



Precise Morrey regularity of the weak solutions to a kind of quasilinear systems with discontinuous data

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Abstract. We consider the Dirichlet problem for a class of quasilinear elliptic systems in domain with irregular boundary. The principal part satisfies componentwise coercivity condition and the nonlinear terms are Carathéodory maps having Morrey regularity in x and verifying controlled growth conditions with respect to the other variables. We have obtained boundedness of the weak solution to the problem that permits to apply an iteration procedure in order to find optimal Morrey regularity of its gradient.

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1 Introduction

We are interested in the regularity properties of a kind of quasilinear elliptic operators with discontinuous data acting in a bounded domain Ω , with irregular boundary $\partial\Omega$. Precisely, we consider the following Dirichlet problem

$$\begin{cases} \operatorname{div}(\mathbf{A}(x)D\mathbf{u} + \mathbf{a}(x, \mathbf{u})) = \mathbf{b}(x, \mathbf{u}, D\mathbf{u}), & x \in \Omega \\ \mathbf{u}(x) = 0, & x \in \partial\Omega. \end{cases} \quad (1.1)$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded *Reifenberg-flat domain*, the matrix $\mathbf{A} = \{A_{ij}^{\alpha\beta}(x)\}_{i,j \leq N}^{\alpha,\beta \leq n}$ of the coefficients is essentially bounded in Ω , and the non linear terms

$$\mathbf{a}(x, \mathbf{u}) = \{a_i^\alpha(x, \mathbf{u})\}_{i \leq N}^{\alpha \leq n} \quad \text{and} \quad \mathbf{b}(x, \mathbf{u}, \mathbf{z}) = \{b_i(x, \mathbf{u}, \mathbf{z})\}_{i \leq N}$$

are Carathéodory maps, i.e., they are measurable in $x \in \Omega$ for all $\mathbf{u} \in \mathbb{R}^N$, $\mathbf{z} \in \mathbb{M}^{N \times n}$ and continuous in (\mathbf{u}, \mathbf{z}) for almost all $x \in \Omega$. Since we are going to study the weak solutions of

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(1.1) we need to impose *controlled growth conditions* on the nonlinear terms in order to ensure convergence of the integrals in the definition (2.6). For this aim we suppose that (cf. [17,33])

$$\begin{aligned} a_i^\alpha(x, \mathbf{u}) &= \mathcal{O}(\varphi_1(x) + |\mathbf{u}|^{\frac{n}{n-2}}), \\ b_i(x, \mathbf{u}, \mathbf{z}) &= \mathcal{O}(\varphi_2(x) + |\mathbf{u}|^{\frac{n+2}{n-2}} + |\mathbf{z}|^{\frac{n+2}{n}}) \end{aligned}$$

for $n > 2$. In the particular case $n = 2$, the powers of $|\mathbf{u}|$ could be arbitrary positive numbers, while the growth of $|\mathbf{z}|$ is subquadratic.

Our aim is to study the dependence of the solution from the regularity of the data and to obtain Calderón–Zygmund type estimate in an optimal Morrey space.

There are various papers dealing with the integrability and regularity properties of different kind of quasilinear and nonlinear differential operators. Namely, it is studied the question *how the regularity of the data influences on the regularity of the solution*. In the scalar case $N = 1$ the celebrated result of De Giorgi and Nash asserts that *the weak solution of linear elliptic and parabolic equations with only L^∞ coefficients is Hölder continuous* [12].

Better integrability can be obtained also by the result of Gehring [16] relating to *functions satisfying the inverse Hölder inequality*. Later Giaquinta and Modica [18] noticed that certain power of the gradient of a function $u \in W^{1,p}$ satisfies locally the reverse Hölder inequality. Modifying Gehring’s lemma they obtained *better integrability for the weak solutions of some quasilinear elliptic equations*. Their pioneer works have been followed by extensive research dedicated to the regularity properties of various partial differential operators using the Gehring–Giaquinta–Modica technique, called also a “*direct method*” (cf. [3,27,28] and the references therein.) Recently *the method of A -harmonic approximation* permits to study the regularity without using Gehring’s lemma (see for example [1]).

The theory for linear divergence form operators defined in Reifenberg’s domain was developed firstly in [8,10]. In [4,5] the authors extend this technique to quasilinear uniformly elliptic equations in the Sobolev–Morrey spaces. Making use of the Adams inequality [2] and the Hartmann–Stampacchia maximum principal they obtain Hölder regularity of the solution while in [7] it is obtained generalized Hölder regularity for regular and nonregular nonlinear elliptic equations.

