Владикавказский математический журнал
Апрель–июнь, 2005, Том 7, Выпуск 2

UDC **517.9**

BERNSTEIN–NIKOLSKIĬ TYPE INEQUALITY IN LORENTZ SPACES AND RELATED TOPICS

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Dedicated to academician S. M. Nikolskiĭ
on the occasion of his 100th-birthday

In this paper we study the Bernstein–Nikolskiĭ type inequality, the inverse Bernstein theorem and some properties of functions and their spectrum in Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.

1. Introduction

The study of properties of functions in the connection with their spectrum has been implemented by many authors (see, for example, [1–16] and their references). Some geometrical properties of spectrums of functions and relations with the sequence of norms of derivatives (in Orlicz spaces and N_Φ-spaces) were studied in [1–9]. In this paper we give some results on the Bernstein–Nikolskiĭ type inequality, the inverse Bernstein theorem and some properties of functions and their spectrum in Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.

Let us recall some notations. If $f \in S'$ then the spectrum of f is defined to be the support of its Fourier transform $\hat{f}$ (see [14, 15]). Denote $\operatorname{sp}(f) = \operatorname{supp}\hat{f}$ and $|E|$ the Lebesgue measure of E. For an arbitrary measurable function $f : \mathbb{R}^n \to \mathbb{C}$ (or $\overline{\mathbb{R}}$), one defines (see [17–22])

$$\lambda_f(y) := \bigl|\{x \in \mathbb{R}^n : |f(x)| > y\}\bigr|, \quad y > 0,$$
$$f^*(t) := \inf\{y > 0 : \lambda_f(y) \leqslant t\}, \quad t > 0,$$

$$\|f\|_{p,q} := \begin{cases} \Bigl(\frac{q}{p}\int\limits_0^\infty \bigl(t^{1/p}f^*(t)\bigr)^q \frac{dt}{t}\Bigr)^{1/q}, & 0 < p < \infty,\ 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p}f^*(t), & 0 < p \leqslant \infty,\ q = \infty. \end{cases}$$

Then the Lorentz spaces $L^{p,q}$ (on $\mathbb{R}^n$) are by definition the collection of all measurable functions f such that $\|f\|_{p,q} < \infty$. The case $p = \infty$, $0 < q < \infty$ is not considered since $\int\limits_0^\infty \bigl(f^*(t)\bigr)^q \dfrac{dt}{t} < \infty$ implies $f = 0$ a. e. (see [17]). Furthermore, there is an alternative representation of $\|\cdot\|_{p,q}$ (see, for example, [17, 20])

$$\|f\|_{p,q} = \begin{cases} \Bigl(q\int\limits_0^\infty y^{q-1}\lambda_f^{q/p}(y)dy\Bigr)^{1/q}, & 0 < p < \infty,\ 0 < q < \infty, \\ \sup\limits_{y>0} y\lambda_f^{1/p}(y), & 0 < p \leqslant \infty,\ q = \infty. \end{cases}$$

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In this paper, for p, q fixed, we always let r such that $0 < r \leqslant 1$, $r \leqslant q$, and $r < p$. There are two useful analogues of f^* used in some below proofs: Let (see [17])

$$f^{**}(t) = f^{**}(t,r) := \sup_{|E| \geqslant t} \left(\frac{1}{|E|} \int_E |f(x)|^r dx \right)^{1/r}, \quad t > 0.$$

Then, $(f^{**})^* = f^{**}$, and

$$(f^*)^{**}(t) = \left(\frac{1}{t} \int_0^t (f^*(y))^r dy \right)^{1/r} =: f^{***}(t), \quad t > 0.$$

It is known that f^*, f^{**} and f^{***} are non-negative, non-increasing, and

$$f^* \leqslant f^{**} \leqslant f^{***}.$$

If f^* is replaced by f^{**} or f^{***} in the expression of $\|f\|_{p,q}$ then one gets by definition $\|f\|_{p,q}^{**}$ or $\|f\|_{p,q}^{***}$ respectively. It is well-known that $\|\cdot\|_{p,q}^{**}$ is a norm when $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ (set $r = 1$ in this case), and moreover, $L^{p,q}$ can be considered as Banach spaces if and only if $p = q = 1$ or $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ (see [17]). In particular there is at that an useful relation among $\|\cdot\|_{p,q}$, $\|\cdot\|_{p,q}^{**}$ and $\|\cdot\|_{p,q}^{***}$ (see [17])

$$\|f\|_{p,q} \leqslant \|f\|_{p,q}^{**} \leqslant \|f\|_{p,q}^{***} \leqslant \big(p/(p-r)\big)^{1/r} \|f\|_{p,q}.$$

Henceforth, Ω is a compact subset of $\mathbb{R}^n$, and

$$\Delta_\nu = \big\{ \xi \in \mathbb{R}^n : |\xi_j| \leqslant \nu_j, \ j = 1, \dots, n \big\},$$

where $\nu = (\nu_1, \dots, \nu_n)$, $\nu_j > 0$, $j = 1, \dots, n$. Denote by

$$L_\Omega^{p,q} = \big\{ f \in L^{p,q} \cap S' : \operatorname{sp}(f) \subset \Omega \big\}.$$

When $\Omega = \Delta_\nu$, $L_\Omega^{p,q}$ is denoted again by $L_\nu^{p,q}$. Similarly one has S_Ω or S_ν respectively.

