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A NOTE ON WEAKLY $\aleph_1$-SEPARABLE p-GROUPS

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It is well-known by Hill-Griffith that there exist $\aleph_1$-separable p-primary groups which are not direct sums of cycles. A problem of challenging interest, mainly due to Hill (Rocky Mount. J. Math., 1971), is under what extra circumstances on the group structure this holds untrue, that is every $\aleph_1$-separable p-group is a direct sum of cyclic groups. We prove here that any weakly $\aleph_1$-separable p-group of cardinality not exceeding $\aleph_1$ is quasi-complete precisely when it is a bounded direct sum of cycles, thus partly answering the posed question in the affirmative.

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1. Preliminaries

Throughout this brief article, let it be agreed that all groups under consideration are p-torsion abelian, for some fixed but arbitrary prime number p, written additively as is the custom when studying commutative groups.

All our notation and notions used in the text are standard and follow essentially those from [13]. For instance, for a group A, the letter $A^1 = p^\omega A = \bigcap_{i<\omega} p^i A$ will always denote in the sequel the first Ulm subgroup of A where $p^i A = \{p^i a | a \in A\}$ is the p^i-th series of powers of A.

A group is said to be by [21] $\aleph_1$-*separable* if each its countable subgroup can be embedded in a countable summand of the whole group which direct summand is a direct sum of cycles. These groups are necessarily separable, that is they have zero first Ulm subgroup. In an equivalent form, in conjunction with the second classical Prufer's theorem, a separable group is $\aleph_1$-separable if each its subgroup is contained in a countable direct summand of the full group. Also in accordance with [21], the group A is called *weakly* $\aleph_1$-*separable* provided that every countable subgroup of A is contained in a countable pure and closed subgroup of A. It is a routine matter to observe that every $\aleph_1$-separable group is weakly $\aleph_1$-separable, whereas the converse implication is demonstrably false [31].

Moreover, a group A is called in [25] a *Fuchs five-group* if each its infinite subgroup is embedded in a direct summand of A with the same cardinality as the subgroup. In this way, following [25], we say that the group A is a Q-*group* if for each infinite subgroup G of A the first Ulm subgroup of the factor-group A modulo G has cardinality no more than that of G. It is self-evident that every separable Fuchs five-group is $\aleph_1$-separable, every separable Fuchs five-group is a Q-group and every Q-group is weakly $\aleph_1$-separable. It is a long time known that the both opposite claims fail, i. e. there is a $\aleph_1$-separable group which is not Fuchs five as well as there is by Kamalov [26, 27] a Q-group which is not Fuchs five.

Utilizing a criterion due to Megibben [31] for weakly $\aleph_1$-separability, namely that the separable group A is weakly $\aleph_1$-separable $\iff$ $\forall\ G \leqslant A$: $|G| = \aleph_0 \Rightarrow |(A/G)^1| \leqslant \aleph_0$, it is elementary to see that the classes of weakly $\aleph_1$-separable groups of cardinality $\aleph_1$ and Q-groups of cardinality $\aleph_1$ do coincide. The same holds true for the classes of $\aleph_1$-separable groups of cardinality $\aleph_1$ and the separable Fuchs five-groups of cardinality $\aleph_1$. That is why, we observe that the cardinal $\aleph_1$ plays a key role in the set-theoretical aspect of these sorts of groups.

As in [28] or [25], a group A is termed *starred* if $G \leqslant A$ with A/G divisible implies $|A| = |G|$, i. e. $G \leqslant A$ with A/G divisible implies $|A/G| \leqslant |G|$. (Note that if G is finite, hence bounded, A/G being divisible simple yields that A is algebraically compact. Thus, if A is reduced, it must be bounded and thereby $A = G$.) With this in hand, it is easily seen that each (weakly) $\aleph_1$-separable group of cardinality $\aleph_1$ is starred, while the reverse implication is wrong as well as the previous one for cardinalities greater than $\aleph_1$. Even more, every Q-group is of necessity starred. Thus the classes of $\aleph_1$-separable groups and Q-groups are absolutely different. Does their intersection is precisely the Fuchs five-groups? For cardinality $\aleph_1$ this is obviously so.

