Directed graphs related to association schemes

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A $d$-class association scheme consists of (relations with adjacency) $\{0, 1\}$ matrices $A_0 = I, A_1, \ldots, A_d$ satisfying

- $A_0 + A_1 + \ldots + A_d = J$

- for each $i$ there exists $i'$ so that $A^t_i = A_{i'}$

- there exist intersection numbers $p^h_{ij}$ so that

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h$$

Each $A_i, i > 0$ is adjacency matrix of a directed or undirected graph.
The association scheme is said to primitive if all these graphs are connected. Otherwise it is imprimitive.
In a symmetric 2-class association scheme $A_1$ is adjacency matrix of a strongly regular graph satisfying

$$A_1^2 = p_{11}^0 A_0 + p_{11}^1 A_1 + p_{11}^2 A_2 = kI + \lambda A_1 + \mu(J - I - A_1),$$

and $A_2$ is adjacency matrix of complementary graph (also strongly regular).

Strongly regular graph means:

- every vertex has degree $k$
- two adjacent vertices have $\lambda$ common neighbours
- two non-adjacent vertices have $\mu$ common neighbours
In a non-symmetric 2-class association scheme $A_1$ is the adjacency matrix of doubly regular tournament:

A tournament (orientation of complete graph) of order $n$ is called doubly regular if every vertex has in-degree and out-degree $k$, and

The number of directed paths of length 2 from $x$ to $y$ is

$$
\begin{cases}
\lambda & \text{if } x \to y, \\
\lambda + 1 & \text{if } x \leftarrow y.
\end{cases}
$$

\[ n = 2k + 1 = 4\lambda + 3. \]

**Paley (1933).** If $n = 4\lambda + 3$ is a prime power then there is a doubly regular tournament on vertices.
An Hadamard matrix of order $n$ is an $n \times n$ \{1, −1\} matrix $H$ such that $HH^t = nI$.

An Hadamard matrix is called skew if $H + H^t = 2I$.

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1
\end{pmatrix}
\]

Reid and Brown (1972)
There exists a doubly regular tournament of order $n$ if and only if there exists a skew Hadamard matrix of order $n + 1$. 
Duval 1988 considered what he called directed strongly regular graphs.

These graphs have adjacency matrix $A$ satisfying

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

But other directed graph variations of strongly regular graphs could be natural. In particular, we can have equations involving $A^t$. 
Non-symmetric 3-class association schemes.

Matrices:

\[ A_0 = I \]
\[ A_1 = A \]
\[ A_2 = A^t \]
\[ A_3 = J - I - A - A^t \]

\( A_3 \) is adjacency matrix of a strongly regular graph. 
\( A_1 \) and \( A_2 \) are adjacency matrices of opposite orientations of the complementary strongly regular graph.
\[ AJ = JA = \kappa J \quad (1) \]

\[ AA^t = \kappa I + \lambda (A + A^t) + \mu (J - I - A - A^t) \quad (2) \]

\[ A^t A = \kappa I + \lambda (A + A^t) + \mu (J - I - A - A^t) \quad (3) \]

\[ A^2 = \alpha A + \beta A^t + \gamma (J - I - A - A^t), \quad (4) \]

where \( \kappa = p_{12}^0, \lambda = p_{12}^1 = p_{21}^1, \mu = p_{12}^3 = p_{21}^3, \alpha = p_{11}^1, \beta = p_{11}^2, \gamma = p_{11}^3 \).
(1): every vertex has in-degree and out-degree $\kappa$.
(2/3): the number of common out-/in- neighbours of $x$ and $y$ is exactly

$$\begin{cases} 
\lambda & \text{if } x \to y \text{ or } x \leftarrow y \\
\mu & \text{otherwise.}
\end{cases}$$

(4): the number of directed paths of length 2 from $x$ to $y$ is exactly

$$\begin{cases} 
\alpha & \text{if } x \to y, \\
\beta & \text{if } x \leftarrow y, \\
\gamma & \text{otherwise.}
\end{cases}$$
Imprimitive case

Assume:
(Directed) graph with matrix $A_1$ is connected,
Graph with matrix $A_3$ disconnected.

Vertices partitioned in $m$ blocks of size $r$.
Directed graph with matrix $A_1$ is an orientation the complete $m$-partite graph $K_{r,...,r}$. 

Conjecture. Every orientation of the complete $m$-partite graph $K_{r,...,r}$ with adjacency matrix $A$ satisfying

$$AJ = JA = \kappa J$$

and

$$AA^t = A^tA = \kappa I + \lambda (A + A^t) + \mu (J - I - A - A^t)$$

is a relation of a non-symmetric 3-class association scheme.

Such a matrix has 4 or 5 distinct eigenvalues. Conjecture is true for 4 eigenvalues.
Definition (JJKS)

A directed graph $\Gamma$ with adjacency matrix $A$ is said to be a doubly regular $(m, r)$-team tournament if

- $\Gamma$ is an orientation of the complete $m$-partite graph $K_{r, \ldots, r}$.

- $AJ = JA = \kappa J$

- $A^2 = \alpha A + \beta A^t + \gamma (J - I - A - A^t)$. 
Theorem (JJKS)

Every doubly regular \((m, r)\)-team tournament is of one of the following three types.

Type 1 A cocliqueextension of a doubly regular tournament.
   Association scheme.