2. Results

First we give some results on the Bernstein—Nikolskiĭ type inequality for Lorentz spaces.

Lemma 1. *Let $0 < p_1 < p_2 \leqslant \infty$, $0 < q_1, q_2 \leqslant \infty$. Then for each multi-index α, there exists a positive constant c such that for all $\varphi \in S_\Omega$*

$$\|D^\alpha \varphi\|_{p_2,q_2} \leqslant c \|\varphi\|_{p_1,q_1}. \tag{1}$$

$\lhd$ **Step 1** ($p_2 = q_2 = \infty$ and $\alpha = (0, \dots, 0)$). Let $\psi \in S$ such that $\hat{\psi}(x) = 1$ in some neighbourhood of Ω. Then for any $x \in \mathbb{R}^n$

$$\begin{aligned} |\varphi(x)| = |\varphi * \psi(x)| &\leqslant \int_{\mathbb{R}^n} |\varphi(x-y)\psi(y)| dy \leqslant \int_0^\infty \varphi(x - \cdot)^*(t) \psi^*(t) dt \\ &= \int_0^\infty \varphi^*(t)\psi^*(t)dt \leqslant \|\varphi\|_\infty^{1-r} \int_0^\infty \big(t^{1/p_1} \varphi^*(t)\big)^r t^{-r/p_1} \psi^*(t) dt \\ &\leqslant \|\varphi\|_\infty^{1-r} \|\varphi\|_{p_1,\infty}^r \int_0^\infty t^{-r/p_1} \psi^*(t) dt = \frac{p_1}{p_1 - r} \|\psi\|_{p_1/(p_1-r),1} \|\varphi\|_\infty^{1-r} \|\varphi\|_{p_1,\infty}^r. \end{aligned}$$

This deduces at once

$$\|\varphi\|_\infty \leqslant \Big(\frac{p_1}{p_1-r}\|\psi\|_{p_1/(p_1-r),1}\Big)^{1/r}\|\varphi\|_{p_1,\infty}.$$

Step 2 ($\alpha=(0,\dots,0)$). We only have to show that there is a constant c such that

$$\|\varphi\|_{p_2,q_2} \leqslant c\|\varphi\|_{p_1,\infty}, \qquad \varphi\in S_\Omega, \tag{2}$$

where $0<p_1<p_2<\infty$, $0<q_2<\infty$.

Indeed, using the alternative representation of $\|\cdot\|_{p,q}$, we have

$$\begin{aligned}\|\varphi\|_{p_2,q_2}^{q_2} &= q_2\int_0^\infty y^{q_2-1}\lambda_\varphi^{q_2/p_2}(y)dy = q_2\int_0^{\|\varphi\|_\infty} y^{q_2-1}\lambda_\varphi^{q_2/p_2}(y)dy\\ &= q_2\int_0^{\|\varphi\|_\infty}\Big(y\lambda_\varphi^{1/p_1}(y)\Big)^{\frac{q_2}{p_2}p_1} y^{q_2-1-\frac{q_2}{p_2}p_1}dy \leqslant q_2\|\varphi\|_{p_1,\infty}^{q_2p_1/p_2}\int_0^{\|\varphi\|_\infty} y^{\frac{q_2(p_2-p_1)}{p_2}-1}dy\\ &= \frac{p_2}{p_2-p_1}\|\varphi\|_{p_1,\infty}^{q_2p_1/p_2}\|\varphi\|_\infty^{q_2(p_2-p_1)/p_2} \leqslant C\frac{p_2}{p_2-p_1}\|\varphi\|_{p_1,\infty}^{q_2},\end{aligned}$$

where the last inequality follows from Step 1. Therefore (2) is obtained.

Step 3. We prove (1) when $p_1=p_2=p$, $q_1=q_2=q$. If $\varphi\in S_\Omega$ then $D^\alpha\varphi\in S_\Omega$ for every multi-index α. Denote by $\mathcal{M}\varphi$ the Hardy—Littlewood maximal function of φ, then (see [14, p. 16]) for all $x\in\mathbb{R}^n$

$$|D^\alpha\varphi(x)| \leqslant c_1\big((\mathcal{M}|\varphi|^r)(x)\big)^{1/r},$$

where c_1 is a constant depending only on Ω. Moreover it is known that for every measurable function f (see, for example, [18, 19])

$$(\mathcal{M}f)^*(t) \sim \frac{1}{t}\int_0^t f^*(s)ds.$$

Hence,

$$(D^\alpha\varphi)^* \leqslant c_1\big((\mathcal{M}|\varphi|^r)^{1/r}\big)^* = c_1\big((\mathcal{M}|\varphi|^r)^*\big)^{1/r} \leqslant c_2\varphi^{***},$$

and consequently,

$$\|D^\alpha\varphi\|_{p,q} \leqslant c_2\|\varphi\|_{p,q}^{***} \leqslant c_3\|\varphi\|_{p,q}. \tag{3}$$

Step 4. The general case follows immediately from (2), (3) and the property $\|\cdot\|_{p,\infty}\leqslant\|\cdot\|_{p,q}$. The proof so has been fulfilled. ⊳

The theorem below is an extension of the Theorems 1.4.1(i) and 1.4.2 in [16].

