$. Obviously $\tilde{v}_t(t) = H_K(t)$ for all $t \in S$.

Let $t \in S_{l+1} \backslash S_{l+2}$. For $n \leqslant l$ and n', s and s' as above we get

$$\tilde{v}_t \leqslant \big(H_K + H_Q - H_n - |\cdot|/s' - H_Q(t) + H_{n'}(t) + 1/s\big)/2 + H_K/2.$$

By the strict convexity of Q at $\partial_r \omega$ (see [21], the proof of Proposition 3.6), there is n'' such that $(H_Q + H_{n'})/2 \leqslant H_{n''}$ and thus $(H_Q - H_{n'})/2 \geqslant H_Q - H_{n''}$. This gives

$$\tilde{v}_t \leqslant H_K + H_Q - H_n - |\cdot|/(2s') - H_Q(t) + H_{n''}(t) + 1/(2s).$$

For $n \geqslant l$ and n', s and s' as above we get

$$\tilde{v}_t \leqslant H_K/2 + \big(H_K + H_Q - H_n - |\cdot|/s' - H_Q(t) + H_{n'}(t) + 1/s\big)/2.$$

As above we get the desired estimate.

For $t \in S_0 = \bigcap_{l\in\mathbb{N}} S_l$, we see as in the first part of the previous arguing that $\tilde{v}_t = v_\infty^0$ satisfies these estimates for all n $(\leqslant l = \infty)$.

For $t \in S \backslash S_2$, as in the second part of the arguing just done, we see that these estimates hold for all n $(\geqslant l = 1)$.

Finally we put $v_t := \tilde{v}_t - H_K(t)$, $t \in S$ and are done. $\triangleright$

Remark 2.6. Let Q be strictly convex at the $\partial_r \omega$. By [21, Proposition 3.6] the following are equivalent:

(i) There are plurisubharmonic functions u_t $(t \in S)$ on $\mathbb{C}^N$ with $u_t(t) \geqslant 0$ such that: $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ with

$$u_t(z) \leqslant H_{n'}(z) - H_n(t) + |z|/s - 1/s' \quad (\forall\, z \in \mathbb{C}^N,\ t \in S).$$

(ii) C_Q^∞ is bounded on some neighborhood of S_0 and $1/C_Q^0$ is bounded on each compact subset of S_ω.

Theorem 2.7. *Let Q be strictly convex at the $\partial_r\omega$ and suppose that $0 \in \operatorname{int} K$ and L is a function as in 2.2 (a). For $N > 1$ assume that K is smooth in the directions of ∂S_ω. If C_Q^∞ and $1/C_K^0$ are bounded on some neighborhood of S_0, $1/C_Q^0$ and C_K^∞ are bounded on each compact subset of S_ω then the representation operator $R : \Lambda_1(Q) \to A(Q)$ has a continuous linear right inverse.*

$\lhd$ The assertion hold by Theorem 2.3, Proposition 2.5 and Remark 2.6. $\rhd$

The equivalent conditions of Theorem 2.7 are fulfilled if ∂Q and ∂K are of Hölder class $C^{1,\lambda}$ for some $\lambda > 0$. They are not fulfilled if Q or K is a polyedra, and for $N = 1$ if ∂Q or ∂K has a corner [24].

3. The case of one complex variable

In this section we consider the case $N = 1$ for which the results of the previous sections can be improved.

Convention 3.1. Further L is an entire function on $\mathbb{C}$ satisfying following conditions:

(i) The zero set of L is a sequence of pairwise distinct simple zeros $\lambda_k, k \in \mathbb{N}$, such that $|\lambda_k| \leqslant |\lambda_{k+1}|$ for each $k \in \mathbb{N}$.

(ii) L is a function of completely regular growth with indicator $H_Q + H_K$.

(iii) The asymptotic equality holds:

$$|L'(\lambda_k)| = \exp(H_Q(\lambda_k) + H_K(\lambda_k) + \bar{o}(|\lambda_k|)) \text{ as } k \to \infty.$$

Such function L exists (see for example [10]).