Type 2 For every vertex \(x \in V_i\), exactly half of the vertices in \(V_j\) \((j \neq i)\) are out-neighbours of \(x\), and \(\alpha = \beta = \frac{(m-2)r}{4}\), and \(\gamma = \frac{(m-1)r^2}{4(r-1)}\).
   These graphs are association schemes.
   Necessary condition:
   \(r - 1\) divides \(m - 1\). \(m\) and \(r\) are even.
Type 3 For every pair \( \{i, j\} \) either \( V_i \) is partitioned in two sets \( V_i^' \) and \( V_i^'' \) of size \( \frac{r}{2} \) so that all edges between \( V_i \) and \( V_j \) are directed from \( V_i^' \) to \( V_j \) and from \( V_j \) to \( V_i^'' \), or similarly with \( i \) and \( j \) interchanged.

The parameters are
\[
\alpha = \frac{(m-1)r}{4} - \frac{3r}{8}, \quad \beta = \frac{(m-1)r}{4} + \frac{r}{8}, \quad \gamma = \frac{(m-1)r^2}{8(r-1)}.
\]

Not association scheme.
No examples are known.
8 divides \( r \) and \( 4(r - 1) \) divides \( m - 1 \).
Smallest possible case: \( r = 8, m = 29 \).

The results for association schemes was first proved by Goldbach and Claasen (1996).
Type 2

Three cases:

Type 2a: $r = m$ even

Type 2b: $r = 2, m \equiv 0 \mod 4$

Type 2c: $4 \leq r < m$, $r - 1$ divides $m - 1$, $m$ and $r$ are even. There exist examples with $r = 4, m = 16$. 
Type 2a: \( r = m \)

An Hadamard matrix \( H \) of order \( m^2 \) (\( m \) even) is said to be Bush-type if \( H \) is block matrix with \( m \times m \) blocks \( H_{ij} \) of size \( m \times m \) such that

\[
H_{ii} = J_m
\]

and

\[
H_{ij}J_m = J_mH_{ij} = 0, \quad \text{for} \ i \neq j.
\]

\[
\begin{pmatrix}
1 & 1 & 1 & - \\
1 & 1 & - & 1 \\
-1 & 1 & 1 & 1 \\
1 & - & 1 & 1 \\
\end{pmatrix}
\]
Theorem
A doubly regular \((m, m)\)-team tournament of type 2 is equivalent to
a Bush-type Hadamard matrix of order \(m^2\) with the property that
\(H_{ij} = -H_{ji}^t\) for all pairs \(i, j\) with \(i \neq j\).
$m \equiv 0 \mod 4$:

If there exists a Hadamard matrix of order $m$
then there exists
a Bush-type Hadamard matrix of order $m^2$.

**Ionin and Kharaghani (2003)** This construction can be modified so that the Bush-type Hadamard matrix is “skew”.
$m \equiv 2 \mod 4$:


**Theorem (J 2009)** There exists “skew” Bush-type Hadamard matrices of order 36.

In an incomplete computer search I found 4 association schemes with $r = m = 6$. These four schemes all have trivial automorphism groups.
Doubly regular \((m,m)\)-team tournament of type 2 are also known to be DRAD (doubly regular asymmetric digraphs) as

\[ AA^t = aI + bJ \]

N. Ito 1987 claimed that imprimitive DRAD’s can not have rank 4 automorphism group, except for \(m = 4\).

J. 2010 found a counterexample with \(m = 8\)

**Davis and Polhill 2010** There exist such DRAD’s (doubly regular \((m,m)\)-team tournament of type 2) with a rank 4 automorphism group when \(m\) is power of 2.
Type 2b

\( r = 2 \): The condition \( r - 1 \) divides \( m - 1 \) is satisfied for every (even) \( m \).

It can be proved that \( m \) is multiple of 4.
Let $T$ be a tournament with adjacency matrix $A$. Then graph with adjacency matrix

$$
\begin{pmatrix}
0 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & 1 \\
\vdots & A & \vdots & A^t \\
0 & 1 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 1 \\
1 & 0 \\
\vdots & A^t & \vdots & A \\
1 & 0
\end{pmatrix}
$$

is denoted by $D(T)$.

**Theorem**
Let $\Gamma$ be a doubly regular $(m,2)$-team tournament of type 2. Then $\Gamma$ is isomorphic to $D(T)$ for some doubly regular tournament $T$. 
\(\mathcal{D}(T)\) has a unique automorphism of order 2. It interchanges pairs of non-adjacent vertices.

**Theorem**

Suppose that for some doubly regular tournament \(T\), the graph \(\Gamma = \mathcal{D}(T)\) is vertex transitive and has automorphism group \(G\). Then the Sylow 2-subgroup \(S\) of \(G\) is the generalized quarternion group of order \(2^n\), where \(2^n\) is the highest power of 2 that divides the order of \(\Gamma\).

**Theorem**

Suppose that for some doubly regular tournament \(T\), the graph \(\Gamma = \mathcal{D}(T)\) is vertex transitive and has order \(2^n\). Then \(\Gamma\) is a Cayley graph of the generalized quarternion group of order \(2^n\).
Theorem

Let \( q \equiv 3 \mod 4 \) be a prime power and \( P_q \) be the Paley tournament of order \( q \). Let \( \Gamma = \mathcal{D}(P_q) \).

Then \( SL(2, q) \) acts as a group of automorphisms on \( \Gamma \) and this group contains a dicyclic subgroup acting regularly on \( V(\Gamma) \).

The dicyclic group of order \( 4n \):

\[
\langle x, y \mid x^n = y^2, y^4 = 1, yxy^{-1} = x^{-1} \rangle
\]