Theorem 1. *Let $0<p_1<p_2\leqslant\infty$, $0<q_1,q_2\leqslant\infty$.*

(i) *If α is a multi-index, then there exists a constant c such that for all $f\in L_\Omega^{p_1,q_1}$*

$$\|D^\alpha f\|_{p_2,q_2} \leqslant c\|f\|_{p_1,q_1}.$$

(ii) *$L_\Omega^{p,q}$ is a quasi-Banach space for arbitrary $0<p,q\leqslant\infty$, and the following topological embeddings hold*

$$S_\Omega \subset L_\Omega^{p_1,q_1} \subset L_\Omega^{p_2,q_2} \subset S'.$$

$\lhd$ (i): Without loss of generality, one can assume that $q_1 = \infty$ and $0 < p_1, p_2, q_2 < \infty$ (note that the case $p_1 = \infty$ and so, $p_1 = p_2 = q_1 = q_2 = \infty$, was proved in [16, Theorem 1.4.1]). Let $p_1 < p < \infty$, and let $\varphi \in S$ such that $\varphi(0) = 1$ and $\mathrm{sp}(\varphi) \subset \{x : |x| \leqslant 1\}$. For each $f \in L_\Omega^{p_1,\infty}$ and $0 < \delta < 1$, put $f_\delta(x) = \varphi(\delta x) f(x)$. Then $f_\delta \to f$ on $\mathbb{R}^n$ and $f_\delta \in S_{\Omega_1}$, where

$$\Omega_1 = \Big\{ y \in \mathbb{R}^n : \exists\, x \in \Omega \text{ such that } |x-y| \leqslant 1 \Big\}.$$

Consequently, it follows from Lemma 1 that

$$\|f\|_p \leqslant \varliminf_{\delta\searrow 0} \|f_\delta\|_p \leqslant c_1 \varliminf_{\delta\searrow 0} \|f_\delta\|_{p_1,\infty} \leqslant c_1 \|\varphi\|_\infty \|f\|_{p_1,\infty},$$

where c_1 is independent of δ and f. Hence $f \in L_\Omega^p$. Now the argument in [16, Theorem 1.4.1] implies that $D^\alpha f_\delta \longrightarrow D^\alpha f$ in L^∞ (and this show that the conclusion is true if $p_2 = q_2 = \infty$). Lemma 1 therefore deduces again that

$$\|D^\alpha f\|_{p_2,q_2} \leqslant \varliminf_{\delta\searrow 0} \|D^\alpha f_\delta\|_{p_2,q_2} \leqslant c \varliminf_{\delta\searrow 0} \|f_\delta\|_{p_1,\infty} \leqslant c \|\varphi\|_\infty \|f\|_{p_1,\infty} \leqslant c\, \|\varphi\|_\infty \|f\|_{p_1,q_1},$$

where c depends only on p_1, p_2, q_2 and Ω.

(ii): First, we show that $L_\Omega^{p,q}$ is a quasi-Banach space for any $0 < p, q \leqslant \infty$. Let $\{f_j\}$ be any fundamental sequence in $L_\Omega^{p,q}$. Then there is a function $f \in L^{p,q}$ such that $f_j \to f$ in $L^{p,q}$ as $j \to \infty$.

Moreover, part (i) above with $\alpha = (0, \dots, 0)$ and $p_2 = q_2 = \infty$ shows that $\{f_j\}$ is also a fundamental sequence in L^∞. Then it implies by standard arguments that $f_j \to f$ in L^∞, and consequently, $f_j \to f$ in S'. Hence $\hat f_j \to \hat f$ in S' and this yields that $\mathrm{sp}(f) \subset \Omega$. Therefore $f \in L_\Omega^{p,q}$ and $f_j \to f$ in $L^{p,q}$, and it follows that $L_\Omega^{p,q}$ is a quasi-Banach space.

Part (i) deduces immediately that $L_\Omega^{p_1,q_1} \subset L_\Omega^{p_2,q_2}$. Moreover, if $0 < \theta < p < \kappa \leqslant \infty$, then for any $q > 0$ (see [16, Theorem 1.4.2])

$$S_\Omega \subset L_\Omega^\theta \subset L_\Omega^{p,q} \subset L_\Omega^\kappa \subset S'. \ \rhd$$

It is difficult to get concrete and good constants for Nikolskiĭ inequality for Lorentz spaces $L_\Omega^{p,q}$. Following some ideas in [13], we have a version of the Nikolskiĭ inequality for Lorentz spaces.

Theorem 2. **(i)** *If* $0 < p_1 < 2$, *then for* $p_2 > p_1, q_2 > 0$,

$$\|f\|_{p_2,q_2} \leqslant \Big(\frac{p_2}{p_2 - p_1}\Big)^{1/q_2} \Big(\frac{|\Omega|}{2 - p_1}\Big)^{1/p_1 - 1/p_2} \|f\|_{p_1,q_1}, \quad f \in L_\Omega^{p_1,q_1};$$

(ii) *If* $0 < p_1 < \infty$, *then for* $p_2 > p_1, q_2 > 0$,

$$\|f\|_{p_2,q_2} \leqslant \Big(\frac{p_2}{p_2 - p_1}\Big)^{1/q_2} \Big(\frac{p_0^2 |\mathrm{co}(\Omega)|}{2p_0 - p_1}\Big)^{1/p_1 - 1/p_2} \|f\|_{p_1,q_1}, \quad f \in L_\Omega^{p_1,q_1},$$

where $\mathrm{co}(\Omega)$ *denotes the convex hull of* Ω *and* p_0 *is the smallest integer number such that* $p_0 > p_1/2$.