As mentioned above, the group A is named *separable* if $A^1 = 0$. Besides, a group A is called *$\aleph_1$-Σ-cyclic* if each its countable subgroup is a direct sum of cycles. It follows directly from the second Prufer's theorem that any separable group is $\aleph_1$-Σ-cyclic, while the converse claim is untrue.

Generally, the following inclusions of group classes hold fulfilled:
{direct sums of cyclic groups} $\subseteq$ {separable Fuchs five-groups} $\subseteq$ {$\aleph_1$-separable groups} $\cup$ {Q-groups} $\subseteq$ {weakly $\aleph_1$-separable groups} $\subseteq$ {separable groups} $\subseteq$ {$\aleph_1$-Σ-cyclic groups}.

Moreover, under (MA+$\neg$ CH), the classes of $\aleph_1$-separable groups, Q-groups and weakly $\aleph_1$-separable groups all of cardinality $\aleph_1$ do coincide (see, e. g., [31]). Unfortunately, it is well-known by Hill [17] and Griffith [14] that the first inclusion concerning the direct sums of cycles is proper, indeed. P. Hill has first asked in [18, p. 348] when Fuchs five-groups plus something else imply that they are direct sums of cyclic groups; as a matter of fact see [18, Theorem 3.2].

The purpose of the present paper is to answer in some aspect Hill's question by finding under what additional circumstances on the group structure the inclusion «$\subseteq$» becomes «$=$». In all that follows, we will illustrate that the intersection of the classes of weakly $\aleph_1$-separable groups and direct sums of quasi-complete groups each of which is with cardinality $\aleph_1$ is precisely the direct sums of cycles.

We shall also discuss here certain related concepts concerning the relationships between some major group classes. Foremost, we mention that the thick, respectively essentially finitely indecomposable, direct sums of cyclic p-groups are bounded and hence the semi-complete thick groups, respectively essentially finitely indecomposable groups, are torsion-complete (see also [10]).

In [6] we have proved that the $p^{\omega+1}$-projective thick (respectively the $p^{\omega+1}$-projective essentially finitely indecomposable) groups are bounded; however this is not the case for $p^{\omega+2}$-projectives. We also have shown there that the $p^{\omega+1}$-projective Σ-p-groups are direct sums of countables of length not exceeding $\omega + 1$; unfortunately this is not true for $p^{\omega+2}$-projectives.

It is clear that every torsion-complete Σ-group is bounded, whence the Σ-group which is a direct sum of torsion-complete p-groups is a direct sum of cycles. The same property is fulfilled even for pure-complete Σ-groups because they are separable. Note that the finite direct sums of torsion-complete groups (i. e. the torsion-complete groups) are thick while the infinite ones cannot be so. A group is said to be pillared if its first Ulm factor is a direct sum of cyclic groups. It is apparent to see that the pillared groups are Σ-groups while the reverse claim is demonstrable wrong. Finally, we remark that the thick pillared groups are algebraically compact in contrast with the thick Σ-groups as will be observed. In fact, in conjunction with [5], their first Ulm factor should be a thick direct sum of cycles, hence bounded. Thus this group is of necessity algebraically compact.

Moreover, it is elementary to see that the separable thick Σ-groups are bounded, contrasting with the inseparable situation. Indeed, Megibben has constructed (e. g. [30]) a Σ-group G such that G/G^1 is unbounded torsion-complete; thus G is unbounded too. Since G/G^1 is thick, appealing to [5] it must be that G is thick. Thereby the thick Σ-p-groups are not algebraically compact. In particular, the thick simply presented groups are algebraically compact and the thick IT-groups (isotype subgroups of totally projective groups) are bounded; note that every IT-group is pillared. In this aspect, we ask whether or not, for any ordinal α, each p^α-projective thick group is bounded and each p^α-projective pillared group is totally projective — we know via [6] that each $p^{\omega+1}$-projective thick group is bounded and that for $n \in \mathbb{N}$ each $p^{\omega+n}$-projective pillared group (even each pillared group of length $\leqslant \omega + n$) is a direct sum of countable groups of length $\leqslant \omega + n$.

2. The main result

We now recollect the major instruments needed for verification of the central theorem (see, for example, [21] and [19]).