Leont'ev (see [10]) introduced an interpolating function, which is defined with the help of an entire function of one complex variable. Leont'ev's interpolating function is a functional from $A(\operatorname{cl} Q + K)' \backslash A(Q)'$ for every K (if $Q \neq \operatorname{cl} Q$). With the help of an entire function of two complex variables we give the analogous definition of an interpolating functional from $A(Q)'$.

Definition 3.2. Let $\tilde{L}$ be an entire function on $\mathbb{C}^2$ such that $\tilde{L}(\cdot, \mu) \in A_Q$ for each $\mu \in \mathbb{C}$. Q-interpolating functional we shall call a functional

$$\Omega_{\tilde{L}}(z, \mu, f) := \mathscr{F}^{-1}\big(\tilde{L}(\cdot, \mu)\big)_t \bigg(\int_0^t f(t - \xi) \exp(z\xi)\, d\xi \bigg), \quad z, \mu \in \mathbb{C},\ f \in A(Q),$$

where the integral is taken along the interval $[0, t]$.

We show certain properties of $\Omega_{\tilde{L}}$.

Lemma 3.3. (a) *$\Omega_{\tilde{L}}(\cdot, \mu, f) \in A_Q$ for all $\mu \in \mathbb{C}$ and $f \in A(Q)$.*

(b) *For all $z, \mu \in \mathbb{C}$ the equality $\Omega_{\tilde{L}}(z, z, e_\mu) = \tilde{l}(\mu, z)$ holds where a function $\tilde{l} \in A(\mathbb{C}^2)$ is such that $\tilde{L}(\mu, z) - \tilde{L}(z, z) = \tilde{l}(\mu, z)(\mu - z)$.*

(c) *$\Omega_{\tilde{L}}(\mu, z, \cdot) \in A(Q)'$ for all $z, \mu \in \mathbb{C}$.*

$\lhd$ (a): We fix $\mu \in \mathbb{C}$, $f \in A(Q)$ and a domain G with $Q \subset G$ and $f \in A(G)$. We choose a contour C in G which contains in its interior the conjugate diagram of $\tilde{L}(\cdot, \mu)$. If $\gamma(\cdot, \mu)$ is Borel conjugate of $\tilde{L}(\cdot, \mu)$, we have:

$$\Omega_{\tilde{L}}(z, \mu, f) = \frac{1}{2\pi i} \int\limits_C \gamma(t, \mu) \left(\int\limits_0^t f(t - \xi) \exp(z\xi) d\xi \right) dt, \quad z \in \mathbb{C}.$$

Since the function $(t, \mu) \mapsto \gamma(t, \mu) \left(\int_0^t f(t - \xi) \exp(z\xi)\, d\xi \right)$ is continuous by $t \in C$ and entire by z, the function $\Omega_{\tilde{L}}(z, \mu, f)$ is entire (with respect to z). From direct upper bounds for $|\Omega_{\tilde{L}}(z, \mu, f)|$ it follows that $\Omega_{\tilde{L}}(\cdot, \mu, f) \in A_Q$.

(b): Obvious.

(c): Since the map $f \mapsto \int_0^t f(t - \xi) \exp(z\xi)\, d\xi$, $t \in Q$, is continuous and linear in $A(Q)$ and $\mathscr{F}^{-1}(Q(\cdot, \mu))$ is a continuous and linear on $A(Q)$, the functional $\Omega_Q(z, \mu, \cdot)$ is continuous and linear on $A(Q)$, too. $\rhd$

Lemma 3.4. *We assume that a function $\tilde{L}$, as in 3.2, satisfies in addition the following conditions: $\tilde{L}(z, z) = L(z)$ for each $z \in \mathbb{C}$ and $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ $(\exists\, C)$ with*

$$|\tilde{L}(z, \mu)| \leqslant C \exp \left(H_{n'}(z) + H_K(\mu) + H_Q(\mu) - H_n(\mu) + |z|/s - |\mu|/s' \right) \quad (\forall\, z, \mu \in \mathbb{C}).$$

Then $\Pi(f) := \left(\Omega_{\tilde{L}}(\lambda_k, \lambda_k, f)/L'(\lambda_k) \right)_{k \in \mathbb{N}}$, $f \in A(Q)$, is continuous linear operator from $A(Q)$ into $\Lambda_1(Q)$.

