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WEAK INSERTION OF A CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE FUNCTIONS[1]

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We give a sufficient condition for insertion of a continuous function between two comparable real-valued functions in terms of lower cut sets.

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1. Introduction

The concept of a pre-open set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [3]. A subset A of a topological space (X, τ) is called *pre-open* or *locally dense* or *nearly open* if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. A set A is called *pre-closed* if its complement is pre-open or equivalently if $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq A$. The term, pre-open, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [11], while the concept of a locally dense set was introduced by H. H. Corson and E. Michael [3].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [10]. A subset A of a topological space (X, τ) is called *semi-open* [10] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq A$.

Recall that a real-valued function f defined on a topological space X is called A-continuous [12] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subset of X. Most of the definitions of continuous function used throughout this paper are the particular cases of the definition of A-continuity. However, for unknown concepts the reader may refer to [4, 5].

Hence, a real-valued function f defined on a topological space X is called *precontinuous* (resp. *semi-continuous*) if the preimage of every open subset of $\mathbb{R}$ is pre-open (resp. semi-open) subset of X.

Precontinuity was called by V. Pták *nearly continuity* [13]. Nearly continuity or precontinuity is known also as *almost continuity* by T. Husain [6]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [1].

Results of Katětov [7, 8] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a sufficient condition for the insertion of a continuous function between two comparable real-valued functions.

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If g and f are real-valued functions defined on a space X, we write $g \leqslant f$ in case $g(x) \leqslant f(x)$ for all x in X. The following definitions are modifications of conditions considered in [9].

A property P defined relative to a real-valued function on a topological space is a *c-property* provided that any constant function has property P and provided that the sum of a function with property P and any continuous function also has property P. If P_1 and P_2 are c-properties, the following terminology is used: A space X has the *weak c-insertion property* for (P_1, P_2) if, given any functions g and f on X such that $g \leqslant f$, g has property P_1 and f has property P_2, there exists a continuous function h such that $g \leqslant h \leqslant f$.

In this paper, a sufficient condition for the weak c-insertion property is given. Also several insertion theorems are obtained as corollaries of this result.

2. The Main Result

Before giving a sufficient condition for insertability of a continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space. The family of all semi-open, semi-closed, pre-open and pre-closed will be denoted by $s\,O(X, \tau)$, $s\,C(X, \tau)$, $p\,O(X, \tau)$ and $p\,C(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). Respectively, we define the *s-closure, s-interior, p-closure* and *p-interior* of A, denoted by $s\,\mathrm{Cl}(A)$, $s\,\mathrm{Int}(A)$, $p\,\mathrm{Cl}(A)$ and $p\,\mathrm{Int}(A)$ as follows:

$s\,\mathrm{Cl}(A) = \bigcap\{F : F \supseteq A,\ F \in s\,C(X, \tau)\}$,

$s\,\mathrm{Int}(A) = \bigcup\{O : O \subseteq A,\ O \in s\,O(X, \tau)\}$,

$p\,\mathrm{Cl}(A) = \bigcap\{F : F \supseteq A,\ F \in p\,C(X, \tau)\}$,

$p\,\mathrm{Int}(A) = \bigcup\{O : O \subseteq A,\ O \in p\,O(X, \tau)\}$.

Respectively, we have: $s\,\mathrm{Cl}(A)$ and $p\,\mathrm{Cl}(A)$ are semi-closed and pre-closed, $s\,\mathrm{Int}(A)$ and $p\,\mathrm{Int}(A)$ are semi-open and pre-open.

The following two definitions are modifications of conditions considered in [7, 8].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\bar{\rho}y$ if and only if $y\rho v$ implies $x\rho v$ and $u\rho x$ implies $u\rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ if ρ satisfies each of the following conditions:

1) If $A_i \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A\bar{\rho}B$.

3) If $A\rho B$, then $\mathrm{Cl}(A) \subseteq B$ and $A \subseteq \mathrm{Int}(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leqslant \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. *Let g and f be real-valued functions on a topological space X with $g \leqslant f$. Suppose that there exist a strong binary relation ρ on the power set of X and lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that $t_1 < t_2$ implies $A(f, t_1)\rho A(g, t_2)$. Then there exists a continuous function h defined on X such that $g \leqslant h \leqslant f$.*

◁ Let g and f be real-valued functions defined on X such that $g \leqslant f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that $t_1 < t_2$ implies $A(f,t_1)\rho A(g,t_2)$.

Define two set-valued functions F and G from the rationals $\mathbb{Q}$ into the power set of X by $F(t) := A(f,t)$ and $G(t) := A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\bar{\rho}F(t_2)$, $G(t_1)\bar{\rho}G(t_2)$, and $F(t_1)\rho G(t_2)$. By Lemmas 1 and 2 of [8] it follows that there exists a function H from $\mathbb{Q}$ into $P(X)$ such that if t_1 and t_2 are any rationals with $t_1 < t_2$, then $F(t_1)\rho H(t_2)$, $H(t_1)\rho H(t_2)$, and $H(t_1)\rho G(t_2)$.

Given $x \in X$, put $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We now verify that $g \leqslant h \leqslant f$: if x is in $H(t)$ then x is also in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leqslant t'$, it follows that $g(x) \leqslant t$. Hence $g \leqslant h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geqslant t$. Hence $h \leqslant f$.

Moreover, for any rationals t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = \operatorname{Int}(H(t_2)) \setminus \operatorname{Cl}(H(t_1))$. Hence $h^{-1}(t_1,t_2)$ is an open subset of X, i. e. h is a continuous function on X. ▷

The above proof uses the technique similar to the proof of Theorem 1 in [7].

3. Applications

The abbreviations *pc* and *sc* are used for precontinuous and semicontinuous, respectively.

