On permutation tableaux of type $A$ and $B$

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(Joint work with Sylvie Corteel)

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First introduced by Postnikov in his study of totally nonnegative Grassmanian.
Permutation tableaux

- First introduced by Postnikov in his study of totally nonnegative Grassmannian.
- There are many bijections between permutation tableaux and permutations.
Permutation tableaux

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- There are many bijections between permutation tableaux and permutations.
- A connection with partially asymmetric exclusion process (PASEP)
Permutation tableaux

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- There are many bijections between permutation tableaux and permutations.
- A connection with partially asymmetric exclusion process (PASEP)
- Type $B$ Permutation tableaux defined by Lam and Williams
Ferrers diagram
Ferrers diagram
Permutation tableau

- Each column has at least one 1.
Permutation tableau

- Each column has at least one 1.
- There is no configuration like

\[
1 \quad \vdots \\
1 \quad \cdots 
\]
Permutation tableau

- Each column has at least one 1.
- There is no configuration like

```
1
.
.
1  ...  0
```
Permutation tableau

- Each column has at least one 1.
- There is no configuration like

\[
\begin{array}{cccc}
1 & \cdots & 0 \\
\vdots \\
1 & \cdots & 0 \\
\end{array}
\]
Permutation tableau

- Each column has at least one 1.
- There is no configuration like

\[ \begin{array}{ccccccc}
1 & \cdots & 0 \\
\vdots \\
1 & \cdots & 0 \\
\end{array} \]

\[ \begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 \\
1 \\
0 \\
\end{array} \]

\[ \begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 \\
1 \\
0 \\
\end{array} \]