$\lhd$ (i): Suppose that $0 < p_1 < 2$, $0 < q_1 \leqslant \infty$ and $f \in L_\Omega^{p_1,q_1}$, then by Theorem 1, $f \in L^2$, so it follows from [13, Theorem 3] that

$$\|f\|_\infty \leqslant |\Omega|^{1/2} \|f\|_2 = |\Omega|^{1/2} \Bigg(\int\limits_0^{\|f\|_\infty} y \lambda_f(y) dy \Bigg)^{1/2}$$

$$= |\Omega|^{1/2} \Bigg(\int\limits_0^{\|f\|_\infty} \big(y \lambda_f^{1/p_1}(y)\big)^{p_1} y^{1-p_1} dy \Bigg)^{1/2} \leqslant |\Omega|^{1/2} \|f\|_{p_1,\infty}^{p_1/2} \Bigg(\frac{\|f\|_\infty^{2-p_1}}{2 - p_1} \Bigg)^{1/2}.$$

Therefore,

$$\|f\|_\infty \leqslant \left(\frac{|\Omega|}{2-p_1}\right)^{1/p_1}\|f\|_{p_1,\infty}.$$

Applying now the argument in Step 2 of the proof of Lemma 1, we can obtain a similar inequality

$$\|f\|_{p_2,q_2} \leqslant \Big(\frac{p_2}{p_2-p_1}\Big)^{1/q_2}\|f\|_{p_1,\infty}^{\frac{p_1}{p_2}}\|f\|_\infty^{\frac{1-p_1}{p_2}}.$$

Hence,

$$\|f\|_{p_2,q_2} \leqslant \Big(\frac{p_2}{p_2-p_1}\Big)^{1/q_2}\Big(\frac{|\Omega|}{2-p_1}\Big)^{1/p_1-1/p_2}\|f\|_{p_1,q_1}.$$

(ii): Since $0<p_1/p_0<2$, we get immediately

$$\|f\|_{p_2,q_2} = \|f^{p_0}\|^{1/p_0}_{p_2/p_0,q_2/p_0} \leqslant \Big(\frac{p_2/p_0}{p_2/p_0-p_1/p_0}\Big)^{\frac{1}{q_2}}\Big(\frac{|\mathrm{co}(\mathrm{sp}(f^{p_0}))|}{2-p_1/p_0}\Big)^{\frac{1}{p_1}-\frac{1}{p_2}}\|f^{p_0}\|_{p_1/p_0,q_1/p_0}$$

$$\leqslant \Big(\frac{p_2}{p_2-p_1}\Big)^{\frac{1}{q_2}}\Big(\frac{p_0|\mathrm{co}(\mathrm{sp}(f))|}{2-p_1/p_0}\Big)^{\frac{1}{p_1}-\frac{1}{p_2}}\|f\|_{p_1,q_1} \leqslant \Big(\frac{p_2}{p_2-p_1}\Big)^{\frac{1}{q_2}}\Big(\frac{p_0^2|\mathrm{co}(\Omega)|}{2p_0-p_1}\Big)^{\frac{1}{p_1}-\frac{1}{p_2}}\|f\|_{p_1,q_1}.$$

The theorem is proved. $\rhd$

Lemma 2. *Let $1<p\leqslant\infty, 0<q\leqslant\infty$. If $f\in L^{p,q}$, then $f\in S'$ and for any $g\in L^1$*

$$\|f*g\|_{p,q}\leqslant c\|f\|_{p,q}\|g\|_1,$$

where c is a constant depending only on p,q.

$\lhd$ Firstly, we show that $f\in S'$. Let $E\subset\mathbb{R}^n$ such that $0<|E|<\infty$. Then the Hölder inequality implies

$$\int\limits_E |f(x)|dx \leqslant \int\limits_0^{|E|} f^*(t)dt = \int\limits_0^{|E|}\big(t^{1/p}f^*(t)\big)t^{-1/p}dt \leqslant \|f\|_{p,\infty}\int\limits_0^{|E|} t^{-1/p}dt = c(E)\|f\|_{p,\infty}.$$

This deduces easily that $f\in S'$.

