

Stability of positive equilibrium of a Nicholson blowflies model with stochastic perturbations

Le Van Hien  and Nguyen Thi Lan-Huong

Department of Mathematics, Hanoi National University of Education, Hanoi, Vietnam

Received 20 August 2019, appeared 6 April 2020

Communicated by Leonid Berezhansky

Dedicated to Vietnamese people who are being at frontline in the battle of repelling the spread of Covid-19

Abstract. This paper is concerned with the stability problem of the positive equilibrium of a Nicholson's blowflies model with nonlinear density-dependent mortality rate subject to stochastic perturbations. More specifically, the existence of a unique positive equilibrium of a Nicholson's blowflies model described by the delay differential equation

$$N'(t) = -\left(a - be^{-N(t)}\right) + \beta N(t - \tau)e^{-\gamma N(t-\tau)}$$

is first quoted. It is assumed that the underlying model in noisy environments is exposed to stochastic perturbations, which are proportional to the derivation of the state from the equilibrium point. Then, by utilizing a stability criterion formulated for linear stochastic differential delay equations, explicit stability conditions are obtained. An extension to models with multiple delays is also presented.

Keywords: Nicholson's blowflies model, nonlinear mortality rate, stochastic perturbations, asymptotic stability.

2010 Mathematics Subject Classification: 34C25, 34K11, 34K25.

1 Introduction

Delay differential equations (DDEs) are typically used to describe dynamics of biology and ecology systems [3,4]. For example, Gurney et al. [5] proposed the following DDE

$$N'(t) = -\alpha N(t) + \beta N(t - \tau)e^{-\gamma N(t-\tau)} \quad (1.1)$$

to model the laboratory population of the Australian sheep-blowfly, where $N(t)$ represents the population size at time t , α is the per capita daily adult mortality rate, β is the maximum per capita daily egg production rate, $\frac{1}{\gamma}$ is the size at which the population reproduces at its

 Corresponding author. Email: hienlv@hnue.edu.vn

maximum rate and $\tau > 0$ is the generation time (i.e. the time taken from birth to maturity). This equation is known as the celebrated Nicholson's blowflies equation.

In the past four decades, Nicholson equation and its extensions have been extensively studied (see, for example, [2, 7, 12, 14] and the references therein). In particular, Wang et al. [13] considered a stochastic variant of model (1.1) where the mortality rate α is affected by environmental noises, $\alpha \rightsquigarrow \alpha - \sigma dB(t)$, which is presented by the following Itô-type differential equation

$$dN(t) = \left[-\alpha N(t) + \beta N(t - \tau) e^{-\gamma N(t - \tau)} \right] dt + \sigma N(t) dB(t) \quad (1.2)$$

with initial condition $N(s) = \phi(s)$, $s \in [-\tau, 0]$, $\phi \in C([-\tau, 0], [0, \infty))$ and $\phi(0) > 0$. Finite ultimate estimations for $\limsup_{t \rightarrow \infty} E[N(t)]$ and $\limsup_{t \rightarrow \infty} \frac{1}{t} E \left[\int_0^t N(s) ds \right]$ were obtained under condition $\alpha > \sigma^2/2$. The results of [13] were later extended to stochastic Nicholson's blowflies differential equations with regime switching

$$dN(t) = \left[-\alpha_{r_t} N(t) + \beta_{r_t} N(t - \tau_{r_t}) e^{-\gamma_{r_t} N(t - \tau_{r_t})} \right] dt + \sigma_{r_t} N(t) dB(t) \quad (1.3)$$

in [17], where $(r_t)_{t \geq 0}$ is a finite state continuous-time Markov chain. An extension of (1.2) to include a patch structure was also investigated in recent work [6].

However, the aforementioned works only dealt with stochastic Nicholson-type models with linear density-dependent mortality rates of the form $D(N) = \alpha N$ with some positive constant α . As discussed in [2], a model of linear density-dependent mortality rate will only be most accurate for populations at low densities. In addition, according to marine ecologists, many models in fishery such as marine protected areas or models of B-cell chronic lymphocytic leukemia dynamics are described by Nicholson-type delay differential equations of the form