NO

NO

NO
Permutation tableau

- Each column has at least one 1.
- There is **no** configuration like

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</tbody>
</table>
```

NO  NO  YES!
Permutation tableau

- Each column has at least one 1.
- There is no configuration like

\[
\begin{array}{ccc}
1 & & \\
\vdots & & \\
1 & \cdots & 0
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 1 & & & \\
1 & & & & & \\
0 & & & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & \\
0 & 1 & 0 & 0 & 1 & \\
1 & & & & & \\
0 & & & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & \\
0 & 1 & 1 & & & \\
1 & & & & & \\
0 & & & & & \\
\end{array}
\]

- A restricted 0 is

\[
\begin{array}{ccc}
1 & & \\
\vdots & & \\
0 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & & & 1 \\
0 & & & 0 \\
1 & & & 0 \\
1 & & & 0 \\
0 & & & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & & & 1 \\
0 & & & 0 \\
1 & & & 0 \\
1 & & & 0 \\
0 & & & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & & & 1 \\
0 & & & 0 \\
1 & & & 0 \\
1 & & & 0 \\
0 & & & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & & & 1 \\
0 & & & 0 \\
1 & & & 0 \\
1 & & & 0 \\
0 & & & 0 \\
\end{array}
\]

YES!
Permutation tableau

- Each column has at least one 1.
- There is no configuration like

\[
\begin{array}{cccc}
1 & \cdots & 0 \\
1 & \cdots & 0 \\
\vdots & & \vdots \\
0 & & 0 \\
\end{array}
\]

- A restricted 0 is

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

- An unrestricted row has no restricted 0.

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

YES!
The alternative representation

- Topmost 1 is ↑
The alternative representation

- Topmost 1 is \( \uparrow \)
- Rightmost restricted 0 is \( \leftarrow \)
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ←

In the alternative representation,
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ←

In the alternative representation,
- Each column has exactly one ↑
The alternative representation

- Topmost 1 is \( \uparrow \)
- Rightmost restricted 0 is \( \downarrow \)

- In the alternative representation,
  - Each column has exactly one \( \uparrow \)
  - No arrow points to another.
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ⇐

In the alternative representation,
- Each column has exactly one ↑
- No arrow points to another.
- Unrestricted row ⇔ row without ⇐
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ↔

In the alternative representation,
- Each column has exactly one ↑
- No arrow points to another.
- Unrestricted row ⇔ row without ↔
- First introduced by Viennot (alternative tableau) and studied more by Nadeau
The bijection $\Phi$ of Corteel and Nadeau

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$1, 10, 11, 13$
The bijection $\Phi$ of Corteel and Nadeau

1, 12, 10, 11, 13
The bijection $\Phi$ of Corteel and Nadeau

$9, 1, 12, 10, 11, 13$
The bijection $\Phi$ of Corteel and Nadeau

9, 2, 7, 8, 1, 12, 10, 11, 13
The bijection $\Phi$ of Corteel and Nadeau

9, 4, 6, 2, 7, 8, 1, 12, 10, 11, 13
The bijection $\Phi$ of Corteel and Nadeau

9, 4, 6, 5, 2, 7, 8, 1, 12, 10, 11, 13
The bijection $\Phi$ of Corteel and Nadeau

$9, 4, 6, 5, 2, 7, 8, 3, 1, 12, 10, 11, 13$
Decompose $\pi$ as $\sigma 1 \tau$.

\[
\pi = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13
\]
Decompose $\pi$ as $\sigma \tau$.

$\pi = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$

The **RL-minima** (right-to-left mimina) of $\pi$:

$4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$
Decompose $\pi$ as $\sigma 1\tau$.

$$\pi = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$$

The **RL-minima** (right-to-left mimina) of $\pi$:

4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13

The **RL-maxima** (right-to-left maxima) of $\sigma$:

$$\sigma$$

4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13
Decompose $\pi$ as $\sigma_1\tau$.

The \textbf{RL-minima} (right-to-left mimina) of $\pi$:

$4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$

The \textbf{RL-maxima} (right-to-left maxima) of $\sigma$:

$4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$

Proposition (Corteel & Nadeau, Nadeau)

Let $\pi = \sigma_1\tau$ and $\Phi(\pi) = T$. Then
Decompose $\pi$ as $\sigma_1\tau$.

\[ \pi = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13 \]

The **RL-minima** (right-to-left mimina) of $\pi$:

4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13

The **RL-maxima** (right-to-left maxima) of $\sigma$:

4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13

**Proposition (Corteel & Nadeau, Nadeau)**

Let $\pi = \sigma_1\tau$ and $\Phi(\pi) = T$. Then

- the unrestricted rows of $T \iff$ the RL-minima of $\pi$
Decompose $\pi$ as $\sigma \tau$.

$$\pi = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$$

The **RL-minima** (right-to-left mimina) of $\pi$:

$4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$

The **RL-maxima** (right-to-left maxima) of $\sigma$:

$$\sigma = 4, 6, 5, 2, 8, 3, 1, 9, 7, 12, 10, 11, 13$$

Proposition (Corteel & Nadeau, Nadeau)

Let $\pi = \sigma \tau$ and $\Phi(\pi) = T$. Then

- the unrestricted rows of $T$ $\Leftrightarrow$ the RL-minima of $\pi$
- the columns with 1 in the first row of $T$ $\Leftrightarrow$ the RL-maxima of $\sigma$
Nadeau’s bijective proof of a theorem of Corteel and Nadeau

Theorem

\[ \sum_{T \in \mathcal{PT}(n)} x^{\text{urr}(T)-1} y^{\text{topone}(T)} = (x + y)_{n-1} = (x + y)(x + y + 1) \cdots (x + y + n - 2). \]
Nadeau’s bijective proof of a theorem of Corteel and Nadeau

Theorem

$$\sum_{T \in \mathcal{PT}(n)} x^{\text{urr}(T) - 1} y^{\text{topone}(T)} = (x + y)_{n-1} = (x + y)(x + y + 1) \cdots (x + y + n - 2).$$

- $c(n, k)$: the number $\pi \in S_n$ with $k$ cycles

$$c(n, k) = \sum_{i, j} c(n - 1, i + j) \binom{i + j}{i} x^i y^j$$

$$\# \{ T \in \mathcal{PT}(n) : \text{urr}(T) - 1 = i, \ \text{topone}(T) = j \} = c(n - 1, i + j) \binom{i + j}{i}$$
Nadeau’s bijective proof of a theorem of Corteel and Nadeau

Theorem

\[ \sum_{T \in \mathcal{PT}(n)} x^{\text{urr}(T) - 1} y^{\text{topone}(T)} = (x + y)^{n-1} = (x + y)(x + y + 1) \cdots (x + y + n - 2). \]

- \( c(n, k) \): the number \( \pi \in S_n \) with \( k \) cycles

\[ (x + y)^{n-1} = \sum_{i, j} c(n - 1, i + j) \binom{i + j}{i} x^i y^j \]

\#\{T ∈ \mathcal{PT}(n) : \text{urr}(T) - 1 = i, \text{topone}(T) = j\} = c(n - 1, i + j) \binom{i + j}{i}

- If \( T \leftrightarrow \pi = \sigma 1 \tau \),

\text{urr}(T) - 1 = \text{RLmin}(\tau) = i
\text{topone}(T) = \text{RLmax}(\sigma) = j
Nadeau’s bijective proof of a theorem of Corteel and Nadeau

**Theorem**

\[
\sum_{T \in \mathcal{P}T(n)} x^{\text{urr}(T) - 1} y^{\text{topone}(T)} = (x + y)_{n-1} = (x + y)(x + y + 1) \cdots (x + y + n - 2).
\]

- \( c(n, k) \): the number \( \pi \in S_n \) with \( k \) cycles

\[
(x + y)_{n-1} = \sum_{i,j} c(n - 1, i + j) \binom{i + j}{i} x^i y^j
\]

\[
\#\{T \in \mathcal{P}T(n) : \text{urr}(T) - 1 = i, \ \text{topone}(T) = j\} = c(n - 1, i + j) \binom{i + j}{i}
\]

- If \( T \leftrightarrow \pi = \sigma 1 \tau \),

\[
\text{urr}(T) - 1 = \text{RLmin}(\tau) = i \\
\text{topone}(T) = \text{RLmax}(\sigma) = j
\]

- \( \tau \) is a set of \( i \) cycles and \( \sigma \) is a set of \( j \) cycles.
Nadeau’s bijective proof of a theorem of Corteel and Nadeau

Theorem