Criterion 1 (Huber, 1983). *The separable group A of cardinality $\aleph_1$ is weakly $\aleph_1$-separable $\iff A = \bigcup_{\alpha<\omega_1} A_\alpha$, where, $\forall\ \alpha < \omega_1$, $A_\alpha \subseteq A_{\alpha+1}$, $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ whenever $\beta < \omega_1$ is limit, $|A_\alpha| = \aleph_0$, $A_{\alpha+1}$ is pure in A and $A/A_{\alpha+1}$ is separable.*

Criterion 2 (Hill, 1995). *The group A of cardinality $\aleph_1$ is a direct sum of cyclic groups $\iff A = \bigcup_{\alpha<\omega_1} A_\alpha$, where, $\forall\ \alpha < \omega_1$, $A_\alpha \subseteq A_{\alpha+1}$, $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ whenever $\beta < \omega_1$ is limit, A_α is a direct sum of cyclic groups, A_α is pure in A and A/A_α is separable.*

We would like to emphasize the similarity of the foregoing two criteria, namely if all of the specific limitations in Criterion 1 are taken for each ordinal number α, not only for $\alpha + 1$ which excludes the limit ordinals, we obtain by Criterion 2 that such a weakly $\aleph_1$-separable group of cardinality $\aleph_1$ has to be a direct sum of cycles.

Imitating [13, V. II] the reduced group A is said to be *quasi-complete* (see [15, 16, 20] as well) if for each pure subgroup H of A the quotient $(A/H)^1$ is divisible. These groups are necessarily separable by taking $H = 0$. Moreover, a criterion from [20] illustrates that a separable group A is quasi-complete if and only if A/G is the direct sum of a divisible group and a torsion-complete group whenever G is an unbounded pure subgroup of A. Note that with the aid of [13, V. II, p. 58, Property e] every torsion-complete group is quasi-complete whereas the converse implication is not true.

The following technicality is well-known in the existing literature and is our crucial tool.

Lemma 3. *A balanced subgroup of a quasi-complete group is quasi-complete.*

$\lhd$ Assume that G is a balanced subgroup of the quasi-complete group A. Let C be a pure subgroup of G. It is then pure in A. Thus $(A/C)^1$ is divisible. But G/C is isotype in A/C, so $(G/C)^1$ is pure in $(A/C)^1$ (see [13, V. II]). Therefore, $(G/C)^1$ is divisible, and thereby we are finished. $\rhd$

Before stating the chief affirmation, the following comments are of interest.

We know via [15] (see also [13, V. II, p. 62, Exercise 2]) that each quasi-complete group is a direct sum of cycles only when it is bounded. In what follows, we intend to give its nontrivial generalization. Even, in virtue of the classical second Prufer's theorem, every countable quasi-complete group is bounded; thus every unbounded quasi-complete group is uncountable.

We also have shown in [3] that under (¬ CH) any starred torsion-complete group is bounded; even assuming (¬ CH) every torsion-complete group of cardinality $\aleph_1$ is bounded (e. g. [13, V. II, p. 29, Exercise 7]). Nevertheless, in [4], we have constructed in (CH) an unbounded torsion-complete group which is starred. However, it is consistent in (ZFC) that each starred quasi-complete (in particular, torsion-complete) group with a countable basic subgroup must be bounded (compare with Proposition 6 proved below).

It is worth noting that other related results of this type the reader can see in [7–10].

In [12] it was proved that under the assumption of (MA+¬CH), every weakly $\aleph_1$-separable group of cardinality $\aleph_1$ possesses a direct summand which is a direct sum of cyclic groups of final rank as the whole group; whence this group must be bounded provided that it is quasi-complete. We shall demonstrate further that the latter claim is consistent in (ZFC) and thus the condition (MA+¬CH) can be removed.

We are now prepared to prove the following. The next two assertions, closely related to the results alluded to above and especially to Criterion 2 due to Hill, are of independent interest as well (about other attainments of this kind we refer the readers to the existing literature in this direction).

