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BANACH LATTICES WITH TOPOLOGICALLY FULL CENTRE

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After some general background discussion on the notion of a *topologically full* centre in a Banach lattice, we study two problems in which it has featured. In 1988 Orhon showed that if the centre is topologically full then it is also a maximal abelian algebra of bounded operators and asked if the converse is true. We give a short proof of his result and a counterexample to the converse. After noting that every non scalar central operator has a hyperinvariant band, we show that any hyperinvariant subspace must be an order ideal, provided the centre is topologically full and conclude with a counterexample to this in a general vector lattice setting.

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1. Definitions and history

The *centre* $Z(E)$ of a vector lattice E consists of those linear operators on E which lie between two multiples of the identity. I. e. there is $\lambda \in \mathbb{R}_+$ such that $-\lambda x \leqslant Tx \leqslant \lambda x$ for all $x \in E_+$. Provided that E is Archimedean then the centre is a commutative algebra and lattice with an order unit which is also an algebra unit (the identity operator) which may identified with a dense sublattice and subalgebra of $C(K)$ for some compact Hausdorff space K. If E is represented as a lattice of continuous extended real valued functions on some topological space S then the centre may be identified with the space of bounded continuous real valued functions on S which map (the representation of) E into itself by pointwise multiplication. For example, if μ is a σ-finite measure then the centre of $L^p(\mu)$ may be identified with $L^\infty(\mu)$, where the identification $Z\big(L^p(\mu)\big) \ni T \leftrightarrow \phi \in L^\infty(\mu)$ is described by $Tf(x) = \phi(x)f(x)$ μ-almost everywhere. If Σ is a locally compact Hausdorff space then the centre of $C_0(\Sigma)$ may similarly be identified with $C^b(\Sigma)$, all continuous bounded functions on Σ. This functional description of the centre makes many simple computations involving the centre easy to comprehend.

Band projections are certainly central operators. In fact they are precisely the idempotents in $Z(E)$ and are also precisely the components of I in $Z(E)$. If E has the principal projection property then E contains many projection bands so the centre will certainly be large. In fact if E is Dedekind σ-complete and $x, y \in E$ with $0 \leqslant x \leqslant y$ then there is $T \in Z(E)$ with $Ty = x$ (see [11, Chapter 3, Theorem 7.6] and [22, Example 1.4 (b)] for two very different proofs). Representing E as an ideal in a space $C^\infty(\Sigma)$, for suitable Σ, as may always be done, provides a simple proof. Hart, in [7], terms this property of $Z(E)$, *transitivity.* At the other extreme there are examples of vector lattices for which the centre consists only of multiples of the identity, which must always lie in $Z(E)$. The simplest example of such is due to Zaanen [25] and comprises the continuous real-valued functions on $[0, 1]$ which are piecewise linear, i.e.

there is a finite dissection (depending on the choice of function) of $[0,1]$ into subintervals on each of which the function is linear. Rather less well understood, but undoubtedly more important, is the example due to Goullet de Rugy [6, Theorem 2.28 and Proposition 3.42] of an AM-space with trivial centre. See [23] for a slightly simpler exposition.

We call the centre $Z(E)$ of a Banach lattice E *topologically full* [22, Definition 1.3] if whenever $0 \leqslant x \leqslant y$, $x, y \in E$, there is a sequence (T_n) in $Z(E)$ such that $T_n y \to x$. This notion can, and has, been generalized to other topologies including the weak topology induced on a vector lattice by its order dual. My account will be restricted to Banach lattices. It is often useful to impose a restriction on central operators that we use in this approximation. This is simple given the very special nature of central operators. If $0 \leqslant x \leqslant y$ and $T_n y \to x$ then $(T_n^+ \wedge I)y = (T_n y)^+ \wedge y \to x \wedge y = x$, so we may assume that $0 \leqslant T_n \leqslant I$ for all $n \in \mathbb{N}$. I will use the notation $Z(E)_{1+}$ for the set $\{T \in Z(E) : 0 \leqslant T \leqslant I\}$.

Clearly Banach lattices which are Dedekind σ-complete have a topologically full centre, but there are plenty of other examples. A positive element u in a Banach lattice is a *topological order unit* (= *quasi-interior point* of the positive cone) if the closed ideal generated by u is the whole space. A Banach lattice with a topological order unit u may be represented as an order ideal in $C^\infty(K)$, the continuous extended real-valued functions on some compact Hausdorff space K which are finite on a dense subset of K, with u corresponding to the constantly one function on K, $\mathbf{1}_K$ [1]. The centre may be identified with $C(K)$ and is topologically full. A detailed proof of this does not seem to be easily accessible, the only one that the author knows of being in Example 1 of [18] and even that is unnecessarily complicated. For completeness, we include here a proof based on that in [18].