\[ \sum_{T \in \mathcal{PT}(n)} x^\text{urr}(T) - 1 y^\text{topone}(T) = (x + y)^{n-1} = (x + y)(x + y + 1) \cdots (x + y + n - 2). \]

- \( c(n, k) \): the number \( \pi \in S_n \) with \( k \) cycles

\[ (x + y)^{n-1} = \sum_{i,j} c(n - 1, i + j) \binom{i+j}{i} x^i y^j \]

\[ \#\{T \in \mathcal{PT}(n) : \text{urr}(T) - 1 = i, \text{topone}(T) = j\} = c(n - 1, i + j) \binom{i+j}{i} \]

- If \( T \leftrightarrow \pi = \sigma 1 \tau \),

\[ \text{urr}(T) - 1 = \text{RLmin}(\tau) = i \]
\[ \text{topone}(T) = \text{RLmax}(\sigma) = j \]

- \( \tau \) is a set of \( i \) cycles and \( \sigma \) is a set of \( j \) cycles.
- \( \tau \cup \sigma \) is a permutation of \( \{2, 3, \ldots, n\} \) with \( i + j \) cycles.
Another bijective proof

**Theorem**

*We have*

\[ \sum_{T \in \mathcal{P}T(n)} x^{\text{urr}(T)} - 1 y^{\text{topone}(T)} = (x + y)^{n-1}. \]
Another bijective proof

**Theorem**

*We have*

\[
\sum_{T \in \mathcal{P}T(n)} x^{u_{\text{rr}}(T)-1} y^{\text{topone}(T)} = (x + y)_{n-1}.
\]

- Let \(x, y\) be any positive integers and let \(N = n + x + y - 2\).
Another bijective proof

Theorem

We have

\[ \sum_{T \in \mathcal{P}T(n)} x^{\text{urr}(T) - 1} y^{\text{topone}(T)} = (x + y)^{n - 1}. \]

- Let \( x, y \) be any positive integers and let \( N = n + x + y - 2. \)
- Given \( T \in \mathcal{P}T(n), \) we construct \( T' \in \mathcal{P}T(N) \) as follows.
Another bijective proof

\[ \begin{align*}
T' &\text{ satisfies the following.} \\
x - 1 &\quad y - 1
\end{align*} \]
Another bijective proof

\[ \{ x - 1 \} \times \{ y - 1 \} \]

\[ T' \] satisfies the following.

- The first \( y \) steps are south and the first \( y \) rows are unrestricted.
Another bijective proof

$T'$ satisfies the following.

1. The first $y$ steps are south and the first $y$ rows are unrestricted.
2. The last $x - 1$ steps are west and the last $x - 1$ columns have ↑’s in the first row.
Another bijective proof

$T'$ satisfies the following.

1. The first $y$ steps are south and the first $y$ rows are unrestricted.
2. The last $x - 1$ steps are west and the last $x - 1$ columns have ↑’s in the first row.

$\pi' = \Phi(T')$ satisfies the following. ($\pi' = \sigma 1\tau$)
Another bijective proof

\( T' \) satisfies the following.
1. The first \( y \) steps are south and the first \( y \) rows are unrestricted.
2. The last \( x - 1 \) steps are west and the last \( x - 1 \) columns have ↑’s in the first row.

\( \pi' = \Phi(T') \) satisfies the following. \( (\pi' = \sigma 1\tau) \)
1. 1, 2, \ldots, \( y \) are RL-minima of \( \pi' \)
Another bijective proof

$\{x - 1\} \{y - 1\}$

$\pi' = \Phi(T')$ satisfies the following. ($\pi' = \sigma \tau$)

1. $1, 2, \ldots, y$ are RL-minima of $\pi'$
2. $N, N - 1, \ldots, N - x + 2$ are RL-maxima of $\sigma$
Another bijective proof

\[ T' \] satisfies the following.

1. The first \( y \) steps are south and the first \( y \) rows are unrestricted.
2. The last \( x - 1 \) steps are west and the last \( x - 1 \) columns have ↑'s in the first row.

\[ \pi' = \Phi(T') \] satisfies the following. \((\pi' = \sigma 1 \tau)\)

1. 1, 2, \ldots, \( y \) are RL-minima of \( \pi' \)
2. \( N, N - 1, \ldots, N - x + 2 \) are RL-maxima of \( \sigma \)

\( N, N - 1, \ldots, N - x + 2, 1, 2, \ldots, y \) are arranged in this order in \( \pi' \).
An unrestricted column of $T \in \mathcal{PT}(n)$ is a column without 0 as follows.

\[
1 \quad \ldots \quad 0
\]
An **unrestricted column** of $T \in \mathcal{PT}(n)$ is a column without 0 as follows.

\[
\begin{array}{cccc}
1 & \cdots & 0 \\
\end{array}
\]

$\text{urc}(T)$ : the number of unrestricted columns of $T$. 
An unrestricted column of $T \in \mathcal{PT}(n)$ is a column without 0 as follows.