Concerning nonlinear nonvariational operators we can mention the results of Campanato [11] related to basic systems of the form $F(D^2u) = 0$ in the Morrey spaces. Afterwards Marino and Maugeri in [24] have contributed to this theory with their own research on the boundary regularity of the solutions of basic systems. Imposing differentiability of the operator F they obtain, via immersion theorems, Morrey regularity of the second derivatives $D^2u \in L^{2,2-\frac{2}{q},q} > 2$. These studies have been extended in [15] to nonlinear equations of a kind $F(x, D^2u)$ without any differentiability assumptions on F . It is obtained global Morrey regularity via the *Korn trick* and the *near operators theory* of Campanato. Moreover, in the variational case it is established a Caccioppoli-type inequality for a second-order degenerate elliptic systems of p -Laplacian type [14]. Exploiting the classical Campanato’s approach and the *hole-filling technique* due to Widman, it is proved a global regularity result for the gradient of \mathbf{u} in the Morrey and Lebesgue spaces.

In the present work we consider quasilinear systems in divergence form with a principal part satisfying *componentwise coercivity condition*. This condition permits to apply the results of [29,33] that gives L^∞ estimate of the weak solution. In addition the *controlled growth conditions* imposed on the nonlinear terms allow to apply the integrability result from [31]. Making use of step-by-step technique we show optimal Morrey regularity of the gradient depending explicitly on the regularity of the data.

In what follows we use the standard notation:

- Ω is a bounded domain in \mathbb{R}^n , with a Lebesgue measure $|\Omega|$ and boundary $\partial\Omega$;
- $\mathcal{B}_\rho(x) \subset \mathbb{R}^n$ is a ball, $\Omega_\rho(x) = \Omega \cap \mathcal{B}_\rho(x)$ with $\rho \in (0, \text{diam } \Omega]$, $x \in \Omega$;
- $\mathbb{M}^{N \times n}$ is the set of $N \times n$ -matrices;
- $\mathbf{u} = (u^1, \dots, u^N) : \Omega \rightarrow \mathbb{R}^N$, $D_\alpha u^j = \partial u^j / \partial x_\alpha$,
 $|\mathbf{u}|^2 = \sum_{j \leq N} |u^j|^2$, $D\mathbf{u} = \{D_\alpha u^j\}_{\substack{\alpha \leq n \\ j \leq N}} \in \mathbb{M}^{N \times n}$, $|D\mathbf{u}|^2 = \sum_{\substack{\alpha \leq n \\ j \leq N}} |D_\alpha u^j|^2$;
- For $\mathbf{u} \in L^p(\Omega; \mathbb{R}^N)$ we write $\|\mathbf{u}\|_{p, \Omega}$ instead of $\|\mathbf{u}\|_{L^p(\Omega; \mathbb{R}^N)}$;
- The spaces $W^{1,p}(\Omega; \mathbb{R}^N)$ and $W_0^{1,p}(\Omega; \mathbb{R}^N)$ are the classical Sobolev spaces as they are defined in [19].

Throughout the paper the standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various positive constants and may change from one occurrence to another.

2 Definitions and auxiliary results

In [34] Reifenberg introduced a class of domains with rough boundary that can be approximated locally by hyperplanes.

Definition 2.1. The domain Ω is (δ, R) Reifenberg-flat if there exist positive constants R and $\delta < 1$ such that for each $x \in \partial\Omega$ and each $\rho \in (0, R)$ there is a local coordinate system $\{y_1, \dots, y_n\}$ with the property

$$\mathcal{B}_\rho(x) \cap \{y_n > \delta\rho\} \subset \Omega_\rho(x) \subset \mathcal{B}_\rho(x) \cap \{y_n > -\delta\rho\}. \quad (2.1)$$

Reifenberg arrived at this concept of flatness in his studies on the Plateau problem in higher dimensions and he proved that such a domain is locally a topological disc when δ is small enough, say $\delta < 1/8$. It is easy to see that a C^1 -domain is a Reifenberg flat with $\delta \rightarrow 0$ as $R \rightarrow 0$. A domain with Lipschitz boundary with a Lipschitz constant less than δ also verifies the condition (2.1) if δ is small enough, say $\delta < 1/8$, (see [10, Lemma 5.1]). But the class of Reifenberg's domains is much more wider and contains domains with fractal boundaries. For instance, consider a self-similar snowflake S_β . It is a flat version of the Koch snowflake $S_{\pi/3}$ but with angle of the spike β such that $\sin \beta \in (0, 1/8)$. This kind of flatness exhibits minimal geometrical conditions necessary for some natural properties from the analysis and potential theory to hold. For more detailed overview of these domains we refer the reader to [35] (see also [8, 27] and the references therein).

In addition (2.1) implies the (A)-property (cf. [17, 28]). Precisely, there exists a positive constant $A(\delta) < 1/2$ such that

$$A(\delta)|\mathcal{B}_\rho(x)| \leq |\Omega_\rho(x)| \leq (1 - A(\delta))|\mathcal{B}_\rho(x)| \quad (\text{A})$$

for any fixed $x \in \partial\Omega$, $\rho \in (0, R)$ and $\delta \in (0, 1)$. This condition excludes that Ω may have sharp outward and inward cusps. As consequence, the Reifenberg domain is $W^{1,p}$ -extension domain, $1 \leq p \leq \infty$, hence the usual extension theorems, the Sobolev and Sobolev-Poincaré inequalities are still valid in Ω up to the boundary.