Proposition 4. *Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$, where $\{A_\alpha\}_{\alpha<\omega_1}$ is a smooth well-ordered ascending chain of pure torsion-complete subgroups of A. Then A is torsion-complete.*

◁ Clearly $p^\omega A = \bigcup_{\alpha<\omega_1} p^\omega A_\alpha = 0$. Further, assume that $(a_n)_{n=1}^\infty$ is a bounded Cauchy sequence in A. Consequently, since it has countable length, there is an index, say γ, so that $(a_n)_{n=1}^\infty \in A_\gamma$. Because of the purity of A_γ in A, it trivially follows that $(a_n)_{n=1}^\infty$ is a bounded Cauchy sequence in A_γ. Invoking a criterion of Kulikov for torsion-completeness from [29] (see [13, V. II, p. 38, Theorem 70.7 too]), we deduce that $(a_n)_{n=1}^\infty$ is convergent in A_γ, whence immediately in A. Finally, again appealing to the topological criterion of Kulikov [29] (see [13, V. II, p. 38, Theorem 70.7]), we infer that A is, in fact, torsion-complete. ▷

Remark. If we have a countable ascending union of pure torsion-complete subgroups, simple examples show that the same statement is no longer available.

Proposition 5. *Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$, where $\{A_\alpha\}_{\alpha<\omega_1}$ is a smooth well-ordered ascending chain of bounded subgroups of A. Then A is bounded.*

◁ Assume in a way of contradiction that the countable set of all different exponents $n_1, n_2, \cdots, n_k, \cdots$ is infinite, where $p^{n_1}A_1 = 0$, $p^{n_2}A_2 = 0, \cdots, p^{n_k}A_k = 0, \cdots$ and $n_1, n_2, \cdots, n_k, \cdots$ are minimal with this property. Since $A_\omega = \bigcup_{m<\omega} A_m$ is bounded, this is a contradiction with the infinity of the considered set. Thus, because $|\omega_1| = \aleph_1$ is uncountable, it follows that there is a natural number, say k, so that $p^k A_\alpha = 0$ for almost all, hence for all, indices α. Finally, we derive that $p^k A = 0$ and thereby we are finished. ▷

Problem 1: Determine whether the first of the preceding two propositions remain true for quasi-complete groups.

Notice also the interesting fact that if $A = \bigcup_{\alpha<\omega_1} A_\alpha$, where $\{A_\alpha\}_{\alpha<\omega_1}$ is a smooth well-ordered ascending chain of pure and closed in A weakly $\aleph_1$-separable subgroups, then A is weakly $\aleph_1$-separable. Indeed, if C is a countable subgroup of A, we observe that $C \subseteq A_\gamma$ for some countable ordinal γ. Thus $C \subseteq C_\gamma$ where C_γ is countable and C_γ is pure and closed in A_γ. Therefore, C_γ is pure in A. On the other hand, $A/A_\gamma \cong$

$A/C_\gamma/A_\gamma/C_\gamma$ is separable, so $p^\omega(A/C_\gamma) \subseteq A_\gamma/C_\gamma$. But A_γ/C_γ being pure in A/C_γ ensures that $p^\omega(A/C_\gamma) = p^\omega(A_\gamma/C_\gamma) = 0$, hence A/C_γ is separable, as desired, and we are done.

In that aspect, find analogous results for Fuchs five-groups, $\aleph_1$-separable groups, Q-groups, starred groups etc.

Proposition 6. *Each weakly $\aleph_1$-separable group with a countable basic subgroup is quasi-complete if and only if it is bounded.*

$\lhd$ Let B be a countable basic subgroup of such a group A. Since A/B is divisible, it follows by the Megibben's criterion, listed in the introduction, that $(A/B)^1 = A/B$ is also at most countable, hence A is countable. Therefore, the second Prufer's theorem implies that the separable group A must be a direct sum of cycles, whence it is bounded by [15], as promised. $\rhd$

Note. As we have aforementioned, the preceding assertion was proved for torsion-complete groups in [10].

So, all the groundwork has now been laid to proceed by proving the main result motivated the present exploration.