Proposition 1.1. *A Banach lattice with a topological order unit has a topologically full centre.*

$\lhd$ Suppose that $x, y \in E$ with $0 \leqslant x \leqslant y$. If u is a topological order unit for E then so is $u \vee y$. Represent E as an ideal in $C^\infty(K)$ with $u \vee y$ corresponding to $\mathbf{1}_K$. from now on, we work with the image of E in $C^\infty(K)$. As $0 \leqslant x \leqslant y \leqslant \mathbf{1}_K$, both x and $y \in C(K)$ and so, for all $n \in \mathbb{N}$, is $x/(y + \frac{1}{n}\mathbf{1}_K)$ which we may thus regard as an element $T_n \in Z(E)$. Considering values pointwise, we see that

$$0 \leqslant T_n y - x = \left(\frac{x}{y + \frac{1}{n}\mathbf{1}_K}\right) y - x = \frac{1}{n}\left(\frac{x}{y + \frac{1}{n}\mathbf{1}_K}\right) \leqslant \frac{1}{n}\mathbf{1}_K$$

so that $\|T_n y - x\| \leqslant \frac{1}{n}\|\mathbf{1}_K\|$. $\rhd$

Piecing together such spaces, it can be seen that Banach lattices with a *topological order system* also have a topologically full centre. It is also easily seen that the centre of $C_0(\Sigma)$ is always topologically full. If Σ is not σ-compact then $C_0(\Sigma)$ does not have a topological order unit and if it is non-σ-compact and connected then it certainly can not have a topological

[1] The representation theorem for Banach lattices with a topological order unit was originally proved independently by Davies in [5] and Lotz in [10] and most references to it quote Theorem III.4.5 of [19]. However, that version omits one important feature of Davies' version, namely that the image of the representation is an ideal in $C^\infty(K)$. This observation makes the identification of the centre with $C(K)$ trivial, as compared with Nagel's proof in [15]. Part of the unnecessary complications in the proof that the centre of a Banach lattice with topological order unit has topologically full centre in [18] is because it virtually repeats that part of the proof of the representation theorem which shows that the image is an ideal. On the other hand the proof of Theorem 2.6 in [7], stating that the centre of a Banach lattice with a topological order unit is algebraically rich, seems to implicitly assume the ideal property of the image even though Schaefer's version of the representation is quoted.

orthogonal system. On the other hand, Goullet de Rugy's example certainly does not have a topologically full centre.

Let me remind the reader that a linear operator on a vector lattice is *band preserving* if $x \perp y \Rightarrow Tx \perp y$. Central operators are always band preserving but the converse is false in general. However on Banach lattices Abramovic, Veksler and Koldunov [2] showed that the two classes of operator coincide. See [14, Theorem 3.1.12] for an alternative proof based on work of de Pagter.

Although I introduced the term «topologically full» in 1981, the concept predates that by some considerable time, originating in a paper by Solomon Leader in 1959. In slightly more generality, he introduced the *separation property* for $Z(E)$ which says that if $x, y \in E_+$ with $y \perp x$ then there is a sequence $T_n \in Z(E)$ such that $T_n y \to y$ and $T_n x \to 0$. His *Fundamental Lemma* in § 4 of [9] shows that if $Z(E)$ has the separation property then it is topologically full. Although this is a significant result, it does seem that topological fullness is often easier to prove directly than the separation property. The converse on the other hand is quite straightforward to prove and, in view of the equivalence of the centre with the space of band preserving operators, is of more significance in the study of the centre than has previously been acknowledged.

Lemma 1.2. *If the centre of a Banach lattice E is topologically full then it has the separation property.*

$\lhd$ If $x, y \in E_+$ and $x \perp y$ then $0 \leqslant y \leqslant x+y$ so the topological fullness gives us a sequence (T_n) in $Z(E)$ such that $T_n(x+y) \to y$. As $x \perp y$ we see that $T_n x \perp y$ and hence $T_n x \perp T_n y$, so that $T_n x \perp T_n y - y$. It follows that $|T_n x|, |T_n y - y| \leqslant |T_n x| + |T_n y - y| = |T_n x + (T_n y - y)| = |T_n(x+y) - y|$. As $\|T_n(x+y) - y\| \to 0$ we have $\|T_n y - y\| \to 0$, so that $T_n y \to y$, and $\|T_n x\| \to 0$. $\rhd$

Topological fullness (and transitivity) are by no means the only conditions which state that the centre is large. Indeed, the centre being maximal abelian, which is the subject of the next section, may be regarded as one such condition. There are others in the literature, many due to Meyer in [12]. The most closely related is perhaps worthy of mention here.