Definition 2.2. A real valued function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ with $p \in [1, \infty)$, $\lambda \in (0, n)$, if

$$\|f\|_{p,\lambda;\Omega} = \left(\sup_{\mathcal{B}_\rho(x)} \frac{1}{\rho^\lambda} \int_{\Omega_\rho(x)} |f(y)|^p dy \right)^{1/p} < \infty$$

where $\mathcal{B}_\rho(x)$ ranges in the set of all balls with radius $\rho \in (0, \text{diam } \Omega]$ and $x \in \Omega$.

In [25] Morrey obtained local Hölder regularity of the solutions to second order elliptic equations. His new approach consisted in estimating the growth of the integral function $g(\rho) = \int_{\mathcal{B}_\rho} |Du(y)|^p dy$ via a power of the radius of the same ball, i.e., $C\rho^\lambda$ with $\lambda > 0$. Although he did not talk about function spaces, his paper is considered as the starting point for the theory of the *Morrey spaces* $L^{p,\lambda}$.

The family of the $L^{p,\lambda}$ spaces is partially ordered (cf. [30]).

Lemma 2.3. For $1 \leq r' \leq r'' < \infty$ and $\sigma', \sigma'' \in [0, n)$ the following embedding holds

$$L^{r''\sigma''}(\Omega) \hookrightarrow L^{r'\sigma'}(\Omega) \quad \text{iff} \quad \frac{n - \sigma'}{r'} \geq \frac{n - \sigma''}{r''}.$$

Furthermore, we have the continuous inclusion

$$L^{\frac{nr'}{n-\sigma'}}(\Omega) \hookrightarrow L^{r'\sigma'}(\Omega).$$

For $x \in \mathbb{R}^n$, I_α is the *Riesz potential operator* whose convolution kernel is $|x|^{\alpha-n}$, $0 < \alpha < n$. Suppose that f is extended as zero in \mathbb{R}^n and consider its Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

In [2] Adams obtained the following inequality.

Lemma 2.4. Let $f \in L^{r,\sigma}(\mathbb{R}^n)$, then $I_\alpha : L^{r,\sigma} \rightarrow L^{r_\sigma^*,\sigma}$ is continuous and

$$\|I_\alpha f\|_{L^{r_\sigma^*,\sigma}(\mathbb{R}^n)} \leq C \|f\|_{L^{r,\sigma}(\mathbb{R}^n)}, \quad (2.2)$$

where C depends on $n, r, \sigma, |\Omega|$, and r_σ^* is the *Sobolev–Morrey conjugate*

$$r_\sigma^* = \begin{cases} \frac{(n-\sigma)r}{n-\sigma-r} & \text{if } r + \sigma < n \\ \text{arbitrary large number} & \text{if } r + \sigma \geq n. \end{cases} \quad (2.3)$$

The nonlinear terms $\mathbf{a}(x, \mathbf{u})$ and $\mathbf{b}(x, \mathbf{u}, \mathbf{z})$ satisfy *controlled growth conditions*

$$|\mathbf{a}(x, \mathbf{u})| \leq \Lambda(\varphi_1(x) + |\mathbf{u}|^{\frac{2^*}{2}}), \quad (2.4)$$

$$\varphi_1 \in L^{p,\lambda}(\Omega), \quad p > 2, \quad p + \lambda > n, \quad \lambda \in [0, n),$$

$$|\mathbf{b}(x, \mathbf{u}, \mathbf{z})| \leq \Lambda(\varphi_2(x) + |\mathbf{u}|^{2^*-1} + |\mathbf{z}|^{2\frac{(2^*-1)}{2^*}}), \quad (2.5)$$

$$\varphi_2 \in L^{q,\mu}(\Omega), \quad q > \frac{2^*}{2^*-1}, \quad 2q + \mu > n, \quad \mu \in [0, n)$$

with a positive constant Λ . Here 2^* is the Sobolev conjugate of 2, i.e. $2^* = \frac{2n}{n-2}$ if $n > 2$ and it is arbitrary large number if $n = 2$ (cf. [17, 22, 31, 33]).

A weak solution to (1.1) is a function $\mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ satisfying

$$\begin{aligned} & \int_{\Omega} A_{ij}^{\alpha\beta}(x) D_{\beta} u^j(x) D_{\alpha} \chi^i(x) dx + \int_{\Omega} a_i^{\alpha}(x, \mathbf{u}(x)) D_{\alpha} \chi^i(x) dx \\ & + \int_{\Omega} b_i(x, \mathbf{u}(x), D\mathbf{u}(x)) \chi^i(x) dx = 0, \quad j = 1, \dots, N \end{aligned} \quad (2.6)$$

for all $\chi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ where the convergence of the integrals is ensured by (2.4) and (2.5).

3 Main result

The general theory of elliptic systems does not ensure boundedness of the solution if we impose only growth conditions as (2.4) and (2.5) (see for example [21, 23]). For this goal we need some additional structural restrictions on the operator as *componentwise coercivity* similar to that imposed in [23, 29, 32, 33].