$\lhd$ We define $\tilde{L}_k(z) := \tilde{L}(z, \lambda_k)/(L'(\lambda_k)(z - \lambda_k))$, $k \in \mathbb{N}$. By using upper bounds for $|\tilde{L}|$, 3.1 (iii) and 3.3 (b), we obtain, that $\tilde{L}_k$ is entire function on $\mathbb{C}$ and $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ $(\exists\, C_1, C_2)$ such that for all $z \notin U(\lambda_k, (1 + |\lambda_k|)^{-2})$

$$|\tilde{L}_k(z)| \leqslant C_1 \exp \big(H_{n'}(z) + H_K(\lambda_k) + H_Q(\lambda_k) - H_n(\lambda_k) + |z|/s - |\lambda_k|/(s' - 1)$$
$$+ 2\log(1 + |\lambda_k|) - \log|L'(\lambda_k)| \big) \leqslant C_2 \exp \left(H_{n'}(z) - H_n(\lambda_k) + |z|/s - |\lambda_k|/s' \right) \quad (\forall\, k \in \mathbb{N}).$$

Applying the maximum principle we get that $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ $(\exists\, C_3)$ with

$$|\tilde{L}_k(z)| \leqslant C_3 \exp \left(H_{n'}(z) - H_n(\lambda_k) + |z|/s - |\lambda_k|/s' \right) \quad (\forall\, z \in \mathbb{C},\ k \in \mathbb{N}).$$

From this it follows that the series $\sum_{k \in \mathbb{N}} c_k \tilde{L}_k$ converges absolutely in A_Q for each $c = (c_k)_{k \in \mathbb{N}} \in K_\infty(Q)$ and $\kappa : c \mapsto \sum_{k \in \mathbb{N}} c_k \tilde{L}_k$ is continuous linear operator from $K_\infty(Q)$ into A_Q. We shall find its adjoint operator $\kappa' : A(Q) \to \Lambda_1(Q)$:

$$\langle c, \kappa'(e_\mu) \rangle = \langle \kappa(c), f \rangle = \left\langle \sum_{k \in \mathbb{N}} c_k \tilde{L}_k, e_\mu \right\rangle$$
$$= \sum_{k \in \mathbb{N}} c_k \tilde{L}_k(\mu) = \sum_{k \in \mathbb{N}} c_k \Omega_{\tilde{L}}(\lambda_k, \lambda_k, e_\mu)/L'(\lambda_k) \quad (\forall\, \mu \in \mathbb{C},\ c \in \Lambda_1(Q)).$$

Hence $\kappa'(e_\mu) = \left(\Omega_{\tilde{L}}(\lambda_k, \lambda_k, e_\mu)/L'(\lambda_k) \right)_{k \in \mathbb{N}}$, $\mu \in \mathbb{C}$. Let $\mathbb{C}^{\mathbb{N}}$ be a space of all number sequence with its natural topologie. The maps $\kappa' : A(Q) \to \mathbb{C}^{\mathbb{N}}$ and $\Pi : A(Q) \to \mathbb{C}^{\mathbb{N}}$ are continuous and linear. Since the set $\{e_\mu : \mu \in \mathbb{C}\}$ is total in $A(Q)$, we have $\Pi = \kappa'$ on $A(Q)$ and Π is continuous and linear from $A(Q)$ into $\Lambda_1(Q)$. $\rhd$