The centre of a Banach lattice E is *topologically rich* if for all $x \in E_+$ there is a sequence (T_n) in $Z(E)_{1+}$ such that, $T_n x \to x$ and, for each $n \in \mathbb{N}$, $T_n(E) \subseteq E_x$ where E_x denotes the closed ideal in E generated by x.

Many equivalents of this property may be found in [12]. In particular he proves that the existence of a topological order unit suffices to guarantee that the centre is topologically rich (in the proof of Proposition 1.1 note that T_n vanishes where x does, so that $T_n z$ also vanishes where x does — now use the fact that closed ideals J in $E \subset C^\infty(K)$ satisfy $J = \{f \in E : f|_{J^o} \equiv 0\}$ where $J^o = \{k \in K : j(k) = 0\ \forall j \in J\}$) and that topological richness of the centre implies topological fullness. He gives an example [13, Example 4.4], in the setting of locally convex Riesz spaces, to show that the converse is false. In fact even Dedekind complete Banach lattices, which have transitive centres, need not have topologically rich centres. The author has vague memories of having seen the following example, or something very similar, in the literature but is unable to track down the reference. He apologizes to the author of that work for not giving the appropriate credit here.

Example 1.3. The Dedekind complete Banach lattice $E = \ell_\infty(\ell_1(n))$, where $\ell_1(n)$ denotes n-dimensional space with the ℓ_1-norm, has a centre which is transitive, and therefore topologically full, but is not topologically rich.

$\lhd$ Let $e = \big((1/n, 1/n, \dots, 1/n)\big)_{n=1}^{\infty}$, which is a well known example of a weak order unit for E which is not a topological order unit. Suppose that there is $T \in Z(E)$ with $\|Te - e\|_\infty < 1/2$

and $T(E) \subset E_e$. We may describe T as pointwise multiplication by $\big((\alpha_1^n, \alpha_2^n, \dots, \alpha_n^n)\big) \in \ell_\infty\big(\ell_\infty(n)\big)$. Looking at the n'th entry in $Te - e$, we see that

$$\left\| \frac{1}{n}\big[\big((\alpha_1^n, \alpha_2^n, \dots, \alpha_n^n)\big) - (1, 1, \dots, 1)\big] \right\|_1 < 1/2,$$

which implies that $\sum_{k=1}^n |\alpha_k^n - 1| < n/2$. It follows that $\alpha_{k_0}^n \geqslant 1/2$ for at least one $k_0 \in \{1, 2, \dots, n\}$.

Now take v to be that element of E in which the n'entry is the n-tuple consisting of all zeros except for a 1 in the k_0'th position, so that the n'th entry of Tv has k_0'th entry equal to $\alpha_{k_0}^n$ which is at least 1/2. If Tv were in the closed ideal generated by e then $Tv \wedge (je) \to Tv$ in norm as $j \to \infty$. But the norm of the n'th entry in the difference $Tv \wedge (je) - Tv$ is

$$\big\|(0, \dots, \alpha_{k_0}^n - \alpha_{k_0}^n \wedge (j/n), 0, \dots)\big\|_1 = \big|\alpha_{k_0}^n - \alpha_{k_0}^n \wedge (j/n)\big| \geqslant \left|\frac{1}{2} - \frac{1}{2} \wedge (j/n)\right| \to \frac{1}{2},$$

as $n \to \infty$, so that $\|Tv - Tv \wedge (je)\| \geqslant \frac{1}{2}$ for all j. Thus $Tv \notin E_e$ and hence $Z(E)$ is not topologically rich. $\rhd$

2. When is the centre maximal abelian?

Given that the centre of $L^2(\mu)$ may be identified with the operators of pointwise multiplication by functions in $L^\infty(\mu)$, there are obvious analogies to be drawn between properties of the centre of a Banach lattice and those of maximal abelian self adjoint algebras of bounded operators on a Hilbert space. However, Goullet de Rugy's AM-space shows that the centre need not be a maximal abelian subalgebra of the space of bounded operators on a Banach lattice in general. Nevertheless, identifying when the centre is maximal abelian was an early concern of those working in this area. We will denote the space of all bounded linear operators on E by $\mathscr{L}(E)$.