Suppose that $\|\mathbf{A}\|_{\infty, \Omega} \leq \Lambda_0$ and for each fixed $i \in \{1, \dots, N\}$ there exist positive constants θ_i and $\gamma(\Lambda_0)$ such that for $|u^i| \geq \theta_i$ we have

$$\begin{cases} \gamma |\mathbf{z}|^2 - \Lambda |\mathbf{u}|^{2^*} - \Lambda \varphi_1(x)^2 \leq \sum_{\alpha=1}^n \left(A_{ij}^{\alpha\beta}(x) z_{\beta}^j + a_i^{\alpha}(x, \mathbf{u}) \right) z_{\alpha}^i \\ b_i(x, \mathbf{u}, \mathbf{z}) \operatorname{sign} u^i(x) \geq -\Lambda \left(\varphi_2(x) + |\mathbf{u}|^{2^*-1} + |\mathbf{z}|^{2 \frac{2^*-1}{2^*}} \right) \end{cases} \quad (3.1)$$

for a.a. $x \in \Omega$ and for all $\mathbf{z} \in \mathbb{M}^{N \times n}$. The functions φ_1 and φ_2 are as in (2.4) and (2.5).

Theorem 3.1. *Let $\mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ be a weak solution of the problem (1.1) under the conditions (2.1), (2.4), (2.5) and (3.1). Then*

$$\mathbf{u} \in W_0^{1,r} \cap L^{\infty}(\Omega; \mathbb{R}^N) \quad \text{with} \quad r = \min\{p, q_{\mu}^*\}.$$

Moreover

$$|D\mathbf{u}| \in L^{r,v}(\Omega) \quad \text{with} \quad v = \min \left\{ n + \frac{r(\lambda - n)}{p}, n + \frac{r(\mu - n)}{q_{\mu}^*} \right\} \quad (3.2)$$

where q_{μ}^* is the Sobolev–Morrey conjugate of q (see (2.3)).

Remark 3.2. *If we take a bounded weak solution of (1.1) $\mathbf{u} \in W_0^{1,r} \cap L^{\infty}(\Omega; \mathbb{R}^N)$ we can substitute the coercivity condition (3.1) with a uniform ellipticity condition. In this case we may suppose the principal coefficients to be discontinuous with small discontinuity controlled by their BMO modulus. Precisely, we suppose that*

$$\begin{aligned} & \sup_{0 < \rho \leq R} \sup_{y \in \Omega} \int_{\Omega_{\rho}(y)} |A_{ij}^{\alpha\beta}(x) - \overline{A_{ij}^{\alpha\beta}}_{\Omega_{\rho}(y)}|^2 dx \leq \delta^2, \\ & \overline{A_{ij}^{\alpha\beta}}_{\Omega_{\rho}(y)} = \int_{\Omega_{\rho}(y)} A_{ij}^{\alpha\beta}(x) dx, \end{aligned}$$

where $\delta \in (0, 1)$ is the same parameter as in (2.1). The small BMO successfully substitute the VMO in the study of PDEs with discontinuous coefficients, harmonic analysis and integral operators studying, geometric measure analysis and differential geometry (see [4, 6, 8, 20, 28, 33] and the references therein). A higher integrability result for such kind of operators can be found in [13, 28, 31] for equations and systems, respectively.

Proof. The essential boundedness of the solution follows by [29] (see also [32, 33]). Precisely, there exists a constant depending on $n, \Lambda, p, q, \|\varphi_1\|_{L^p(\Omega)}, \|\varphi_2\|_{L^q(\Omega)}$ and $\|D\mathbf{u}\|_{L^2(\Omega)}$ such that

$$\|\mathbf{u}\|_{\infty, \Omega} \leq M. \quad (3.3)$$

Let the solution and the functions φ_1 and φ_2 be extended as zero outside Ω . By the Definition 2.2 we have that $\varphi_1 \in L^p(\Omega)$ and $\varphi_2 \in L^q(\Omega)$. In [17] Giaquinta show that there exists an exponent $\tilde{r} > 2$ such that $\mathbf{u} \in W_{\text{loc}}^{1, \tilde{r}}(\Omega; \mathbb{R}^N)$. His approach is based on the reverse Hölder inequality and a version of Gehring's lemma. Since the Cacciopoli-type inequalities hold up to the boundary, this method can be carried out up to the boundary and it is done in [17, Chapter 5] for the Dirichlet problem in Lipschitz domain (see also [3, 11, 13, 31]). In [9] the authors have shown that an inner neighborhood of (δ, R) -Reifenberg flat domain is a Lipschitz domain with the (δ, R) -Reifenberg flat property.