$$1 \quad \cdots \quad 0$$

$\text{urc}(T)$: the number of unrestricted columns of $T$.

Let

$$P_t(x) = \sum_{n \geq 0} \left( \sum_{T \in \mathcal{PT}(n)} t^{\text{urc}(T)} \right) x^n.$$
Unrestricted Columns

- An unrestricted column of $T \in \mathcal{PT}(n)$ is a column without 0 as follows.
  
  \[
  \begin{array}{cccc}
  1 & \cdots & 0 \\
  \end{array}
  \]

- $\text{urc}(T)$: the number of unrestricted columns of $T$.

Let

\[
P_t(x) = \sum_{n \geq 0} \left( \sum_{T \in \mathcal{PT}(n)} t^{\text{urc}(T)} \right) x^n.
\]

**Theorem**

*We have*

\[
P_t(x) = \frac{1 + E_t(x)}{1 + (t - 1)x E_t(x)},
\]

*where*

\[
E_t(x) = \sum_{n \geq 1} n(t)_{n-1} x^n.
\]
The case $t = 2$ : connected permutations

Corollary

\[ \sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} x^n = \frac{1}{x} \left( 1 - \frac{1}{\sum_{n \geq 0} n! x^n} \right). \]
The case $t = 2$ : connected permutations

Corollary

\[
\sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} x^n = \frac{1}{x} \left( 1 - \frac{1}{\sum_{n \geq 0} n! x^n} \right).
\]

- $\pi = \pi_1 \cdots \pi_n \in S_n$ is a connected permutation if there is no $k < n$ satisfying $\pi_1 \cdots \pi_k \in S_k$. 
The case $t = 2$ : connected permutations

**Corollary**

$$
\sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} x^n = \frac{1}{x} \left( 1 - \frac{1}{\sum_{n \geq 0} n!x^n} \right).
$$

- $\pi = \pi_1 \cdots \pi_n \in S_n$ is a **connected permutation** if there is no $k < n$ satisfying $\pi_1 \cdots \pi_k \in S_k$.

- $CP(n)$ : the set of connected permutations in $S_n$

$$
\sum_{n \geq 0} \#CP(n)x^n = 1 - \frac{1}{\sum_{n \geq 0} n!x^n}
$$
The case \( t = 2 \) : connected permutations

Corollary

\[
\sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} x^n = \frac{1}{x} \left( 1 - \frac{1}{\sum_{n \geq 0} n!x^n} \right).
\]

- \( \pi = \pi_1 \cdots \pi_n \in S_n \) is a **connected permutation** if there is no \( k < n \) satisfying \( \pi_1 \cdots \pi_k \in S_k \).

- \( \text{CP}(n) \) : the set of connected permutations in \( S_n \)

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Corollary

\[
\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} = \#\text{CP}(n + 1).
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The case $t = 2$: connected permutations

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Corollary

$$\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} = \#CP(n + 1).$$

- Combinatorial proof?
A **shift-connected permutation** is \( \pi = \pi_1 \cdots \pi_n \in S_n \) with \( \pi_j = 1 \) for some \( j \in [n] \) such that
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**SCP($n$) :** the set of shift-connected permutations in $S_n$

**Proposition**

$$\#CP(n) = \#SCP(n)$$
A bijection between $S_n \setminus CP(n)$ and $S_n \setminus SCP(n)$

- Given $\pi = \pi_1 \cdots \pi_n \in S_n \setminus CP(n)$, define $\pi' \in S_n \setminus SCP(n)$ as follows.
A bijection between $S_n \setminus CP(n)$ and $S_n \setminus SCP(n)$

- Given $\pi = \pi_1 \cdots \pi_n \in S_n \setminus CP(n)$, define $\pi' \in S_n \setminus SCP(n)$ as follows.
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- Define

$$\sigma^+ = \sigma_1^+ \cdots \sigma_k^+, \quad \sigma_i^+ = \sigma_i + 1$$

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Example
A bijection between $S_n \setminus CP(n)$ and $S_n \setminus SCP(n)$

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Example

- Let $\pi \in S_n \setminus CP(n)$ be
  $$\pi = 4, 2, 5, 1, 3, 7, 6, 9, 8$$
A bijection between $S_n \setminus CP(n)$ and $S_n \setminus SCP(n)$

- Given $\pi = \pi_1 \cdots \pi_n \in S_n \setminus CP(n)$, define $\pi' \in S_n \setminus SCP(n)$ as follows.