Theorem 3.5. (I) *Let* $0 \in \mathrm{int}_r K$. *The following assertions are equivalent:*

(i) *The representation operator* $R : \Lambda_1(Q) \to A(Q)$ *has a continuous linear right inverse.*

(ii) *There is an entire function* $\tilde{L}$ *on* $\mathbb{C}^2$ *such that* $\tilde{L}(z,z) = L(z)$ *and* $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ $(\exists\, C)$ *with*

$$|\tilde{L}(z,\mu)| \leqslant C \exp\big(H_{n'}(z) + H_K(\mu) + H_Q(\mu) - H_n(\mu) + |z|/s - |\mu|/s'\big) \quad (\forall\, z, \mu \in \mathbb{C}).$$

(iii) Q *is strictly convex at* $\partial_r\omega$, *the interior of* K *is not empty,* C_Q^∞ *and* $1/C_K^0$ *are bounded on some neighborhood of* S_0, $1/C_Q^0$ *and* C_K^∞ *are bounded on each compact subset of* S_ω.

(II) (iv) *If* $\tilde{L}$ *is a function as in* (*ii*), *the operator*

$$\Pi(f) \mapsto (\Omega_{\tilde{L}}(\lambda_k, \lambda_k, f)/L'(\lambda_k))_{k\in\mathbb{N}}, \quad f \in A(Q),$$

is a continuous linear right inverse for R.

(v) *If* $\Pi : A(Q) \to \Lambda_1(Q)$ *is a continuous linear right inverse for* R, *then there is a unique function* $\tilde{L}$ *as in* (*ii*) *such that* $\Pi(f) = (\Omega_{\tilde{L}}(\lambda_k, \lambda_k, f)/L'(\lambda_k))_{k\in\mathbb{N}}$, $f \in A(Q)$.

$\lhd$ (iv) (and (ii) $\Rightarrow$ (i)): Let $\tilde{L}$ be a function as in (ii). Then

$$\kappa : c \mapsto \sum_{(k)\in\mathbb{N}} c_k \frac{\tilde{L}(\cdot, \lambda_k)}{L'(\lambda_k)(\cdot - \lambda_k)}$$

maps continuously (and linearly) $K_\infty(Q)$ into A_Q. Since for each $f \in A_Q$ the function $f\tilde{L}(z,\cdot)$ belongs to $A_{\mathrm{int}Q+K}$, taking into account the Lagrange interpolation formula (1), we obtain:

$$\begin{aligned} L(z)\sum_{k\in\mathbb{N}} f(\lambda_k)\frac{\tilde{L}(z,\lambda_k)}{L'(\lambda_k)(z-\lambda_k)} &= \sum_{k\in\mathbb{N}} f(\lambda_k)\tilde{L}(z,\lambda_k)\frac{L(z)}{L'(\lambda_k)(z-\lambda_k)} \\ &= L(z,z)f(z) = L(z)f(z) \quad (\forall\, z \in \mathbb{C},\ f \in A_Q). \end{aligned}$$

This implies that $\kappa = \Pi'$ is a left inverse for R'. By the proof of Lemma 3.4 κ is the adjoint to Π for each function $\tilde{L}$ as in (ii). Hence Π is a right inverse for R.

(*i*) $\Rightarrow$ (*ii*): Let Π be a continuous linear right inverse for R. Then $\kappa := \Pi' : K_\infty(A) \to A_Q$ is a left inverse for R'. We put $f_k := \kappa(e_{(k)})$, where $e_{(k)} := (\delta_{k,n})_{n\in\mathbb{N}}$, $k \in \mathbb{N}$. By Grothendieck's factorization theorem, for each n there is n' such that κ maps continuously $\mathrm{proj}_{\leftarrow m} K_{n,m}(Q)$ in $\mathrm{proj}_{\leftarrow m} A_{n',m}$. Hence the following holds: $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, r)$ $(\exists\, C)$ with

$$|f_k(z)| \leqslant C \exp\big(H_{n'}(z) - H_n(\lambda_{(k)}) + |z|/s - |\lambda_{(k)}|/r\big) \quad (\forall\, z \in \mathbb{C},\ k \in \mathbb{N}). \tag{4}$$