The earliest result was used by Nakano in his work on dilatators when he essentially showed that if a vector lattice has the principal projection property then a linear operator is band preserving if and only if it commutes with the band projections. It might be instructive to look at the proof in this case. We couch it in a Banach lattice setting for ease of comparison with our other results.

Proposition 2.1. *If E is a Dedekind σ-complete Banach lattice then $Z(E)$ is a maximal abelian subalgebra of $\mathscr{L}(E)$.*

$\lhd$ It suffices to prove that if T commutes with $Z(E)$ then T is band preserving, so let us suppose that $x \perp y$. If P is the band projection onto the principal band generated by x then $P \in Z(E)$, $Px = x$ and $Pz = 0$ if and only if $z \perp x$. If T commutes with $Z(E)$ then in particular $P(Ty) = T(Py) = T(0) = 0$ so that $Ty \perp x$ as needed. $\rhd$

I obtained in Corollary 3.3 of [21] a rather technical condition on a Banach lattice E which guaranteed that $Z(E)$ is a maximal abelian subalgebra of $\mathscr{L}(E)$. The next development seems to have been by Orhon who showed in Theorem 12 of [17] that the centre remains maximal abelian if we weaken the hypothesis to $Z(E)$ being topologically full. His proof, as well as all subsequent ones that I know of, was rather indirect being a by-product of study of the duality of the centre. He couched the proof in purely vector lattice terms in [18]. Nevertheless, one would have hoped that an approximate version of the Dedekind σ-complete case would work and indeed it does, provided we remember Leader's separation condition.

I first isolate two simple results about convergence of sequences of central operators.

Lemma 2.2. *Let E be a Banach lattice, $0 \leqslant x \leqslant y$ and (T_n) a sequence in $Z(E)_{1+}$ such that $T_n y \to y$. If $T_n x \to z$ then $z = x$.*

$\lhd$ As

$$0 \leqslant T_n x \leqslant x, \qquad 0 \leqslant T_n(y-x) \leqslant y-x,$$

for all $n \in \mathbb{N}$, letting $n \to \infty$ we have

$$0 \leqslant z \leqslant x, \qquad 0 \leqslant y-z \leqslant y-x,$$

from which it follows that also $x \leqslant z$ and hence $z = x$. $\rhd$

Lemma 2.3. *Let E be a Banach lattice, $x, y \in E$ and (T_n) a sequence in $Z(E)_{1+}$. If $T_n y \to y$ and $T_n x \to 0$ then $x \perp y$.*

$\lhd$ Recall that each T_n is a lattice homomorphism so that

$$T_n(|x| \wedge |y|) = T_n|x| \wedge T_n|y| = |T_n x| \wedge |T_n y| \to |0| \wedge |y| = 0.$$

As $0 \leqslant |x| \wedge |y| \leqslant |y|$, Lemma 2.2 tells us that $|x| \wedge |y| = 0$. $\rhd$

Theorem 2.4. *If E is a Banach lattice with topologically full centre, then $Z(E)$ is a maximal abelian subalgebra of $\mathscr{L}(E)$.*

$\lhd$ Suppose that S commutes with $Z(E)$. It suffices to prove that if $0 \leqslant x, y \in E$ with $x \perp y$ then $Sx \perp y$. As $Z(E)$ is topologically full there by Lemma 1.2 is a sequence (T_n) in $Z(E)$ with $T_n y \to y$ and $T_n x \to 0$. As noted above, we may assume that the sequence lies in $Z(E)_{1+}$. Since

$$T_n(Sx) = S(T_n x) \to S(0) = 0,$$

the fact that $Sx \perp y$ follows from Lemma 2.3. $\rhd$

This rather simple proof lends itself easily to generalization in various ways. For example if we start with a vector lattice E with separating order dual and give E the weak topology induced by the order dual and interpret the term «topologically full» accordingly then we see that any order bounded linear operator which commutes with the centre will be band preserving (and, as it is order bounded, an orthomorphism.) This is contained in Corollary 2 of [3] as well as one of the unnumbered propositions in [4]. If we start with any Archimedean vector lattice and, rather than using a topology, use order convergence to define an analogue of topological fullness then we may obtain a result giving a condition under which order continuous linear operators that commute with the centre must be orthomorphisms. This variant does not seem to have been observed before.