Theorem 7. *Any weakly $\aleph_1$-separable group of cardinality $\aleph_1$ is quasi-complete if and only if it is bounded.*

$\lhd$ Assume that A is the weakly $\aleph_1$-separable group in question. Write in conjunction with the Huber's criterion alluded to above that $A = \bigcup_{\alpha<\omega_1} A_\alpha$, where, for each $\alpha < \omega_1$, A_α is a countable direct sum of cycles and $A_{\alpha+1}$ is balanced in A. Since A is quasi-complete and since owing to Lemma 3 any balanced subgroup of A is again quasi-complete, it must be that each $A_{\alpha+1}$ is a quasi-complete direct sum of cycles, hence as aforementioned it is bounded. Thus every $A_\alpha \subseteq A_{\alpha+1}$ is bounded. Next, we can proceed in two different ways. That A has to be bounded follows at once from Proposition 5. Nevertheless, we shall use another idea which is more difficult but however interesting. In fact, since each countable subgroup can be embedded in a countable pure subgroup (see e. g. [13, V. I, p. 137, Proposition 26.2]), one may write A as an ascending well-ordered at ω_1 tower of pure and bounded subgroups. Knowing this, combined with a theorem due to Prufer-Kulikov [29] or [13, V. I, p. 140, Theorem 27.5], all A_α are direct summands of A, henceforth A/A_α are separable. Thus the foregoing criterion of Hill is applicable to conclude that A is a direct sum of cycles. Finally, A must be a quasi-complete direct sum of cyclic groups, whence by [15] it is really bounded as expected. $\rhd$

As an immediate consequence (which is actually an equivalence), we yield the following.

Corollary 8. *Every Q-group of cardinality $\aleph_1$ is quasi-complete only when it is bounded.*

$\lhd$ Any Q-group is weakly $\aleph_1$-separable, so the previous Theorem works. $\rhd$

Because of the important role played by the cardinal $\aleph_1$, we pose the following.

Conjecture: There exists an unbounded weakly $\aleph_1$-separable group (or more generally a Q-group) of cardinality $\aleph_2$ which is quasi-complete.

The next problem arises also naturally.

Problem 2: Decide whether or not in ($\neg$ CH) any starred quasi-complete group is bounded.

If yes, we strengthen Theorem 7 in ($\neg$ CH) since, as already observed, the weakly $\aleph_1$-separable groups of cardinality $\aleph_1$ are starred.

In closing, we raise the following generalized question.

Problem 3: Determine whether or not the weakly $\aleph_1$-separable groups are thick, respectively are essentially finitely indecomposable, only when they are bounded.

This query can be treated as a generalized version of the central theorem because it is a long time known that each torsion-complete group is thick (cf. [1, Corollary 3.4]) and the finite direct sums of thick groups are thick (cf. [10]); the same property for essentially finitely indecomposable groups may be seen in [24]. Even more, employing Lemma 3 and [15], every quasi-complete group is essentially finitely indecomposable. The corresponding fact concerning the thick groups, namely that the quasi-complete groups are thick, is not so easy and follows from [1, Corollary 3.3].

We conjecture that Problem 3 possesses a positive answer only for the cardinal $\aleph_1$; however the complete solution seems to be in the distant future. This our opinion is inspired by the following. Under (MA+$\neg$ CH) in [12] was showed that any unbounded weakly $\aleph_1$-separable group of cardinality $\aleph_1$ is C-decomposable, that is it has an unbounded direct sum of cycles summand with final rank equal to that of the whole group. On the other hand, the thick groups being essentially finitely indecomposable do not have an unbounded direct sum of cycles summand. So, we are finished.

We state the following actual question as well.

PROBLEM 4: Decide whether or not the pure-complete weakly $\aleph_1$-separable groups are direct sums of cyclic groups. Also does it follow that the pure-complete thick groups are precisely the quasi-complete ones?

It is worthwhile noting that in [23] was shown that the $p^{\omega+n}$-projective pure-complete groups are direct sums of cycles for each $n \geqslant 1$; for $n = 1$ see [6] too. Moreover, it can be obtained (see [11] for instance) that the $p^{\omega+n}$-projective thick groups are bounded; note that this is a crucial fact necessary to be showed that in [2] there exists an essentially finitely indecomposable but not thick group, thus answering once again in the negative the classical Irwin's conjecture.