For $f \in A_Q$ let

$$T_z(f)(\mu) := \sum_{k\in\mathbb{N}} \frac{L(\mu)}{\mu - \lambda_k}(z - \lambda_k) f_k(z) f(\lambda_k), \quad \mu \in \mathbb{C}.$$

By 2.2 (b) (ii) and (4) the series converges absolutely in A_Q and converges uniformly (by μ) on compact sets of $\mathbb{C}$. Fix $z \in \mathbb{C}$. Then $T_z(\mu f)(\mu)) = \mu T_z(f)(\mu)$ for all $f \in A_Q$ and $\mu \in \mathbb{C}$. By [12, Lemma 1.7] there is a function $a_z \in A(\mathbb{C})$ such that $T_z(f)(\mu) = a_z(\mu)f(\mu)$ for all $\mu \in \mathbb{C}$, $f \in A_Q$. The function $\tilde{L}(z,\mu) := a_z(\mu)$, $z, \mu \in \mathbb{C}$, satisfies the conditions in (ii) (see the proof of (i) $\Rightarrow$ (ii) in [12, Theorem 1.8] too).

(iii) $\Rightarrow$ (i) holds by Theorem 2.7.

(i) $\Rightarrow$ (iii): Since the operator R has a continuous linear righ inverse, $R : \Lambda_1(Q) \to A(Q)$ is surjectiv. By [13, Theorem 8] the set Q is strictly convex at $\partial_r\omega$.

Since (i) is equivalent to (ii) there is a function $\tilde{L}$ which satisfies the conditions in (*ii*). Let P be the (radial) indicator of $\tilde{L}$, i. e.

$$P(z,\mu) := \left(\limsup_{t\to+\infty} \frac{\log|\tilde{L}(tz,t\mu)|}{t}\right)^*, \quad z,\mu \in \mathbb{C}.$$

Then P is a plurisubharmonic function on $\mathbb{C}^2$ satisfying the conditions in (II) of Theorem 2.3. Hence, by Theorem 2.3, there are subharmonic functions v_t $(t \in S)$ as in (III) (b). We put $g_t(\mu) := |t|v_{t/|t|}(\mu/|t|)$, $\mu, t \in \mathbb{C}$, $t \neq 0$. Then g_t are subharmonic functions on $\mathbb{C}$ such that $g_t(t) \geqslant 0$ and $(\forall\, n)$ $(\exists\, n')$ $(\forall\, s)$ $(\exists\, s')$ with

$$g_t(\mu) \leqslant H_K(\mu) + H_Q(\mu) - H_n(\mu) - H_K(t) - H_Q(t) + H_{n'}(t) - |\mu|/s' + |t|/s$$

for all $\mu, t \in \mathbb{C}$, $t \neq 0$. If $S_\omega = \varnothing$, the set Q is open. Hence the following holds: $(\forall\, n)$ $(\exists\, n')$ with

$$g_t(\mu) \leqslant H_K(\mu) - H_K(t) + |\mu|/s' - |t|/s \quad (\forall\, \mu, t \in \mathbb{C},\ t \neq 0).$$

Then, by [12, Proposition 1.17], an angle with the corner at 0 doesn't exist in which the support function H_K of K is harmonic. Hence $\operatorname{int} K \neq \varnothing$. If $S_\omega \neq \varnothing$, there is an open (with respect to S) subset A of S such that $H_n = H_Q$ on A for large n. Let $\Gamma(A) := \{ra : r > 0\}$. Then for each s there is s' with

$$g_t(\mu) \leqslant H_K(\mu) - H_K(t) + |t|/s - |\mu|/s' \quad (\forall\, \mu, t \in \mathbb{C},\ t \neq 0).$$

As in [12, Proposition 1.17] from the maximum principle for harmonic functions it follows that the interior of K is not empty.