Now, we prove the last conclusion. For an arbitrary $t>0$, we define

$$f^{(*)}(t) = \frac{1}{t}\int\limits_0^t f^*(y)dy.$$

Then for any $E\subset\mathbb{R}^n$ such that $t\leqslant|E|<\infty$ we have by Jensen's inequality

$$\left(\frac{1}{|E|}\int\limits_E |f*g(x)|^r dx\right)^{\frac{1}{r}} \leqslant \frac{1}{|E|}\int\limits_E |f*g(x)|dx \leqslant \int\limits_{\mathbb{R}^n}|g(y)|\left(\frac{1}{|E|}\int\limits_E |f(x-y)|dx\right)dy \leqslant f^{(*)}(t)\|g\|_1.$$

Hence,

$$\|f*g\|_{p,q}\leqslant\|f*g\|^{**}_{p,q}\leqslant\|f^{(*)}\|_{p,q}\|g\|_1.$$

It now yields from [22, Lemma 3.2] the existence of a constant c such that (in the case $p > 1$)

$$\|f^{(*)}\|_{p,q} \leqslant c\|f\|_{p,q}, \qquad f \in L^{p,q},$$

The lemma therefore is proved completely. $\rhd$

Theorem 3. *Let $f \in L^{p,q}$ $(1 < p < \infty,\ 0 < q \leqslant \infty)$ such that $f \not\equiv 0$. Then* sp(f) *contains only points of condensation.*

$\lhd$ Let $\xi_0 \in \mathrm{sp}(f)$ be an arbitrary point, and let V be any neighbourhood of ξ_0. Choose $\hat\varphi(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $\hat\varphi(\xi) = 1$ in V. Then by Lemma 2, $F^{-1}(\hat\varphi \hat f) = \varphi * f \in L^{p,q}$. Hence we can assume that sp(f) is bounded, moreover we merely have to show that sp(f) is uncountable.

It deduces from Theorem 1 that there is a positive integer m such that $f \in L^m(\mathbb{R}^n)$. Hence $(f^m)\hat{} \in C_0(\mathbb{R}^n)$. Since $f \not\equiv 0$, there exists a non-void ball B such that

$$B \subset \mathrm{sp}(f^m) = \mathrm{supp}(\hat f * \cdots * \hat f)\ (m \text{ terms}) \subset \mathrm{sp}(f) + \cdots + \mathrm{sp}(f).$$

Therefore it follows at once that sp(f) is uncountable. $\rhd$

It is noticeable that Theorem 3 is a corollary of the following theorem which can be proved by the same method used in [4, Theorem 1].

Theorem 4. *Let $f \in L^{p,q}$ $(1 < p < \infty, 0 < q \leqslant \infty)$, $f \not\equiv 0$ and $\xi_0 \in \mathrm{sp}(f)$ be an arbitrary point. Then the restriction of $\hat f$ on any neighbourhood of ξ_0 cannot concentrate on any finite number of hyperplanes.*

It is trivial that $\lambda_f(y) < \infty$ for all $y > 0$, $f \in L^{p,q}$ if $p < \infty$. Then by the argument used in [7, Theorem 3] and Theorem 1, a property of such functions can be formulated as follows.

Theorem 5. *If $f \in L^{p,q} \cap S'$ $(0 < p < \infty, 0 < q \leqslant \infty)$ such that* sp(f) *is bounded, then*

$$\lim_{|x|\to\infty} f(x) = 0.$$

Remark 1. In contrast with hyperplanes, $\hat f$ may concentrate on surfaces (see [4, Remark 2]). In addition, Theorems 3–5 are not true when $p = \infty$, i. e., $p = q = \infty$ (see [4, 7]).

To obtain more properties of functions with bounded spectrum, we prove an auxiliary result which is interesting in itself.

Theorem 6. *If $f \in L^{p,q}$ $(0 < p, q < \infty)$, then*

$$\lim_{a\to\mathbf{1}} \|f(a.x) - f(x)\|_{p,q} = 0, \tag{4}$$

where $\mathbf{1} = (1, \ldots, 1)$ and $a.x = (a_1x_1, \ldots, a_nx_n)$ for all $a, x \in \mathbb{R}^n$.

$\lhd$ It is known in [17] that the set A of all measurable simple functions with bounded support is dense in $L^{p,q}$ if $0 < q < \infty$. Therefore, it suffices to show (4) for each $f \in A$. Hence, let $f \in A$ and assume on the contrary that there exist $\{a^k\} \subset \mathbb{R}^n$, $a^k \to \mathbf{1}$, and $\varepsilon > 0$ such that

$$\|f_k - f\|_{p,q} > \varepsilon, \quad k \geqslant 1, \tag{5}$$

where $f_k(x) = f(a^k.x)$. Since $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then for each $K_\ell = [-\ell, \ell]^n$, one obtains

$$\int_{K_\ell} |f_k(x) - f(x)|dx \to 0, \quad \text{as } k \to \infty.$$

So there is a subsequence of $\{a^k\}$, which is still denoted by $\{a^k\}$, such that $f_k \to f$ a. e. on K_ℓ. Therefore, there exists a subsequence, denoted again by $\{a^k\}$, such that $f_k \to f$ a. e. on $\mathbb{R}^n$. Consequently,

$$\varliminf_{k\to\infty} f_k^*(t) \geqslant f^*(t), \quad t > 0.$$

Furthermore, it is easy to verify that

$$\|f_k\|_{p,q} = (a_1^k \cdots a_n^k)^{-1}\|f\|_{p,q}.$$

The Fatou lemma then yields for arbitrary $0 < u < v < \infty$

$$\varlimsup_{k\to\infty} \int_0^u t^{q/p-1} f_k^{*q}(t)dt = \varlimsup_{k\to\infty} \left(\int_0^\infty t^{q/p-1} f_k^{*q}(t)dt - \int_u^\infty t^{q/p-1} f_k^{*q}(t)dt \right)$$

$$\leqslant \frac{p}{q} \varlimsup_{k\to\infty} \|f_k\|_{p,q}^q - \varliminf_{k\to\infty} \int_u^\infty t^{q/p-1} f_k^{*p}(t)dt \leqslant \frac{p}{q}\|f\|_{p,q}^q - \int_u^\infty t^{q/p-1} f^{*q}(t)dt = \int_0^u t^{q/p-1} f^{*q}(t)dt,$$

and similarly,

$$\varlimsup_{k\to\infty} \int_v^\infty t^{q/p-1} f_k^{*q}(t)dt \leqslant \int_v^\infty t^{q/p-1} f^{*q}(t)dt.$$