Lemma 3.3. ([9]) *Let Ω be a (δ, R) -Reifenberg flat domain for sufficiently small $\delta > 0$. Then for any $0 < \varepsilon < \frac{R}{5}$ the set $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ is a Lipschitz domain satisfying (2.1).*

This lemma permits us to extend the results of [17, Chapter 5] in Reifenberg-flat domains. Further $|D\mathbf{u}|$ belongs at least to $L^{r_0}(\Omega)$ with $r_0 = \min\{p, q^*\} > \frac{n}{n+2}$ (cf. [31]).

Let $\boxed{n > 2}$ and $\mathbf{u} \in W_0^{1, r_0}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ be a solution to (1.1). Our *first step* is to improve its integrability. Fixing that solution in the nonlinear terms we obtain linear problem

$$\begin{cases} D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j(x)) = f_i(x) - D_\alpha A_i^\alpha(x), & x \in \Omega \\ \mathbf{u}(x) = 0, & x \in \partial\Omega \end{cases} \quad (3.4)$$

where we have used the notion

$$f_i(x) = b_i(x, \mathbf{u}, D\mathbf{u}), \quad A_i^\alpha(x) = a_i^\alpha(x, \mathbf{u}).$$

By (2.4), (2.5) and (3.3) we get

$$|A_i^\alpha(x)| \leq \Lambda \left(\varphi_1(x) + |\mathbf{u}(x)|^{\frac{n}{n-2}} \right) \quad (3.5)$$

that gives $A_i^\alpha(x) \in L^{p, \lambda}(\Omega)$ with $p > 2$ and $p + \lambda > n$. Analogously

$$|f_i(x)| \leq \Lambda \left(\varphi_2(x) + |\mathbf{u}|^{\frac{n+2}{n-2}} + |D\mathbf{u}|^{\frac{n+2}{n}} \right). \quad (3.6)$$

Since $|D\mathbf{u}| \in L^{r_0}(\Omega)$ we get $|D\mathbf{u}|^{\frac{n+2}{n}} \in L^{\frac{r_0 n}{n+2}}(\Omega)$ that gives $f_i \in L^{q_1}(\Omega)$ where $q_1 = \min\{q, \frac{r_0 n}{n+2}\}$.

Let Γ be the fundamental solution of the Laplace operator. Recall that the *Newtonian potential* of $f_i(x)$ is given by

$$\mathcal{N}f_i(x) = \int_{\Omega} \Gamma(x-y)f_i(y) dy, \quad \Delta \mathcal{N}f_i(x) = f_i(x) \text{ for a.a. } x \in \Omega$$

and by [19, Theorem 9.9] we have that $\mathcal{N}f_i \in W^{2, q_1}(\Omega)$. Denote by

$$F_i^\alpha(x) = D_\alpha \mathcal{N}f_i(x) = C(n) \int_{\Omega} \frac{(x-y)_\alpha f_i(y)}{|x-y|^n} dy \quad \text{for a.a. } x \in \Omega$$

and $\mathbb{F}_i = (F_i^1, \dots, F_i^n) = \text{grad } \mathcal{N}f_i$. Hence $\text{div } \mathbb{F}_i = f_i$ and

$$\begin{cases} D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j(x)) = D_\alpha(F_i^\alpha(x) - A_i^\alpha(x)), & x \in \Omega \\ \mathbf{u}(x) = 0, & x \in \partial\Omega. \end{cases} \quad (3.7)$$

By (3.5) and (3.6) we get

$$\begin{aligned}
 |F_i^\alpha(x) - A_i^\alpha(x)| &\leq C(n, \Lambda) \int_{\Omega} \frac{\varphi_2(y) + |\mathbf{u}(y)|^{\frac{n+2}{n-2}} + |D\mathbf{u}(y)|^{\frac{n+2}{n}}}{|x-y|^{n-1}} dy \\
 &\quad + \Lambda \left(\varphi_1(x) + |\mathbf{u}(x)|^{\frac{n}{n-2}} \right) \\
 &\leq C \left(1 + \varphi_1(x) + I_1 \varphi_2(x) + I_1 |D\mathbf{u}(x)|^{\frac{n+2}{n}} \right)
 \end{aligned} \tag{3.8}$$

with a constant depending on n, Λ , and $\|\mathbf{u}\|_{\infty, \Omega}$. By (2.2) we get

$$\begin{aligned}
 \|I_1 \varphi_2\|_{L^{q_\mu^*}(\Omega)} &\leq C \|\varphi_2\|_{L^{q_\mu}(\Omega)} \\
 \|I_1 |D\mathbf{u}|^{\frac{n+2}{n}}\|_{L^{(\frac{r_0 n}{n+2})^*}(\Omega)} &\leq C \| |D\mathbf{u}|^{\frac{n+2}{n}} \|_{L^{\frac{r_0 n}{n+2}}(\Omega)} \leq C \|D\mathbf{u}\|_{L^{\frac{n+2}{n}}(\Omega)}
 \end{aligned}$$

where q_μ^* is the Sobolev–Morrey conjugate of q and

$$\left(\frac{r_0 n}{n+2} \right)^* = \begin{cases} \frac{r_0 n}{n+2-r_0} & \text{if } r_0 < n+2, \\ \text{arbitrary large number} & \text{if } r_0 \geq n+2. \end{cases}$$