By Theorem 2.3 , Proposition 2.5 and Remark 2.6 C_Q^∞ and $1/C_K^0$ are bounded on some neighborhood of S_0, $1/C_Q^0$ and C_K^∞ are bounded on each compact subset of S_ω.

(v): By the proof of (i) $\Rightarrow$ (ii) there is an entire function $\tilde{L}$ satisfying the conditions in (ii) and such that $\Pi'(e_{(k)}) = \frac{\tilde{L}(\cdot,\lambda_k)}{L'(\lambda_k)(\cdot-\lambda_k)}$ for each $k \in \mathbb{N}$. Hence $\Pi'(c) = \sum_{k\in\mathbb{N}} c_k \frac{\tilde{L}(\cdot,\lambda_k)}{L'(\lambda_k)(\cdot-\lambda_k)}$ for each $c \in K_\infty(Q)$ and $\Pi(f) = (\Omega_{\tilde{L}}(\lambda_k, \lambda_k, f)/L'(\lambda_k))_{k\in N}$ for all $f \in A(Q)$ (see the proof of Lemma 3.4). We shall show uniqueness of such function $\tilde{L}$. Let $\tilde{L}_1$, $\tilde{L}_2$ be two such functions. Then $\tilde{L}_1(z,\lambda_k) = \tilde{L}_2(z,\lambda_k)$ for all $k \in \mathbb{N}$, $z \in \mathbb{C}$. Since $\{\lambda_k : k \in \mathbb{N}\}$ is the uniqueness set for $A_{\operatorname{int}Q+K}$ (see 2.2 (b)) and $\tilde{L}_1(z,\cdot), \tilde{L}_2(z,\cdot) \in A_{\operatorname{int}Q+K}$, we get $\tilde{L}_1(z,\cdot) \equiv \tilde{L}_2(z,\cdot)$ for each $z \in \mathbb{C}$ and, consequently, $\tilde{L}_1 \equiv \tilde{L}_2$ on $\mathbb{C}^2$.

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References

1. *Азарин В. С.* О субгармонических функциях вполне регулярного роста в многомерном промтранстве // Докл. АН СССР.—1961.—Т. 146, № 4.—С. 743–746.
2. *Коробейник Ю. Ф.* Представляющие системы // Успехи мат. наук.—1981.—Т. 36, вып. 1.—С. 73–126.
3. *Коробейник Ю. Ф., Ле Хай Хой.* Представляющие системы экспонент в полицилиндрических областях // Мат. сб.—1983.—Т. 122(164), № 4.—С. 458–474.
4. *Коробейник Ю. Ф., Мелихов С. Н.* Линейный непрерывный правый обратный для оператора представления и конформные отображения // Докл. АН.—1992.—Т. 323, № 5.—С. 826–829.
5. *Коробейник Ю. Ф., Мелихов С. Н.* Линейный непрерывный правый обратный для оператора представления и приложения к операторам свертки // Сиб. мат. журн.—1993.—Т. 34, № 1.—С. 70–84.