Hence, if $u < v/2$ are chosen such that for $c = \max(2^{q-1}, 1)$

$$\int_0^u t^{q/p-1} f^{*q}(t)dt < \delta, \qquad \int_{v/2}^\infty t^{q/p-1} f^{*q}(t)dt < \delta, \tag{6}$$

where $\delta = p\varepsilon^q/(3.2^{q/p}.q.c)$, then there is a positive constant N_1 such that for all $k > N_1$

$$\int_0^u t^{q/p-1} f_k^{*q}(t)dt < \delta, \qquad \int_{v/2}^\infty t^{q/p-1} f_k^{*q}(t)dt < \delta. \tag{7}$$

Therefore, it follows from (6), (7), and the inequality $(f+g)^*(t) \leqslant f^*(t/2) + g^*(t/2)$, that for all $k > N_1$

$$\begin{aligned} \int_0^u t^{q/p-1} \big(f_k - f\big)^{*q}(t)dt &\leqslant c\left(\int_0^u t^{q/p-1} f_k^{*q}(t/2)dt + \int_0^u t^{q/p-1} f^{*q}(t/2)dt \right) \\ &\leqslant 2^{q/p-1}c\left(\int_0^u t^{q/p-1} f_k^{*q}(t)dt + \int_0^u t^{q/p-1} f^{*q}(t)dt \right) < 2^{q/p}c\delta. \end{aligned} \tag{8}$$

Similarly, one obtains for all $k > N_1$

$$\begin{aligned} \int_v^\infty t^{q/p-1} \big(f_k - f\big)^{*q}(t)dt &\leqslant c\left(\int_v^\infty t^{q/p-1} f_k^{*q}(t/2)dt + \int_v^\infty t^{q/p-1} f^{*q}(t/2)dt \right) \\ &= 2^{q/p-1}c\left(\int_{v/2}^\infty t^{q/p-1} f_k^{*q}(t)dt + \int_{v/2}^\infty t^{q/p-1} f^{*q}(t)dt \right) < 2^{q/p}c\delta. \end{aligned} \tag{9}$$

Next, since $a^k \to \mathbf{1}$ and supp$f$ is bounded, there is a ball B including supp$f$ such that supp$f_k \subset B$, for all $k \geqslant 1$. Thus taking account of $f_k \to f$ a. e. on $\mathbb{R}^n$, it deduces that $f_k \to f$ in measure. Then the definition of the non-increasing rearrangement of a measurable function yields for every $t > 0$ that

$$(f_k - f)^*(t) \longrightarrow 0, \quad \text{as } k \to \infty.$$

Applying the dominated convergence theorem, one arrives at

$$\int_u^v t^{q/p-1}(f_k - f)^{*q}(t)dt \to 0, \quad \text{as } k \to \infty.$$

Consequently, there exists a number $N_2 > N_1$ such that for all $k > N_2$

$$\int_u^v t^{q/p-1}(f_k - f)^{*q}(t)dt < \frac{p}{3q}\varepsilon^q. \tag{10}$$

Combining (8), (9) and (10), it is evident that for all $k > N_2$

$$\frac{p}{q}\|f_k - f\|_{p,q}^q = \int_0^\infty t^{q/p-1}(f_k - f)^{*q}(t)dt < 2^{q/p+1}c\delta + \frac{p}{q}\varepsilon^q/3 = \frac{p}{q}\varepsilon^q.$$

This contradicts (5). $\triangleright$

Remark 2. It is well-known that $L^{p,q}$ can be considered as Banach spaces if and only if $p = q = 1$ or $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Using Theorem 1 and the method of [14], one can obtain the Bernstein inequality for $L^{p,q}$ spaces in these cases: If $f \in L^{p,q}_\nu$, then there is a constant $1 \leqslant c \leqslant e^{1/p}$ such that

$$\|D^\alpha f\|_{p,q} \leqslant c\, \nu^\alpha \|f\|_{p,q} \tag{11}$$

holds for any multi-index α. Moreover this inequality still holds when $p = 1$. Indeed, it yields at once from the dominated convergence theorem when $p = 1, 1 \leqslant q < \infty$ that $\|f\|_{p,q} \to \|f\|_{1,q}$ as $p \searrow 1$, and the claim follows. Therefore we have only to show that this convergence is also true when $q = \infty$ and imply directly the desired. Suppose that $\|f\|_{p,\infty} \not\to \|f\|_{1,\infty}$ as $p \searrow 1$. Then there is $\epsilon > 0$ and $\{p_n\}$, $p_n \searrow 1$, such that:

Case 1. $\|f\|_{p_n,\infty} < \|f\|_{1,\infty} - \epsilon$, $n \geqslant 1$. Thus there exists $0 < u < \|f\|_\infty$ such that

$$\sup_{0<y<\|f\|_\infty} y\lambda_f^{1/p_n}(y) < u\lambda_f(u) - \epsilon/2,$$

and hence, $u\lambda_f^{1/p_n}(u) < u\lambda_f(u) - \epsilon/2$. Let $n \to \infty$, we get a contradiction.