Corollary 3.1. *If for each pair of disjoint preclosed* (*resp. semi-closed*) *sets F_1, F_2 in X, there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$, and $G_1 \cap G_2 = \varnothing$ then X has the weak c-insertion property for* (pc, pc) (*resp.* (sc, sc)).

◁ Let g and f be real-valued functions defined on X such that f and g are *pc* (resp. *sc*) and $g \leqslant f$. If a binary relation ρ is defined by $A\rho B$ if and only if $p\operatorname{Cl}(A) \subseteq p\operatorname{Int}(B)$ (resp. $s\operatorname{Cl}(A) \subseteq s\operatorname{Int}(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any rationals with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leqslant t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2).$$

Since $\{x \in X : f(x) \leqslant t_1\}$ is a pre-closed (resp. semi-closed) set and $\{x \in X : g(x) < t_2\}$ is a pre-open (resp. semi-open) set, it follows that $p\operatorname{Cl}(A(f,t_1)) \subseteq p\operatorname{Int}(A(g,t_2))$ (resp. $s\operatorname{Cl}(A(f,t_1)) \subseteq s\operatorname{Int}(A(g,t_2))$). Hence $t_1 < t_2$ implies that $A(f,t_1)\rho A(g,t_2)$. The proof follows from Theorem 2.1. ▷

Corollary 3.2. *If for each pair of disjoint preclosed* (*resp. semi-closed*) *sets F_1, F_2, there exist open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$, and $G_1 \cap G_2 = \varnothing$ then every precontinuous* (*resp. semi-continuous*) *function is continuous.*

◁ Let f be a real-valued precontinuous (resp. semi-continuous) function defined on the X. Set $g = f$, then by Corollary 3.1, there exists a continuous function h such that $g = h = f$. ▷

Corollary 3.3. *If for each pair of disjoint subsets F_1, F_2 of X, such that F_1 is pre-closed and F_2 is semi-closed, there exist open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X have the weak c-insertion property for* (pc, sc) *and* (sc, pc).

◁ Let g and f be real-valued functions defined on X with $g \leqslant f$ such that g is *pc* (resp. *sc*) and f is *sc* (resp. *pc*). If a binary relation ρ is defined by $A\rho B$ if and only if $s\operatorname{Cl}(A) \subseteq p\operatorname{Int}(B)$ (resp. $p\operatorname{Cl}(A) \subseteq s\operatorname{Int}(B)$), then by hypothesis ρ is a strong binary relation in $P(X)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leqslant t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leqslant t_1\}$ is a semi-closed (resp. pre-closed) set and $\{x \in X : g(x) < t_2\}$ is a pre-open (resp. semi-open) set, it follows that $s\,\mathrm{Cl}(A(f,t_1)) \subseteq p\,\mathrm{Int}(A(g,t_2))$ (resp. $p\,\mathrm{Cl}(A(f,t_1)) \subseteq s\,\mathrm{Int}(A(g,t_2))$). Hence $t_1 < t_2$ implies that $A(f,t_1)\rho A(g,t_2)$. The proof follows from Theorem 2.1. $\triangleright$

References

1. *Blumberg H.* New properties of all functions // Trans. Amer. Math. Soc.—1922.—Vol. 24.—P. 113–128.
2. *Brooks F.* Indefinite cut sets for real functions // Amer. Math. Monthly.—1971.—Vol. 78.—P. 1007–1010.
3. *Corson H. H., Michael E.* Metrizability of certain countable unions // Illinois J. Math.—1964.—Vol. 8.—P. 351–360.
4. *Dontchev J.* The characterization of some peculiar topological space via α- and β-sets // Acta Math. Hungar.—1995.—Vol. 69, № 1–2.—P. 67–71.
5. *Ganster M., Reilly I.* A decomposition of continuity // Acta Math. Hungar.—1990.—Vol. 56, № 3–4.—P. 299–301.
6. *Husain T.* Almost continuous mappings // Prace Mat.—1966.—Vol. 10.—P. 1–7.
7. *Katětov M.* On real-valued functions in topological spaces // Fund. Math.—1951.—Vol. 38.—P. 85–91.
8. *Katětov M.* On real-valued functions in topological spaces // Fund. Math.—1953.—Vol. 40.—P. 203–205.—(Correction).
9. *Lane E.* Insertion of a continuous function // Pacific J. Math.—1976.—Vol. 66.—P. 181–190.
10. *Levine N.* Semi-open sets and semi-continuity in topological space // Amer. Math. Monthly.—1963.—Vol. 70.—P. 36–41.
11. *Mashhour A. S., Abd El-Monsef M. E., El-Deeb S. N.* On pre-continuous and weak pre-continuous mappings // Proc. Math. Phys. Soc. Egypt.—1982.—Vol. 53.—P. 47–53.
12. *Przemski M.* A decomposition of continuity and α-continuity // Acta Math. Hungar.—1993.—Vol. 61, № 1–2.—P. 93–98.
13. *Pták V.* Completeness and open mapping theorem // Bull. Soc. Math. France.—1958.—Vol. 86.—P. 41–74.

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О СУЩЕСТВОВАНИИ НЕПРЕРЫВНОЙ ФУНКЦИИ МЕЖДУ ДВУМЯ СРАВНИМЫМИ ВЕЩЕСТВЕННЫМИ ФУНКЦИЯМИ

Мирмиран М.

В терминах лебеговых множеств функции найдены условия, при которых между двумя сравнимыми вещественнозначными функциями можно вставить непрерывную функцию.

Ключевые слова: слабая вставляемость, сильное бинарное отношение, предоткрытое множество, полуоткрытое множество, лебегово множество.