Hence $F_i^\alpha - A_i^\alpha \in L^{r_1}(\Omega)$ with $r_1 = \min\{p, q_\mu^*, (\frac{r_0 n}{n+2})^*\}$. If $r_1 = \min\{p, q_\mu^*\}$ then we have the assertion, otherwise $r_1 = (\frac{r_0 n}{n+2})^*$ and we consider two cases:

1. $r_0 = p$ that leads to $p > (\frac{pn}{n+2})^*$ which is impossible;
2. $r_0 = q^*$ and we consider two sub-cases:
 - 2a) $q^* \geq n+2$ which means that r_1 is arbitrary large number and we arrive to contradiction with the assumption $r_1 < \min\{p, q_\mu^*\}$;
 - 2b) $q^* < n+2$ hence $r_1 = \frac{q^* n}{n+2-q^*}$.

Applying [10, Theorem 1.7] to the linearized system (3.7) we get that for each matrix function $\mathbb{F} - \mathbb{A} \in L^{r_1}(\Omega; \mathbb{M}^{N \times n})$, with $r_1 = \frac{q^* n}{n+2-q^*}$ holds $\mathbf{u} \in W^{1, r_1} \cap L^\infty(\Omega; \mathbb{R}^N)$ with the estimate

$$\|D\mathbf{u}\|_{r_1, \Omega} \leq C \|\mathbb{F} - \mathbb{A}\|_{r_1, \Omega}.$$

Here $\mathbb{A}(x) = \{A_i^\alpha(x)\}_{i \leq N}^{\alpha \leq n}$ and $\mathbb{F}(x) = \{F_i^\alpha(x)\}_{i \leq N}^{\alpha \leq n}$. Let us note that this estimate is valid for each solution of (3.7) including \mathbf{u} .

Repeating the above procedure for $\mathbf{u} \in W^{1, r_1}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ we get that

$$|D\mathbf{u}| \in L^{r_2}(\Omega) \quad r_2 = \min \left\{ p, q_\mu^*, \left(\frac{r_1 n}{n+2} \right)^* \right\}.$$

If $r_2 = \min\{p, q_\mu^*\}$ then we have the assertion, otherwise $r_2 = (\frac{r_1 n}{n+2})^* > r_1$ and we repeat the arguments of the previous case. In such a way we get an increasing sequence of indexes $\{r_k\}_{k \geq 0}$. After k' iterations we obtain $r_{k'} \geq \min\{p, q_\mu^*\}$ and

$$\|D\mathbf{u}\|_{r, \Omega} \leq C \|\mathbb{F} - \mathbb{A}\|_{r, \Omega} \quad \text{with } r = \min\{p, q_\mu^*\}. \tag{3.9}$$

The *second step* consists of showing that the gradient lies in a suitable Morrey space. Suppose that $|D\mathbf{u}| \in L^{r, \theta}(\Omega)$ with arbitrary $\theta \in [0, n)$. Direct calculations give $|D\mathbf{u}|^{\frac{n+2}{n}} \in L^{\frac{rn}{n+2}, \theta}(\Omega)$

$$\left(\frac{1}{\rho^\theta} \int_{\mathcal{B}_\rho} |D\mathbf{u}|^{\frac{n+2}{n} \frac{rn}{n+2}} dx \right)^{\frac{n+2}{rn}} = \left(\frac{1}{\rho^\theta} \int_{\mathcal{B}_\rho} |D\mathbf{u}|^r dx \right)^{\frac{n+2}{rn}} \leq \|D\mathbf{u}\|_{r, \theta; \Omega}^{\frac{n+2}{n}}.$$

Keeping in mind (3.8) and (2.2) we get

$$I_1|D\mathbf{u}|^{\frac{n+2}{n}} \in L^{(\frac{nr}{n+2})_{\theta}^*, \theta}(\Omega)$$

while $\varphi_1 \in L^{p, \lambda}(\Omega)$ and $I_1\varphi_2 \in L^{q_{\mu}^*, \mu}(\Omega)$.

Further by the Hölder inequality we get the estimates

$$\begin{aligned} \left(\frac{1}{\rho^{n-\frac{n-\lambda}{p}r}} \int_{\mathcal{B}_{\rho}} \varphi_1(x)^r dx \right)^{\frac{1}{r}} &\leq C(n) \|\varphi_1\|_{p, \lambda; \Omega}, \\ \left(\frac{1}{\rho^{n-\frac{n-\mu}{q_{\mu}^*}r}} \int_{\mathcal{B}_{\rho}} (I_1\varphi_2(x))^r dx \right)^{\frac{1}{r}} &\leq C(n) \|I_1\varphi_2\|_{q_{\mu}^*, \mu; \Omega} \end{aligned}$$

that implies $\varphi_1 \in L^{r, n-\frac{n-\lambda}{p}r}(\Omega)$ and $I_1\varphi_2 \in L^{r, n-\frac{n-\mu}{q_{\mu}^*}r}(\Omega)$.