In Question 5 in § 4 of [17], Orhon asked whether the converse of Theorem 2.4 holds? I. e. if $Z(E)$ is a maximal abelian subalgebra of $\mathscr{L}(E)$ then must $Z(E)$ be topologically full? Given the simple proof that we have just given, Leader's separation property looks very natural in this setting. This problem seems to have remained open since Orhon posed it. We are able to give a solution here, but possibly not the one that might have been hoped for, as we show that the implication fails.

Example 2.5. There is an AM-space H with centre $Z(H)$ which is not topologically full but which is a maximal abelian subalgebra of $\mathscr{L}(H)$.

$\lhd$ To start, let us recall that for any AM-space E there is a compact Choquet simplex K with extreme point 0, such that E is isometrically order isomorphic to the space $A_0(K)$ of continuous real-valued affine functions on K which vanish at 0, equipped with the pointwise order and supremum norm. K may be taken to be the positive part of the unit ball of E^* equipped with the weak* topology. Let X denote the set of non-zero extreme points of K with the relative topology. The space X is not particularly well-behaved topologically, but that does not matter for our purposes. The restriction map from K to X is an isometric order isomorphism because of the Krein–Milman theorem. The lattice operations in E are pointwise on X but not on the whole of K. There is another topology, the *facial topology*, on X such that the centre of E may be identified with the space of all facially-continuous bounded real-valued functions on X with the action being pointwise multiplication on the set $X \subset K$. Details of these properties of the centre may be found in [6] or [24].

Let H denote the set of all bounded real-valued functions h on $X \times [0, 1]$ with the following properties:

$$h(x, \cdot) \text{ is continuous on } [0, 1] \text{ for each } x \in X,$$
$$h(\cdot, 0) \text{ lies in } E.$$

It is routine to check that H is an AM-space under the sup-norm and the pointwise partial order. The map $f \mapsto h$ of E into H defined by $h(x, t) = f(x)$ is an isometric order isomorphism of E into H. It is routine to verify that the centre of H may be identified with the space of all bounded real-valued functions on $X \times [0, 1]$ which are continuous on each $\{x\} \times [0, 1]$ and are facially continuous on $X \times \{0\}$.

We now assume that E has a trivial centre, from which it follows that the centre of H may be identified with the space of all bounded real-valued functions on $X \times [0, 1]$ that are continuous on each $\{x\} \times [0, 1]$ and are constant on $X \times \{0\}$.

Suppose that a linear operator T on H commutes with every element of the centre. Let $h \in H_+$ and $(x, y) \in X \times [0, 1]$. We intend to show that if $h(x, t) = 0$ then $Th(x, t) = 0$. The proof proceeds by looking at two cases.

In the first place, let us suppose that $h|_{\{x\}\times[0,t]}$ is identically zero. Let b be a non-negative function on $X \times [0, 1]$ that (i) is zero except on $\{x\} \times [0, t]$, (ii) is continuous on $\{x\} \times [0, 1]$ and which is non-zero on the set $\{x\} \times (0, t)$. The function b defines an element of the centre of H with the property that $bh = 0$. It follows that

$$0 = T(bh) = b(Th),$$

so that Th vanishes on $\{x\}\times(0, t)$ and by continuity on $\{x\}\times[0, t]$. In particular, $(Th)(x, t) = 0$ as claimed.

If the last case does not hold, then there is $s \in (0, t)$ with $\alpha = \sqrt{h(x, s)} > 0$. We define two new functions p and q on $X \times [0, 1]$ as follows. On $\{x\} \times [s, 1]$ we set $q(x, k) = \sqrt{h(x, k)}$ and on the rest of $X \times [0, 1]$ q is constantly α. This definition certainly ensures that q corresponds to an element of the centre of H. Note, in particular, that as $t \in [s, 1]$ $q(x, t) = \sqrt{h(x, t)} = 0$. On $\{x\} \times [s, 1]$, we again let $p(x, k) = \sqrt{h(x, k)}$ and on the rest of $X \times [0, 1]$ we define it to be h/α. Again, noting that this is simply a scalar multiple of h on X, this ensures that $p \in H$. It follows that, as T commutes with the centre of H,

$$Th = T(qp) = q(Tp),$$

and, in particular,

$$Th(x, t) = q(x, t)(Tp)(x, t) = 0(Tp)(x, t) = 0,$$

as required.