Case 2. $\|f\|_{p_n,\infty} > \|f\|_{1,\infty} + \epsilon$, $n \geqslant 1$. Then there is a sequence $\{y_n\}$, $0 < y_n < \|f\|_\infty$ such that

$$y_n\lambda_f^{1/p_n}(y_n) > y_n\lambda_f(y_n) + \epsilon/2.$$

It is easy to see from Theorem 5 and the continuity of f that λ_f is continuous. Therefore let v be any accumulative point of $\{y_n\}$ and let $n \to \infty$ in the last inequality, we also have a contradiction and then the claim is proved.

Furthermore, using the argument in [7, Theorem 6], one can get a stronger result.

Theorem 7. *If $\nu_j > 0$, $j = 1, \dots, n$ and $1 \leqslant p, q < \infty$, then for all $f \in L^{p,q}_\nu$*

$$\lim_{|\alpha|\to\infty} \nu^{-\alpha} \|D^\alpha f\|_{p,q} = 0.$$

REMARK 3. Applying the Bernstein inequality we have $\nu^{-\alpha}\|D^\alpha f\|_{p,q} \leqslant \nu^{-\beta}\|D^\beta f\|_{p,q}$ if $\alpha \geqslant \beta$ for such above p, q. Moreover, Theorems 6, 7 fail if $p = q = \infty$. But we still don't know what happens if $p < \infty$, $q = \infty$.

Let us recall some notations about the directional derivatives. Suppose that $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ is an arbitrary real unit vector. Then

$$D_a f(x) = f'_a(x) := \sum_{j=1}^n a_j \frac{\partial f}{\partial x_j}(x)$$

is the derivative of f at the point x in the direction a, and

$$D_a^m f(x) = D_a f_a^{(m-1)} = \sum_{|\alpha|=m} a^\alpha D^\alpha f(x)$$

is the derivative of order m of f at x in the direction a ($m = 1, 2, \dots$).

Denote $h_a(f) = \sup_{\xi \in \mathrm{sp}(f)} |a\xi|$. By an argument similar to the proof of [8, Theorem 2], one can obtain the corresponding results for directional derivatives cases in certain Lorentz spaces.

Theorem 8. *If $1 \leqslant p, q \leqslant \infty$, then there is a constant $1 \leqslant c \leqslant e^{1/p}$ such that for all $f \in L^{p,q} \cap S'$ satisfying $h_a(f) < \infty$*

$$\|D_a f\|_{p,q} \leqslant c\, h_a(f) \|f\|_{p,q}. \tag{12}$$

Theorem 9. *If $f \in L^{p,q} \cap S'$ $(1 \leqslant p, q < \infty)$ is such that $h_a(f) < \infty$, then*

$$\lim_{m\to+\infty} \big(h_a(f)\big)^{-m} \|D_a^m f\|_{p,q} = 0.$$

It is clearly that one can let $c = 1$ in (11) and (12) if $\|\cdot\|_{p,q}$ is a norm, and let $c = e^{1/p}$ in general case.

Finally, we will show that the Bernstein inequality wholly characterizes the spaces $L^{p,q}_\nu$ in the case they are normable.

Theorem 10. *Suppose that $p = q = 1$ or $1 < p \leqslant \infty, 1 \leqslant q \leqslant \infty$ and $f \in S'$. Then in order that $f \in L^{p,q}_\nu$ it is necessary and sufficient that there exists a constant $c = c(f)$ such that*

$$\|D^\alpha f\|_{p,q} \leqslant c\,\nu^\alpha, \quad \alpha \in \mathbb{Z}^n_+. \tag{13}$$

$\lhd$ Only sufficiency hod to be verified. Assume that (13) holds.

Case 1 $(1 < p < \infty, 1 \leqslant q \leqslant \infty)$. If $g \in L^{p,q}(\mathbb{R}^n)$, then $g \in L^1_{loc}(\mathbb{R}^n)$ by the first part of the proof of Lemma 2. It hence deduces from (13) that $D^\alpha f \in L^1_{loc}(\mathbb{R}^n)$ for all $\alpha \geqslant 0$. Consequently, we can assume that $f \in C^\infty(\mathbb{R}^n)$ by virtue of Sobolev embedding theorem.