Concerning the potential $I_1|D\mathbf{u}|^{\frac{n+2}{n}}$ we consider two cases:

1. $n - \theta \leq \frac{rn}{n+2}$ then $(\frac{nr}{n+2})_{\theta}^*$ is arbitrary large number and we can take it such that

$$I_1|D\mathbf{u}|^{\frac{n+2}{n}} \in L^r(\Omega);$$

2. $n - \theta > \frac{rn}{n+2}$ then by the embedding between the Morrey spaces we have

$$L^{(\frac{nr}{n+2})_{\theta}^*, \theta}(\Omega) \subset L^{r, r-2+\theta\frac{n+2}{n}}(\Omega).$$

Then

$$|F_i^{\alpha} - A_i^{\alpha}| \in L^{r, \min\{r-2+\theta\frac{n+2}{n}, n-\frac{n-\lambda}{p}r, n-\frac{n-\mu}{q_{\mu}^*}r\}}(\Omega)$$

which implies via [6, Theorem 5.1] that the gradient of the solution of the linearized problem satisfies

$$|D\mathbf{u}| \in L^{r, \min\{r-2+\theta\frac{n+2}{n}, n-\frac{n-\lambda}{p}r, n-\frac{n-\mu}{q_{\mu}^*}r\}}(\Omega).$$

In order to determine the optimal θ we use step-by-step arguments starting with the result obtained in the first step and taking as $\theta_0 = 0$. Suppose that

$$r - 2 < \min \left\{ n - \frac{n-\lambda}{p}r, n - \frac{n-\mu}{q_{\mu}^*}r \right\},$$

otherwise we have the assertion.

Repeating the above procedure with \mathbf{u} such that $|D\mathbf{u}| \in L^{r, \theta_1}(\Omega)$ with $\theta_1 = r - 2$ we obtain

$$|D\mathbf{u}| \in L^{r, \theta_2}(\Omega)$$

with

$$\theta_2 = \min \left\{ r - 2 + \theta_1 \frac{n+2}{n}, n - \frac{n-\lambda}{p}r, n - \frac{n-\mu}{q_{\mu}^*}r \right\}.$$

If $\theta_2 = \min\{n - \frac{n-\lambda}{p}r, n - \frac{n-\mu}{q_{\mu}^*}r\}$ we have the assertion, otherwise we take

$$\theta_2 = r - 2 + \theta_1 \frac{n+2}{n} = (r - 2) \left(1 + \frac{n+2}{n} \right).$$

Iterating we obtain an increasing sequence $\{\theta_k = (r-2) \sum_{i=0}^{k-1} (\frac{n+2}{n})^i\}_{k \geq 1}$. Then there exists an index k'' for which

$$r-2 + \theta_{k''} \frac{n+2}{n} \geq \min \left\{ n - \frac{n-\lambda}{p} r, n - \frac{n-\mu}{q_\mu^*} r \right\}$$

that gives the assertion.

If $\boxed{n=2}$ then the growth conditions have the form

$$\begin{aligned} |\mathbf{a}(x, \mathbf{u})| &\leq \Lambda(\varphi_1(x) + |\mathbf{u}|^\varkappa), \\ \varphi_1 &\in L^{p,\lambda}(\Omega), \quad p > 2, \quad p + \lambda > n, \quad \lambda \in [0, n), \end{aligned} \quad (3.10)$$

$$\begin{aligned} |\mathbf{b}(x, \mathbf{u}, \mathbf{z})| &\leq \Lambda(\varphi_2(x) + |\mathbf{u}|^{\varkappa-1} + |\mathbf{z}|^{2-\epsilon}), \\ \varphi_2 &\in L^{q,\mu}(\Omega), \quad q > 1, \quad 2q + \mu > n, \quad \mu \in [0, n) \end{aligned} \quad (3.11)$$

with $\varkappa > 1$ arbitrary large number and $\epsilon > 0$ arbitrary small.

Fixing again the solution $\mathbf{u} \in W_0^{1,r_0}(\Omega; \mathbb{R}^N) \cup L^\infty(\Omega; \mathbb{R}^N)$ in the nonlinear terms and using the Lemma 2.3 and Lemma 2.4 we obtain

$$F_i^\alpha - A_i^\alpha \in L^{r_1}(\Omega), \quad r_1 = \min \left\{ p, q_\mu^*, \left(\frac{r_0}{2-\epsilon} \right)^* \right\}.$$

If $r_1 = \left(\frac{r_0}{2-\epsilon} \right)^*$ then the only possible value for r_0 is $r_0 = q^*$ and hence $r_1 = \frac{2q^*}{2(2-\epsilon)-q^*}$, otherwise we reach to contradiction. Then by [10] we get $|D\mathbf{u}| \in L^{r_1}(\Omega)$.