The remainder of the proof that T is central is now routine. For the sake of completeness, and because the details are simpler than usual in this case, we give the argument.

For any $(x, t) \in X \times [0, 1]$ there is $f \in H$ with $f(x, t) \neq 0$. Define $q(x, t) = (Tf)(x,t)/f(x,t)$. If $g \in H$ let $h = g - \big(g(x,t)/f(x,t)\big)f$ so that $h(x,t) = 0$ and hence

$$(Th)(x,t) = Tg(x,t) - \big(g(x,t)/f(x,t)\big)Tf(x,t) = 0.$$

Note first that if $g(x,t) \neq 0$ then we have

$$\frac{(Tg)(x,t)}{g(x,t)} = \frac{(Tf)(x,t)}{f(x,t)} = q(x,t),$$

so that the definition of q does not depend on the choice of f. Second, we may rearrange the equality as

$$Tg(x,t) = \big(g(x,t)/f(x,t)\big)Tf(x,t) = \big(Tf(x,t)/f(x,t)\big)g(x,t) = q(x,t)g(x,t)$$

showing that T simply acts *via* pointwise multiplication by q. From this representation it follows immediately that T is band preserving and therefore central.

We have thus established that the centre of H is maximal Abelian. On the other hand it is not topologically full, for if we take any two linearly independent members of E (f and g, say) and consider the function $f \otimes 1 \in H$ then for any central operator S, $S(f \otimes 1)$ has restriction to $X \times \{0\}$ which is a multiple of f so cannot approximate $g \otimes 1$ arbitrarily well. ▷

In the preceding example, the set $J = \{h \in H : h_{X\times\{0\}} \equiv 0\}$ is a norm closed order dense ideal in H and $Z(H)|_J$ is topologically full on J (making an obvious generalization of the usual definition.) Thus Example 2.5 does not disprove the following conjecture.

Conjecture 2.6. If J is a norm closed order dense ideal in the Banach lattice E such that $Z(E)|_J$ is topologically full on J then $Z(E)$ is a maximal abelian algebra of bounded linear operators on E.

The proof in Example 2.5 did not assume boundedness of T, so that there $Z(H)$ is actually a maximal abelian subalgebra of the algebra of *all* linear operators rather than just $\mathscr{L}(E)$. The proof of Proposition 2.1 did not assume boundedness either. It is also true for $C_0(\Sigma)$ that the centre is a maximal abelian subalgebra of the space of all linear operators. Indeed, if $f \in C_0(\Sigma)$ then we can write $f = \sqrt{|f|}(\operatorname{sgn}(f)\sqrt{|f|})$ and $\operatorname{sgn}(f)\sqrt{|f|} \in C_0(\Sigma)$ whilst $\sqrt{|f|}$ may be identified with an element of the centre of $C_0(\Sigma)$, $C^b(\Sigma)$. If T commutes with the centre then $Tf = T\big(\sqrt{|f|}(\operatorname{sgn}(f)\sqrt{|f|})\big) = \sqrt{|f|}T(\operatorname{sgn}(f)\sqrt{|f|})$. In particular the support of Tf is contained in the support of $\sqrt{|f|}$, which is also that of f, which suffices to prove that T is band preserving and therefore central.

It seems even more difficult to find necessary and sufficient conditions for $Z(E)$ to be maximal abelian in the algebra of *all* linear operators than in the bounded case, but it might be of interest to answer the following:

Question 2.7. Is there an example of a Banach lattice E such that $Z(E)$ is a maximal abelian subalgebra of $\mathscr{L}(E)$, but which is *not* a maximal abelian subalgebra of the algebra of all linear operators on E?

3. Hyperinvariant subspaces for central operators

Recall that an *invariant subspace* for a bounded linear operator $T \in \mathscr{L}(E)$ is a closed linear subspace $\{0\} \neq J \subsetneq E$ such that $TJ \subset J$. A *hyperinvariant subspace* for T is one that is invariant for all bounded linear operators that commute with T. Apart from the recurrence of the theme of operators that commute with a given one, there are historical precedents for looking at hyperinvariance in the context of central operators. Huang [8], proved a result that has its roots in a classical result of Fuglede and Dunford on spectral operators. He showed, in our terminology, that for a central operator T on separable $L^p(\mu)$ the hyperinvariant subspaces are precisely certain bands associated with T. In particular, unless T is a scalar multiple of the identity, then there always are such subspaces. The existence of such hyperinvariant subspaces extends to a much more general setting and, indeed, even in the original setting we can say rather more.