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  \[ \pi' = \tau \sigma^+ 1 \rho \]

Example

- Let $\pi \in S_n \setminus CP(n)$ be
  \[ \pi = 4, 2, 5, 1, 3, \underbrace{7, 6, 9, 8}_\tau \]
- Then $\pi' \in S_n \setminus SCP(n)$ is
  \[ \pi' = \underbrace{7, 5, 3, 6, 2, 4}_\tau, \underbrace{1}_{\sigma^+}, \underbrace{9, 8}_\rho \]
A combinatorial proof of \[
\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} = \#CP(n+1)
\]

**Proposition**

\[
\sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} \text{ is the number of } T \in \mathcal{PT}(n+1) \text{ without a column containing 1 only in the first row.}
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- For \( \pi = \Phi(T) \),
- \( T \) has a column which has a 1 only in the first row if and only if
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A combinatorial proof of \[ \sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} = \#CP(n + 1) \]

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Proposition

\[ \sum_{T \in \mathcal{PT}(n)} 2^{\text{urc}(T)} = \#SCP(n + 1) = \#CP(n + 1) \]
The case $t = -1$: sign-imbalance

Corollary

$$P_{-1}(x) = \sum_{n \geq 0} \sum_{T \in \mathcal{P}T(n)} (-1)^{\text{urc}(T)} x^n = \frac{1 - x}{1 - 2x + 2x^2}$$

$$= \frac{1}{2} \cdot \left( \frac{1}{1 - (1 + i)x} + \frac{1}{1 - (1 - i)x} \right)$$
The case $t = -1$: sign-imbalance

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**Definition**

For $T \in \mathcal{P}T(n)$, define the **sign** of $T$ by

$$\text{sgn}(T) = (-1)^{\text{urc}(T)}.$$
The case $t = -1$ : sign-imbalance

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Definition

For $T \in \mathcal{PT}(n)$, define the **sign** of $T$ by

$$\text{sgn}(T) = (-1)^{\text{urc}(T)}.$$

Corollary

$$\sum_{T \in \mathcal{PT}(n)} \text{sgn}(T) = \frac{(1 + i)^n + (1 - i)^n}{2} = \begin{cases} 
(-1)^k \cdot 2^{2k}, & \text{if } n = 4k \text{ or } n = 4k + 1, \\
0, & \text{if } n = 4k + 2, \\
(-1)^{k+1} \cdot 2^{2k+1}, & \text{if } n = 4k + 3.
\end{cases}$$
The sign of a standard Young tableau is defined as follows.

\[
\text{sgn} \left( \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & \end{array} \right) = \text{sgn}(12534) = 1
\]
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Stanley conjectured

\[
\sum_{T \in \text{SYT}(n)} \text{sgn}(T) = 2^{\left\lfloor \frac{n}{2} \right\rfloor}
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Generalized to skew SYTs by Kim
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Generalized to skew SYTs by Kim

If $n \not\equiv 2 \pmod{4}$,

$$\left| \sum_{T \in \text{PT}(n)} \text{sgn}(T) \right| = \left| \sum_{T \in \text{SYT}(n)} \text{sgn}(T) \right| = 2 \left\lfloor \frac{n}{2} \right\rfloor.$$
The yellow cells are the **diagonal** cells.
The yellow cells are the **diagonal** cells.

The row containing the diagonal cell in Column $d$ is labeled with $-d$. 
Type $B$ permutation tableaux

- Each column has at least one 1.
Type $B$ permutation tableaux

- Each column has at least one 1.
- There is **no** configuration like

\[
\begin{array}{c}
1 \\
\vdots \\
1 \ldots 0 \\
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\vdots \\
1 \ldots 0 \\
\end{array}
\]
Type $B$ permutation tableaux

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\begin{array}{cccccc}
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\vdots \\
1 \\
\end{array}
\quad \text{or} \quad
\begin{array}{cccc}
1 & \cdots & 0
\end{array}
\]
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\vdots \\
1 \quad \ldots \quad 0
\end{array}
\]
Type B permutation tableaux

- Each column has at least one 1.
- There is no configuration like