Repeating the above procedure with $\mathbf{u} \in W_0^{1,r_1} \cap L^\infty(\Omega; \mathbb{R}^N)$ we obtain that

$$|D\mathbf{u}| \in L^{r_2}(\Omega), \quad r_2 = \min \left\{ p, q_\mu^*, \left(\frac{r_1}{2-\epsilon} \right)^* \right\}.$$

If

$$r_2 = \left(\frac{r_1}{2-\epsilon} \right)^* < \min\{p, q_\mu^*\}$$

we repeat the same procedure obtaining an increasing sequence $\{r_k\}_{k \geq 0}$. Hence there exist an index k_0 such that $r_{k_0} \leq \min\{p, q_\mu^*\}$ that gives the assertion.

To obtain Morrey's regularity we take $|D\mathbf{u}| \in L^{r,\theta}(\Omega)$ with arbitrary $\theta \in [0, 2)$. Hence $|D\mathbf{u}|^{2-\epsilon} \in L^{\frac{r}{2-\epsilon}, \theta}(\Omega)$. By Lemma 2.3 and Lemma 2.4 we obtain

$$\begin{aligned} \varphi_1 &\in L^{p,\lambda}(\Omega) \subset L^{r, 2-\frac{2-\lambda}{p}r}(\Omega) \\ I_1 \varphi_2 &\in L^{q_\mu^*, \mu}(\Omega) \subset L^{r, 2-\frac{2-\mu}{q_\mu^*}r}(\Omega) \\ I_1 |D\mathbf{u}|^{2-\epsilon} &\in L^{\left(\frac{r}{2-\epsilon}\right)^*, \theta}(\Omega) \subset L^{r, r-2(1-\epsilon)+\theta(2-\epsilon)}(\Omega). \end{aligned}$$

Hence the Calderón–Zygmund estimate for the linearized problem (see [6]) gives

$$|D\mathbf{u}| \in L^{r, \min\{2-\frac{2-\lambda}{p}r, 2-\frac{2-\mu}{q_\mu^*}r, r-2(1-\epsilon)+\theta(2-\epsilon)\}}(\Omega).$$

To determine the precise Morrey space we apply the step-by-step procedure.

1. Since the last term is minimal when $\theta = 0$ than we start with an this initial value $\theta_0 = 0$. Suppose that

$$r-2(1-\epsilon) < \min \left\{ 2 - \frac{2-\lambda}{p} r, 2 - \frac{2-\mu}{q_\mu^*} r \right\} < 2$$

(otherwise we have the assertion) and denote $\theta_1 = r-2(1-\epsilon)$.

2. Take $|D\mathbf{u}| \in L^{r,\theta_1}(\Omega)$. The above procedure gives $|D\mathbf{u}| \in L^{r,\theta_2}(\Omega)$ with

$$\theta_2 = \min \left\{ 2 - \frac{2-\lambda}{p}r, 2 - \frac{2-\mu}{q_\mu^*}r, r - 2(1-\epsilon) + \theta_1(2-\epsilon) \right\}.$$

If $\theta_2 = r - 2(1-\epsilon) + \theta_1(2-\epsilon)$ (otherwise we have the assertion) then we continue with the same procedure obtaining the sequence defined by recurrence

$$\theta_0 = 0, \quad \theta_k = r - 2(1-\epsilon) + \theta_{k-1}(2-\epsilon).$$

3. Since $r > 2$, hence the sequence is increasing and there exists an index \bar{k} such that

$$\theta_{\bar{k}} \geq \min \left\{ 2 - \frac{2-\lambda}{p}r, 2 - \frac{2-\mu}{q_\mu^*}r \right\}$$

which is the assertion. □

Corollary 3.4. *Let the conditions of Theorem 3.1 hold. Then*

$$u^i \in C^{0,\alpha}(\Omega) \quad \text{with} \quad \alpha = \min \left\{ 1 - \frac{n-\lambda}{p}, 1 - \frac{n-\mu}{q_\mu^*} \right\},$$

and for any ball $\mathcal{B}_\rho(z) \subset \Omega$ we have

$$\text{osc}_{\mathcal{B}_\rho(z)} u^i \leq C\rho^\alpha \quad \forall i = 1, \dots, N.$$

Proof. By (3.2) we have that for each ball $\mathcal{B}_\rho(z) \subset \Omega$

$$\int_{\mathcal{B}_\rho(z)} |Du^i(y)| dy \leq C\rho^{n-\frac{n-\nu}{r}}.$$

Then for any $x, y \in \mathcal{B}_\rho(z)$ and for each fixed $i = 1, \dots, N$ we have

$$\begin{aligned} |u^i(x) - u^i(y)| &\leq 2|u^i(x) - u_{\mathcal{B}_\rho(z)}^i| \leq C \int_{\mathcal{B}_\rho(z)} \frac{Du^i(y)}{|x-y|^{n-1}} dy \\ &\leq C \int_0^\rho \int_{\mathcal{B}_t(z)} |Du^i(y)| dy \frac{dt}{t^n} \leq C\rho^{1-\frac{n-\nu}{r}}. \end{aligned} \quad \square$$

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