6. *Красичков-Терновский И. Ф.* Одна геометрическая лемма, полезная в теории целых функций, и теоремы типа Левинсона // Мат. заметки.—1978.—Т. 24, вып. 4.—С. 531–546.
7. *Кривошеев А. С.* Критерий разрешимости неоднородных уравнений свертки в выпуклых областях пространства $\mathbb{C}^n$ // Изв. АН СССР. Сер. мат.—1990.—Т. 54, № 3.—С. 480–450.
8. *Лелон П., Груман Л.* Целые функции многих комплексных переменных.—М.: Мир, 1989.—350 с.
9. *Леонтьев А. В.* О представлении аналитических функций рядами экспонент в полицилиндрической области // Мат. сб.—1972.—Т. 89, № 4.—С. 364–383.
10. *Леонтьев А. Ф.* Ряды экспонент.—М.: Наука, 1976.—536 с.
11. *Мелихов С. Н.* Продолжение целых функций вполне регулярного роста и правый обратный для оператора представления аналитических функций рядами квазиполиномов // Мат. сб.—2000.—Т. 191, № 7.—С. 105–128.
12. *Мелихов С. Н.* О левом обратном к оператору сужения на весовых пространствах целых функций // Алгебра и анализ.—2002.—Т. 14, вып. 1.—С. 99–133.
13. *Мелихов С. Н., Момм З.* О свойстве внутрь-продолжаемости представляющих систем экспонент на выпуклых локально замкнутых множествах // Владикавк. мат. журн.—2008.—Т. 10, вып. 2.—С. 36–45.
14. *Папуш Д. Е.* О росте целых функций с «плоскими» нулями // Теория функций, функцион. анализ и их прил.—1987.—Вып. 48.—С. 117–125.
15. *Секерин А. Б.* О представлении аналитических функций многих переменных рядами экспонент // Изв. РАН. Сер. мат.—1992.—Т. 56, № 3.—С. 538–565.
16. *Хермандер Л.* Введение в теорию функций нескольких комплексных переменных.—М.: Мир, 1968.—280 с.
17. *Эдвардс Р.* Функциональный анализ. Теория и приложения.—М.: Мир, 1969.—1071 с.
18. *Gruman L.* Entire functions of several variables and their asymptotic growth // Ark. Mat.—1971.—Vol. 9.—P. 141–163.
19. *Matheron G.* Random sets and integral geometry.— New York: Wiley, 1975.—xxiii+261 p.
20. *Melikhov S. N., Momm S.* Solution operators for convolution equations on the germs of analytic functions on compact convex sets on $\mathbb{C}^N$ // Studia Math.—1995.—Vol. 117.—P. 79–99.
21. *Melikhov S. N., Momm S.* Analytic solutions of convolution equations on convex sets with obstacle in the boundary // Math. Scand.—2000.—Vol. 86.—P. 293–319.
22. *Momm S.* Division problem in spaces of entire functions of finite order // Functional Analysis.—1993.—P. 435–457.
23. *Momm S.* The boundary behavior of extremal plurisubharmonic functions // Acta Math.—1994.—Vol. 172.—P. 51–75.
24. *Momm S.* Extremal plurisubharmonic functions for convex bodies in $\mathbb{C}^N$ // Complex analysis, harmonic analysis and appl. / Deville R. (Bordeaux, France, 1995).—Harlow: Longman, Bitman.—1996.—Vol. 347.—P. 87–103.—(Res. Notes Math. Ser.)
25. *Momm S.* An extremal plurisubharmonic functions associated to a convex pluricomplex Green function with pole at infinity // J. Reine Angew. Math.—1996.—Vol. 471.—P.139–163.
26. *Schneider R.* Convex bodies: The Brunn–Minkowski theory.—Cambridge: Cambridge University Press, 1993.—490 p.

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О РАЗЛОЖЕНИИ В РЯДЫ ЭКСПОНЕНТ ФУНКЦИЙ, АНАЛИТИЧЕСКИХ НА ВЫПУКЛЫХ ЛОКАЛЬНО ЗАМКНУТЫХ МНОЖЕСТВАХ

Мелихов С. Н., Момм З.

Пусть Q — ограниченное, выпуклое, локально замкнутое подмножество $\mathbb{C}^N$ с непустой внутренностью. Для $N > 1$ получены достаточные условия того, что оператор представления рядами экспонент функций, аналитических на Q, имеет линейный непрерывный правый обратный. Для $N = 1$ доказаны критерии существования линейного непрерывного правого обратного к оператору представления.

Ключевые слова: локально замкнутое множество, аналитические функции, ряды экспонент, линейный непрерывный правый обратный.