$$N'(t) = -D(N(t)) + \beta N(t - \tau) e^{-\gamma N(t - \tau)}, \quad (1.4)$$

where the mortality rate function $D(N)$ is of the forms $D(N) = a - be^{-N}$ (type-I) or $D(N) = \frac{aN}{b+N}$ (type-II) with positive constants a and b . In the past few years, significant research attention has been devoted to studies of model (1.4) and its extensions. For example, by utilizing some reasoning techniques of the so-called fluctuation lemma combining with the method of using differential and integral inequalities, the problems of existence and global convergence of positive periodic/almost periodic solutions of Nicholson-type models with nonlinear mortality rates of type-I and type-II were investigated in [15] and [16], respectively. In [11], a novel approach based on comparison techniques via differential and integral inequalities and extended Lyapunov functions was developed to establish the existence, uniqueness and global attractivity of a positive periodic solution of Nicholson-type models with type-I mortality rate function. The proposed approach of [11] can also be utilized to derive conditions ensuring the global convergence of a unique positive equilibrium of autonomous (constant coefficients) Nicholson-type models with type-I mortality rates. However, up to date the study of Nicholson-type models as (1.4) subject to certain types of stochastic noises has received considerably less attention. It is noted that in population models, characteristic quantities as growth rates, environmental capacity, competition coefficients and some other parameters are always affected by environmental noises due to which model (1.4) is more suitable to be described by stochastic DDEs [8, 13]. Thus, it is relevant to study model (1.4) and its variants subject to certain type of stochastic noises. This motivates us for the present investigation.

In this paper, we study the problem of asymptotic stability in probability of a stochastic extension of model (1.4). Specifically, we consider Nicholson-type model (1.4) with nonlinear

mortality rate function $D(N) = a - be^{-N}$ for positive scalars a, b and apply the method of Son et al. [11] to establish the existence of a unique positive equilibrium namely N^* . We then consider the case that model (1.4) is exposed to stochastic perturbations which are proportional to the derivation of its state from the equilibrium point N^* . This will be represented in the form of an Itô stochastic differential equation. Based on the linearization method and by utilizing a stability criterion established for linear stochastic differential delay equations [9, Lemma 2.1], explicit delay-dependent stability conditions are obtained. The presented result is then also extended to models with multiple delays.

2 Preliminaries

Consider the following Nicholson-type delay differential equation

$$N'(t) = - \left(a - be^{-N(t)} \right) + \beta N(t - \tau) e^{-\gamma N(t - \tau)}, \quad t > 0, \quad (2.1)$$

with initial condition

$$N(s) = \phi(s) \quad \text{for } s \in [-\tau, 0] \quad \text{and} \quad \phi \in C([-\tau, 0], [0, \infty)), \phi(0) > 0, \quad (2.2)$$

where a, b, β, γ and τ are positive constants. It was shown in [11, Theorem 3.1] that if $b > a$ the initial value problem (IVP) governed by (2.1)-(2.2) has a unique solution $N(t, \phi)$ which is strictly positive on $[0, \infty)$ and satisfies $\liminf_{t \rightarrow \infty} N(t, \phi) \geq \ln(\frac{b}{a})$. Moreover, if $\frac{\beta}{\gamma e} < a < b$ then, for any solution $N(t, \phi)$ of (2.1)-(2.2), it holds that [11, Proposition 5.1]

$$\ln\left(\frac{b}{a}\right) \leq \liminf_{t \rightarrow \infty} N(t, \phi) \leq \limsup_{t \rightarrow \infty} N(t, \phi) \leq \ln\left(\frac{b}{a - \frac{\beta}{\gamma e}}\right). \quad (2.3)$$

2.1 Positive equilibrium

By substituting $N(t) = N^*$, a positive equilibrium point of (2.1) is defined by the following algebraic equation

$$-a + be^{-N^*} + \beta N^* e^{-\gamma N^*} = 0. \quad (2.4)$$

Assume that the parameters β, γ, a and b of model (2.1) satisfy the following condition

$$\beta \left(\frac{1}{\gamma e} + \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma \ln(\frac{b}{a})}{e^{\gamma \ln(\frac{b}{a})}} \right\} \right) < a < b. \quad (2.5)$$

Then, by (2.3), any positive equilibrium point of (2.1) is confined within the range $[r_1, r_2]$, where $r_1 = \ln(\frac{b}{a})$ and $r_2 = \ln\left(\frac{b}{a - \frac{\beta}{\gamma e}}\right)$.