Besides, we give some more definitions: Following [22], a group A is termed *a Crawley group* if and only if $A = B \oplus C$ implies either B or C is finite. Moreover, again in [22], the group A is named *essentially indecomposable* if and only if $A = B \oplus C$ implies either B or C is bounded. It is a trivial fact that every direct sum of cycles is a Crawley group and every Crawley group is essentially indecomposable. Now, we define A as *strongly essentially finitely indecomposable* if and only if $A = B \oplus C$ with C a direct sum of cycles implies that C is finite. It is elementarily that each strongly essentially finitely indecomposable group is essentially finitely indecomposable while the converse need not be true. A question that arises quite usually, inspired by [6], is whether each $p^{\omega+n}$-projective Crawley group or each $p^{\omega+n}$-projective essentially indecomposable group is a direct sum of cycles and whether each $p^{\omega+n}$-projective strongly essentially finitely indecomposable group is bounded; we know that this is not the case for essentially finitely indecomposable groups since consulting with [2] there is an unbounded $p^{\omega+2}$-projective essentially finitely indecomposable group. Evidently, every C-decomposable essentially indecomposable group is a direct sum of cycles; in particular, every $p^{\omega+1}$-projective essentially indecomposable group is a direct sum of cycles. In fact, for such a group A we have that $A = B \oplus C$ where C is a direct sum of cyclic groups and $\mathrm{fin}r(C) = \mathrm{fin}r(A)$. If for a moment C is bounded, it easily follows that so does A. Otherwise, if B is bounded, we derive that A is a direct sum of cyclic groups. Note also that (see [6]) each C-decomposable essentially finitely indecomposable group is bounded.

A group G is defined to be *weakly thick* if and only if $\exists\ K \leqslant G[p^s]$ for some $s \in \mathbb{N}$ such that G/K being a direct sum of cycles implies the existence of $t \in \mathbb{N}$ so that $(p^tG)[p] \subseteq K$. Certainly, each weakly thick group is $p^{\omega+n}$-projective for some $n \in \mathbb{N}$, hence it is of necessity with length $< \omega.2$ or even more precise with length $\leqslant \omega + n$. Although the name, the thick groups need not be weakly thick — they only satisfied a similar property. By what we have shown above they are weakly thick only when they are bounded. Apparently, the direct sums of cycles which are weakly thick are obviously bounded. Even more, pillared groups are weakly thick uniquely when they are bounded (as above observed, we note once again that the pillared groups of length strictly less than $\omega.2$ are direct sums of countable groups as well as pillared thick groups are algebraically compact). In fact, G/G^1 being a direct sum of cycles and $G^1 \subseteq G[p^k]$ for some existing $k \geqslant 1$ yield that there is $m \in \mathbb{N}$: $(p^m G)[p] \subseteq G^1$, i. e. $(p^m G)[p] = G^1[p]$, hence G is algebraically compact. Indeed, letting $g \in p^m G$, it follows that $g \in (p^m G)[p^l]$ for some integer $l \geqslant 1$. Furthermore, $p^{l-1}g \in (p^{m+l-1}G)[p]$ and thus $p^{l-1}g \in p^{m+l}G$. Therefore, $g \in p^{m+1}G + G[p^{l-1}]$, i. e. $g \in p^{m+1}G + (p^m G)[p^{l-1}]$ etc. by induction or a finite number of steps we deduce that $g \in p^{m+1}G + (p^m G)[p] = p^{m+1}G$, that is $p^m G = p^{m+1}G$. But G is reduced, whence it is bounded. We thus ask whether or not every weakly thick Σ-group or every weakly thick essentially finitely indecomposable (or Crawley, or essentially indecomposable, or strongly essentially finitely indecomposable, respectively) group is bounded.

Finally, a group A is said to be *weakly C-decomposable* if $A[p]$ is a C-decomposable valuated vector space, that is it has a free valuated summand of the same final rank itself (see [23]). Clearly, complying with [13] and [23], summable groups as well as C-decomposable groups as well as $p^{\omega+n}$-projective groups for each $n \geqslant 1$ are weakly C-decomposable. So, find necessary and sufficient conditions when every (weakly) C-decomposable pillared group is totally projective; notice that if $A = B \oplus L$ where L is separable then $A/A^1 \cong (B/B^1) \oplus L$ because $A^1 = B^1$.

Remark. In [5] the groups considered in the main theorems are *reduced*.

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