In order conveniently to state this little extra, we reluctantly introduce a new piece of notation. We will say that a linear subspace J of a linear space E is *algebraically hyperinvariant* for a linear operator T on E if J is invariant for *all* linear operators on E which commute with T. The emphasis here is that we do not restrict our attention to bounded operators. A precursor of this result, for Banach function spaces, may be found in Corollary 2.3 of [1]. The proof has no connection to topological fullness of the centre and will be published elsewhere.

Theorem 3.1. *Let E be an Archimedean vector lattice and let $T \in Z(E)$* not *be a scalar multiple of the identity, then there is a proper algebraically hyperinvariant band for T.*

As in [1] it is easily seen that, unless T is a finite sum of multiples of band projections, T has infinitely many disjoint hyperinvariant bands. Of course there is no reason in general to expect the band B in the preceding proof to be a projection band. Even in the oldest known setting, of Hilbert space, the fact that we do not have to restrict to bounded operators seems to be new. To complete the extension of Huang's result we need to prove that every hyperinvariant subspace is actually a band, at least for nice enough spaces. It is at this point that topologically full centres become involved again.

Proposition 3.2. *If E is a Banach lattice with topologically full centre, $T \in Z(E)$ and J is a norm closed hyperinvariant subspace for T then J is an order ideal.*

$\lhd$ If $0 \leqslant |x| \leqslant |y|$ and $y \in J$ then we must show that $x \in J$. As a first step, we show that $|y| \in J$. As $Z(E)$ is topologically full, there is a sequence (U_n) in $Z(E)$ with $U_n|y| \to y^+$. Since U_n is band preserving, $U_n y^- \perp y^+$ and $U_n y^+ \in {y^+}^{dd}$ so that

$$\big|U_n|y| - y^+\big| = |U_n(y^+ + y^-) - y^+| = |(U_n y^+ - y^+) + U_n y^-| = |U_n y^+ - y^+| + |U_n y^-|,$$

and we see that $U_n y^+ - y^+ \to 0$ and $U_n y^- \to 0$ so that $U_n y = U_n(y^+ - y^-) \to y^+$. Similarly, there is a sequence (V_n) in $Z(E)$ with $V_n y \to y^-$ so that $(U_n + V_n)y \to |y|$. As each $(U_n + V_n)$ certainly commutes with T, J is invariant under $U_n + V_n$ so that $(U_n + V_n)y \in J$ and, as J closed, $|y| \in J$. Similarly, we may obtain central sequences P_n, Q_n with $P_n|y| \to x^+$ and $Q_n|y| \to x^-$ which shows that $x^+, x^- \in J$ and hence $x = x^+ - x^- \in J$ as required. $\rhd$

There will be a version of this result, in the setting of Archimedean vector lattices with separating order dual, which will refer to weakly closed hyperinvariant subspaces. Banach lattices with an order continuous norm are Dedekind complete (so have topologically full centre) and have the property that all closed order ideals are actually projection bands, so we have:

Corollary 3.3. *If T is a central operator, which is not a scalar multiple of the identity, on a Banach lattice E with an order continuous norm then T has an algebraically hyperinvariant projection band and furthermore every closed subspace of E which is hyperinvariant for T must be a projection band.*

The role of topological fullness in its various forms is not clear here. After all, for the proof of Proposition 3.2 to work we need only to be able to approximate $x \in [0, y]$ by $T_n y$ where the T_n lie in the commutant of $Z(E)$, which can be much larger than $Z(E)$ in general. Of course if E is nice enough we can certainly do this and if E is Dedekind σ-complete we can even choose T so that $Ty = x$ (of course it will actually be central!) It would be rash at this stage even to suggest that such approximation by elements of the commutant of the centre is possible, but let me propose the following:

CONJECTURE 3.4.If E is any Banach lattice such that $Z(E)$ is Dedekind complete, $x, y \in E$ with $0 \leqslant x \leqslant y$ then there is T in the commutant of $Z(E)$ such that $Ty = x$.

It is tempting to use the Hahn–Banach theorems for modules of Vincent–Smith [20] or Orhon [16] to produce such an operator but, unlike the vector space case, it does not seem possible to write a bounded (in some sense) module homomorphism defined on a principal submodule as a bounded algebra-valued module homomorphism multiplied by an element of the space.