```
1 1 1 ...
```

or

```
0 0 0 ...
```

---

NO

NO

NO

NO
Type B permutation tableaux

1. Each column has at least one 1.
2. There is no configuration like

![Tableaux Diagram]

NO

NO

YES!
The alternative representation

- Topmost 1 is ↑
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ←
The alternative representation

- Topmost 1 is \( \uparrow \)
- Rightmost restricted 0 is \( \leftarrow \)
- Cut off the diagonal cells.
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ←
- Cut off the diagonal cells.
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ←
- Cut off the diagonal cells.

No arrow points to another.
The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ←
- Cut off the diagonal cells.

No arrow points to another.
- The diagonal line acts like a mirror!
A theorem of Lam and Williams

Theorem (Lam and Williams)

$$\sum_{T \in \mathcal{PT}_B(n)} x^\text{urr}(T) - 1 z^\text{diag}(T) = (1 + z)^n (x + 1)^{n-1}$$
A theorem of Lam and Williams

Theorem (Lam and Williams)

\[ \sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T) - 1} z^{\text{diag}(T)} = (1 + z)^n (x + 1)^{n-1} \]

- \( \text{diag}(T) \): the number of diagonal cells with 1
A theorem of Lam and Williams

Theorem (Lam and Williams)

\[ \sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T)} - 1 \cdot z^{\text{diag}(T)} = (1 + z)^n (x + 1)^{n-1} \]

- \( \text{diag}(T) \): the number of diagonal cells with 1
- \( \text{urr}(T) \): the number of unrestricted rows, i.e. without 1.

\[
\begin{align*}
1 \\
\vdots \\
0
\end{align*}
\]

and

0
A theorem of Lam and Williams

Theorem (Lam and Williams)

\[
\sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T) - 1} z^{\text{diag}(T)} = (1 + z)^n (x + 1)^{n-1}
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- \text{diag}(T) : the number of diagonal cells with 1
- \text{urr}(T) : the number of unrestricted rows, i.e. without

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\begin{align*}
1 \\
\vdots \\
0
\end{align*}
\]

and

\[
\begin{align*}
0 \\
\vdots \\
1
\end{align*}
\]

Theorem

\[
\sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T) - 1} y^{\text{top}_0,1(T)} z^{\text{diag}(T)} = (1 + z)^n (x + y)^{n-1}
\]
Generalization of a theorem of Lam and Williams

For $T \in \mathcal{PT}_B(n)$ with the topmost nonzero row labeled $m$,

$$\text{top}_{0,1}(T) = (\# \text{ 1s in Row } m \text{ except in the diagonal})$$

$$+ (\# \text{ rightmost restricted 0s in Column } -m)$$

$$= \# \text{ arrows in Row } m \text{ and Column } -m$$
A type $B$ extension of Corteel and Nadeau’s bijection

\[ \begin{array} {cccccc}
10 & 9 & 8 & 6 & 3 & 2 \\
-10 & & & & & \\
-9 & & & & & \\
-8 & \uparrow & & & & \\
-6 & & & & & \\
-3 & \leftarrow & & & & \\
-2 & & & & & \\
1 & \leftarrow & \uparrow & & & \\
4 & \uparrow & & & & \\
5 & \leftarrow & & & & \\
7 & \leftarrow & \uparrow & & & \\
11 & \leftarrow & \uparrow & \leftarrow & \uparrow & \\
\end{array} \]

\(-8,4,11\)
A type $B$ extension of Corteel and Nadeau’s bijection

7, 10, −8, 4, 11
A type $B$ extension of Corteel and Nadeau’s bijection

9, 7, 10, −8, 4, 11
A type $B$ extension of Corteel and Nadeau’s bijection

9, 7, 10, $-3, 5$ – 8, 4, 11
A type $B$ extension of Corteel and Nadeau’s bijection

9, 7, 10, $-3$, 5, $-8$, 6, 4, 11
A type $B$ extension of Corteel and Nadeau’s bijection

9, 7, 10, 1, $-3$, 5 $- 8, 6, 4, 11$
A type B extension of Corteel and Nadeau’s bijection

9, 7, 10, 2, 1, −3, 5 − 8, 6, 4, 11
Some properties of the type $B$ bijection

**Proposition**

Then,

\[
\begin{array}{ccccccc}
-10 & -9 & -8 & -6 & -3 & -2 & 1 \\
\uparrow & & & & & & \\
10 & 9 & 8 & 6 & 3 & 2 & \\
\end{array}
\]

\[
\pi = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11
\]

\[