Next let $\omega \in C_0^\infty(\mathbb{R}^n)$ such that $\|\omega\|_1 = 1$, and define for each $\varepsilon > 0$

$$f_\varepsilon(x) = f * \omega_\varepsilon(x),$$

where $\omega_\varepsilon(x) = \varepsilon^{-n}\omega(x/\varepsilon)$. Then $f_\varepsilon(x) \to f(x)$ *as* $\varepsilon \downarrow 0$, for every $x \in \mathbb{R}^n$. Moreover, by the argument at the first step of Lemma 1 (recall that $r = 1$ in this case), one has for each multi-index α

$$\sup_{x\in\mathbb{R}^n} |D^\alpha f_\varepsilon(x)| \leqslant b_\varepsilon \|D^\alpha f_\varepsilon\|_{p,\infty} \leqslant b_\varepsilon \|D^\alpha f_\varepsilon\|_{p,q} \leqslant B_\varepsilon\ \nu^\alpha, \tag{14}$$

where $B_\varepsilon > 0$ is a constant depending only on ε. Thus the Taylor series

$$\sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} D^\alpha f_\varepsilon(0).z^\alpha$$

converges for any point $z \in \mathbb{C}^n$ and represents $f_\varepsilon(x)$ in $\mathbb{R}^n$. Hence taking account of (14), we obtain

$$|f_\varepsilon(z)| \leqslant B_\varepsilon \exp\left(\sum_{j=1}^{n} \nu_j |z_j|\right), \quad z \in \mathbb{C}^n,$$

i. e., $f_\varepsilon(z)$ is an entire function of exponential type ν. It therefore follows from the Paley–Wiener–Schwartz theorem that

$$\mathrm{sp}(f_\varepsilon) = \mathrm{supp}\,\hat{f}_\varepsilon \subset \Delta_\nu. \tag{15}$$

Therefore, Theorem 1 and Lemma 2 yield that for each $\varepsilon > 0$

$$\|f_\varepsilon\|_{p+1} \leqslant c_1\ \|f_\varepsilon\|_{p,\infty} \leqslant c_2\ \|\omega_\varepsilon\|_1 \|f\|_{p,\infty} = c_2\ \|f\|_{p,\infty}.$$

The Banach–Alaoglu theorem hence implies that there are a sequence $\{\varepsilon_n\}$ and an $\widetilde{f} \in L^{p+1}(\mathbb{R}^n)$ such that $f_{\varepsilon_n} \to \widetilde{f}$ weakly in $L^{p+1}(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$. Then by standard arguments, one has $f = \widetilde{f}$ a. e., that is, $f_{\varepsilon_n} \to f$ weakly in $L^{p+1}(\mathbb{R}^n)$. Because $S \subset L^{(p+1)/p}(\mathbb{R}^n)$, the dual space of $L^{p+1}(\mathbb{R}^n)$, it follows immediately that $f_{\varepsilon_n} \to f$ in S'. Consequently, $\hat{f}_{\varepsilon_n} \to \hat{f}$ in S' and this deduces at once from (15) that $\mathrm{sp}(f) \subset \Delta_\nu$.

Case 2 ($p = q = 1$). This case can be proved by above manner.

Case 3 ($p = q = \infty$). Let φ and f_δ, $0 < \delta < 1$, as in the proof of Theorem 1. Then it yields from the Leibniz formula, the Bernstein inequality for L^∞ and (13) that for all $\alpha \in \mathbb{Z}_+^n$

$$|D^\alpha f_\delta(x)| \leqslant \sum_{\gamma+\beta=\alpha} \left|D^\gamma(\varphi(\delta x))\right| \left|D^\beta f(x)\right| \leqslant c \sum_{\gamma+\beta=\alpha} \delta^{|\gamma|}\nu^\beta = c(\nu + \boldsymbol{\delta})^\alpha,$$

where $\boldsymbol{\delta} = (\delta, ..., \delta)$. Thus, as in Case 1, $f_\delta(z)$ is an entire function of exponential type $\nu + \boldsymbol{\delta}$ for each $0 < \delta < 1$, and therefore, $\mathrm{sp}(f_\delta) \subset \Delta_{\nu+\boldsymbol{\delta}}$. Moreover, it is clear that $f_\delta \to f$ in S' as $\delta \downarrow 0$. This implies obviously that $\mathrm{sp}(f) \subset \Delta_{\nu+\boldsymbol{\theta}}$ for any $0 < \theta < 1$ and then $\mathrm{sp}(f) \subset \Delta_\nu$. $\rhd$

Theorem 11. *If* $p = q = 1$ *or* $1 < p \leqslant \infty, 1 \leqslant q \leqslant \infty$, *then a function* $f \in S'$ *belongs to* $L_\nu^{p,q}$ *if and only if*

$$\varlimsup_{|\alpha|\to\infty} \left(\nu^{-\alpha}\|D^\alpha f\|_{p,q}\right)^{1/|\alpha|} \leqslant 1. \tag{16}$$

$\lhd$ It is sufficient to prove «only if» part. Given any $\varepsilon > 0$, there is a positive constant $C_\varepsilon > 0$ such that for all $\alpha \geqslant 0$

$$\|D^\alpha f\|_{p,q} \leqslant C_\varepsilon (1+\varepsilon)^{|\alpha|}\nu^\alpha.$$

It hence deduces from Theorem 10 that $\mathrm{sp}(f) = \mathrm{supp}\,Ff \subset \Delta_{(1+\varepsilon)\nu}$. Therefore $\mathrm{sp}(f) \subset \bigcap_{\varepsilon>0} \Delta_{(1+\varepsilon)\nu} = \Delta_\nu$. $\rhd$

Remark 4. It is noticeable that the root $1/|\alpha|$ in (16) cannot be replaced by any $1/|\alpha|\ t(\alpha)$, where $0 < t(\alpha)$, $\lim\limits_{|\alpha|\to\infty} t(\alpha) = +\infty$.

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Received by the editors January 13, 2005.

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