We thus have to leave open the question:

QUESTION 3.5. It J is a hyperinvariant subspace for a central operator on a Banach lattice, must J be an order ideal?

In the algebraic setting though, in spite of the hypotheses being stronger as they involve a wider class of operators, we can disprove the corresponding result.

EXAMPLE 3.6. There is an Archimedean vector lattice E, a central operator Z on E and a subspace H of E which is algebraically hyperinvariant for T but such that H is not an order ideal in E.

$\lhd$ Take

$$K = ([0,1] \times \{0\}) \cup \big((\mathbb{Q} \cap [0,1]) \times \{1/k : \, k \in \mathbb{N}\}\big)$$

and E to be the set of all bounded functions f on K which satisfy the two conditions:

(1) $f|_{[0,1]\times\{0\}}$ is continuous and piecewise affine;

(2) for all $q \in \mathbb{Q} \cap [0,1]$, $f(q, 1/n) \to f(q, 0)$ as $n \to \infty$.

It is routine to verify that E is an Archimedean vector lattice. It is also routine to verify that the centre of E, $Z(E)$, may be identified with the subspace of E consisting of those f which are constant on $[0,1] \times \{0\}$.

The set $A = \{1 + 1/n : \, n \in \mathbb{N}\}$ has only the one accumulation point, namely 1. The sets $A_n = \{1 + 1/\big(2^n(2k-1)\big) : \, k \in \mathbb{N}\}$, for $n \in \mathbb{N}$ are disjoint subsets of A. Thus, if we define $\phi \in Z(E)$ by

$$\phi(q_n, 1/k) = 1 + 1/\big(2^n(2k-1)\big), \quad \phi(x, 0) = 1,$$

where $\{q_n : \, n \in \mathbb{N}\}$ is an enumeration of $\mathbb{Q} \cap [0,1]$, $k \in \mathbb{N}$ and $x \in [0,1]$, then the only accumulation point of $\phi(K)$ is 1 and $\phi(q, 1/k) > 1$ whenever $q \in \mathbb{Q} \cap [0,1]$ and $k \in \mathbb{N}$.

I claim that if any linear operator T (without any assumption of norm or order boundedness) commutes with ϕ then T is actually central.

The first step is to prove that if we take any $f \in E$, $r \in \mathbb{Q} \cap [0,1]$ and $m \in \mathbb{N}$ such that $f(r, 1/m) = 0$ then $Tf(r, 1/m) = 0$. As $\phi(r, 1/m)$ is not an accumulation point of $\phi(K \setminus \{(r, 1/m)\})$, the function $\sigma = \phi - \phi(r, 1/m)\mathbf{1}$ is bounded away from zero, except at the point $(r, 1/m)$ where it vanishes. Thus the function defined by

$$\psi(q, x) = \sigma(q, x)^{-1}, \quad (q, x) \neq (r, 1/m),$$
$$\psi(r, m) = 0,$$

is bounded and it is easily verified that it lies in $Z(E)$. As $\sigma\psi$ is identically equal to one, except at $(r, 1/m)$ where both it and f vanish, we see that $\sigma\psi f = f$.

As T commutes with ϕ and σ differs from ϕ only by a multiple of the identity, T also commutes with σ. Thus

$$Tf = T\big(\sigma(\psi f)\big) = \sigma\big(T(\psi f)\big),$$

and, in particular, as $\sigma(r, 1/m) = 0$ we also have $Tf(r, 1/m) = 0$.

If we now consider an arbitrary element $g \in E$ and $(q, 1/k) \in (\mathbb{Q} \cap [0,1]) \times \{1/n : n \in \mathbb{N}\}$ then $f = g - g(q, 1/k)\mathbf{1}$ vanishes at $(q, 1/k)$ so that

$$0 = Tf(q, 1/k) = Tg(q, 1/k) - g(q, 1/k)T\mathbf{1}(q, 1/k),$$

and hence $Tg(q, 1/k) = T\mathbf{1}(q, 1/k)g(q, k)$. By continuity, we have also $Tg(q, 0) = T\mathbf{1}(q, 0)\times g(q, 0)$ for all $q \in \mathbb{Q} \cap [0, 1]$ and, by continuity again, for all $q \in [0, 1]$. Thus T is actually central, being multiplication by $T\mathbf{1}$.

The subspace $Z(E) \subset E$ is invariant under all central operators, so is hyperinvariant for T, but is clearly not an order ideal in E. $\rhd$

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