Lemma 2.1. *Assume that $\frac{\beta}{\gamma e} < a < b$. Then, for any $x \in [r_1, r_2]$, where $r_1 = \ln(\frac{b}{a})$, $r_2 = \ln\left(\frac{b}{a - \frac{\beta}{\gamma e}}\right)$, it holds that*

$$|1 - \gamma x| e^{-\gamma x} \leq \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma \ln(\frac{b}{a})}{e^{\gamma \ln(\frac{b}{a})}} \right\}.$$

Proof. Let $\varphi(x) = |1 - \gamma x|e^{-\gamma x}$, $-\infty < x < \infty$. Note that $\varphi(x) = (1 - \gamma x)e^{-\gamma x}$ for $x < 1/\gamma$ and $\varphi'(x) = \gamma(\gamma x - 2)e^{-\gamma x} < 0$. Thus, the function $\varphi(x)$ is strictly decreasing on the interval $(-\infty, 1/\gamma)$. On the other hand, for $x > 1/\gamma$, we have $\varphi'(x) = \gamma(2 - \gamma x)e^{-\gamma x}$, $\varphi'(2/\gamma) = 0$, $\varphi'(x) > 0$ for $x \in (1/\gamma, 2/\gamma)$ and $\varphi'(x) < 0$ for $x > 2/\gamma$. Therefore, $\varphi(x) \leq \varphi(2/\gamma) = \frac{1}{e^2}$ for any $x \geq 1/\gamma$. This shows that for any $x \in [r_1, r_2]$, we have

$$\varphi(x) \leq \max \left\{ \frac{1}{e^2}, \varphi(r_1) \right\} = \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma \ln(\frac{b}{a})}{e^{\gamma \ln(\frac{b}{a})}} \right\}.$$

The proof of this lemma is now completed. \square

Lemma 2.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = xe^{-\gamma x}$, $\gamma > 0$. Then, $f(x) \leq (\gamma e)^{-1}$ for all $x \in \mathbb{R}$. Moreover, $f(x) = (\gamma e)^{-1}$ if and only if $x = 1/\gamma$.*

Proof. The derivative $f'(x)$ of $f(x)$ is given by

$$f'(x) = (1 - \gamma x)e^{-\gamma x}.$$

Thus, $f'(1/\gamma) = 0$, $f'(x) > 0$ for $x < 1/\gamma$ and $f'(x) < 0$ for $x > 1/\gamma$. Therefore, the function $f(x)$ is strictly increasing on the interval $(-\infty, 1/\gamma)$ and decreasing on the interval $(1/\gamma, \infty)$. This shows that $f(x)$ attains its maximum $f(1/\gamma) = (\gamma e)^{-1}$ at $x = 1/\gamma$. Consequently, $f(x) \leq (\gamma e)^{-1}$. The proof is completed. \square

It is clear that the function $\Psi(N) = -a + be^{-N} + \beta Ne^{-\gamma N}$ is continuous on $[r_1, r_2]$, $\Psi(r_1) = \beta r_1 e^{-\gamma r_1} > 0$ and $\Psi(r_2) = \beta(r_2 e^{-\gamma r_2} - \frac{1}{\gamma e}) < 0$ according to Lemma 2.2 and the fact $r_2 < 1/\gamma$. Thus, there exists an $N^* \in (r_1, r_2)$ such that $\Psi(N^*) = 0$, which is a positive equilibrium of (2.1). On the other hand, for any $N \in [r_1, r_2]$, by Lemma 2.1, we have $be^{-N} \geq be^{-r_2} = a - \frac{\beta}{\gamma e}$ and $|1 - \gamma N|e^{-\gamma N} \leq \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma \ln(\frac{b}{a})}{e^{\gamma \ln(\frac{b}{a})}} \right\}$. Therefore,

$$\Psi'(N) = -be^{-N} + \beta(1 - \gamma N)e^{-\gamma N} < 0, \quad \forall N \in [r_1, r_2],$$

which implies that the function $\Psi(N)$ is strictly decreasing on $[r_1, r_2]$. By this, we can conclude under condition (2.5) that model (2.1) has a unique positive equilibrium point N^* which is defined by equation (2.4).

2.2 Stochastic perturbations

Considering that equation (2.1) is affected by some white noise of the environment, which is proportional to the derivation of $N(t)$ from the equilibrium N^* [1]. Then, model (2.1) can be represented by the following Itô stochastic differential equation [9]

$$dN(t) = \left[-D(N(t)) + \beta N(t - \tau)e^{-\gamma N(t - \tau)} \right] dt + \sigma(N(t) - N^*)dB(t), \quad (2.6)$$

where $D(N) = a - be^{-N}$, σ denotes the intensity of the white noise and $B(t)$ is an one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Note that the equilibrium point N^* is also a stationary solution of the stochastic differential equation (2.6). We now define $N(t) = N^* + x(t)$ then, by (2.6), we have

$$dx(t) = \left[-D(x(t) + N^*) + \beta(N^* + x(t - \tau))e^{-\gamma(N^* + x(t - \tau))} \right] dt + \sigma x(t)dB(t). \quad (2.7)$$

Remark 2.3. By similar arguments of [17, Theorem 2.1] and [11, Theorem 3.1], it can be verified that for any initial function $\phi \in C([-\tau, 0], \mathbb{R})$, Eq. (2.7) possesses a unique solution $x(t, \phi)$ defined on the interval $[-\tau, \infty)$.