\begin{align*}
\sigma & \quad \tau & \quad m & \quad \rho \\
\end{align*}
\]
Some properties of the type $B$ bijection

Proposition

$$\pi = \Phi_B(T)$$

Then,

$$\pi = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11$$
Some properties of the type $B$ bijection

Proposition

- $\pi = \Phi_B(T)$
- Row $m$ is the topmost nonzero row of $T$

Then,

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Some properties of the type $B$ bijection

Proposition

- $\pi = \Phi_B(T)$
- Row $m$ is the topmost nonzero row of $T$
- Decompose $\pi = \sigma \tau m \rho$

Then,

$\pi = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11$
Some properties of the type $B$ bijection

Proposition

- $\pi = \Phi_B(T)$
- \text{Row } m \text{ is the topmost nonzero row of } T$
- \text{Decompose } \pi = \sigma \tau m \rho$
  \hspace{1cm} $\min(\pi) = m$

Then,

$\pi = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11$
Some properties of the type $B$ bijection

Proposition

- $\pi = \Phi_B(T)$
- Row $m$ is the topmost nonzero row of $T$
- Decompose $\pi = \sigma \tau m \rho$
  - $\min(\pi) = m$
  - the last element of $\sigma$ is $> |m|$

Then,

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$\pi = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11$
Some properties of the type $B$ bijection

**Proposition**

- $\pi = \Phi_B(T)$
- **Row $m$ is the topmost nonzero row of $T$**
- **Decompose** $\pi = \sigma \tau m \rho$
  - $\min(\pi) = m$
  - the last element of $\sigma$ is $> |m|$
  - each element of $\tau$ is $< |m|$

Then,

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\tau$</th>
<th>$m$</th>
<th>$\rho$</th>
</tr>
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<tr>
<td>9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11</td>
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Generalization Theorem

\[ \sum_{T \in \mathcal{PT}_B(n)} x^{\text{urr}(T)} y^{\text{top}_0,1(T)} z^{\text{diag}(T)} = (1 + z)^n (x + y)^{n-1} \]
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\]
### Zigzag maps

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<tr>
<td>13</td>
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Zigzag maps

1 0 0 1 0 0 1
2 0 0 0 1 1 1
4 0 0 0 0 1
7 0 1 1
10 1
11 0
13

1 12 9 8 6 5 3
2 1
4 1
7 1
10 1
11 1
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2 1
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11 1
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10 0 1 0 0 1
0 0 0 1 1 1
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7
10 1
11 0
13

12 9 8 6 5 3
11 10 9 8 7 6
10 9 8 7 6 5
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**Example**
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- $\Phi(T) = 4, 6, 5, 2, 8, 3, 1, 9, 7, 11, 12, 10$
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**Example**

- $\Phi(T) = 4, 6, 5, 2, 8, 3, 1, 9, 7, 11, 12, 10$
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The zigzag map on the type B alternative representation

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**Example**

\[ \Phi_B(T) = 9, 7, 10, 6, 2, 1, -3, 5, -8, 4, 11 \]
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Further study

- Find a combinatorial proof of the following:

\[
\sum_{T \in \mathcal{PT}(n)} \text{sgn}(T) = \frac{(1 + i)^n + (1 - i)^n}{2} = \begin{cases} 
(-1)^k \cdot 2^{2k}, & \text{if } n = 4k \text{ or } n = 4k + 1, \\
0, & \text{if } n = 4k + 2, \\
(-1)^{k+1} \cdot 2^{2k+1}, & \text{if } n = 4k + 3.
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  \left| \sum_{T \in PT(n)} \text{sgn}(T) \right| = \left| \sum_{T \in SYT(n)} \text{sgn}(T) \right| = 2 \left\lfloor \frac{n}{2} \right\rfloor.
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Thank you for your attention!