According to (2.4), we have $\beta N^* e^{-\gamma N^*} = a - be^{-N^*}$. Therefore,

$$\beta N^* e^{-\gamma(N^* + x(t-\tau))} = (a - be^{-N^*}) e^{-\gamma x(t-\tau)}.$$

This, together with (2.7), leads to

$$\begin{aligned} dx(t) = & \left[-a + be^{-N^*} e^{-x(t)} + (a - be^{-N^*}) e^{-\gamma x(t-\tau)} \right. \\ & \left. + \beta e^{-\gamma N^*} x(t-\tau) e^{-\gamma x(t-\tau)} \right] dt + \sigma x(t) dB(t). \end{aligned} \quad (2.8)$$

The asymptotic stability of the equilibrium N^* of (2.6) is equivalent to that of the zero solution $x = 0$ of (2.8) [10]. Thus, together with (2.8), we consider the following linearized equation at the zero point

$$d\tilde{x}(t) = [-\delta \tilde{x}(t) + p \tilde{x}(t-\tau)] dt + \sigma \tilde{x}(t) dB(t), \quad (2.9)$$

where $\delta = be^{-N^*}$ and

$$p = \beta e^{-\gamma N^*} - \gamma(a - be^{-N^*}) = \beta(1 - \gamma N^*) e^{-\gamma N^*}.$$

Note also that $N^* \leq r_2 < 1/\gamma$, thus δ, p are positive coefficients.

2.3 Auxiliary results

In this section, we present some definitions of stability and auxiliary results which will be used to derive stability conditions of the positive equilibrium point N^* of (2.1).

Definition 2.4 ([9]). The zero solution $x = 0$ of (2.7) is said to be stable in probability if for any $\epsilon > 0$, $\eta > 0$, there exists a $\delta > 0$ such that $\mathbb{P}\{\sup_{t \geq 0} |x(t, \phi)| > \epsilon | \mathcal{F}_0\} < \eta$ for any initial function $\phi \in C([-\tau, 0], \mathbb{R})$ with $\mathbb{P}\{\sup_{s \in [-\tau, 0]} |\phi(s)| < \delta\} = 1$.

Definition 2.5 ([9]). The linearized Eq. (2.9) is said to be (i) mean square stable (MSS) if for any given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for any initial function ϕ with $\sup_{s \in [-\tau, 0]} \mathbb{E}|\phi(s)|^2 < \delta$, it holds that $\mathbb{E}|\tilde{x}(t, \phi)|^2 < \epsilon$ for all $t \geq 0$, where $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation on $(\Omega, \mathcal{F}, \mathbb{P})$; and (ii) asymptotically mean square stable (AMSS) if it is MSS and any solution $\tilde{x}(t, \phi)$ of (2.9) satisfies $\lim_{t \rightarrow \infty} \mathbb{E}|\tilde{x}(t, \phi)|^2 = 0$.

Remark 2.6. As mentioned in [9, 10], the AMSS property of (2.9) implies stability in probability of the zero solution of nonlinear equation (2.7). This fact will be used to derive stability conditions for the equilibrium N^* .

In the remaining of this section, let us reformulate an auxiliary result on asymptotic mean square stability of linear stochastic differential equations from [9]. Consider the following linear stochastic differential equation

$$dx = [Ax(t) + Bx(t-\tau)] dt + \sigma x(t) dB(t) \quad (2.10)$$

where $A, B, \sigma, \tau \geq 0$ are known constants.

Lemma 2.7 ([9, Lemma 2.1, p. 44]). *The zero solution of (2.10) is asymptotically mean square stable if and only if*

$$A + B < 0, \quad G^{-1} > \frac{\sigma^2}{2},$$

where

$$G = \frac{2}{\pi} \int_0^\infty \frac{dt}{(A + B \cos \tau t)^2 + (t + B \sin \tau t)^2}.$$

Moreover,

$$G = \begin{cases} \frac{Bq^{-1} \sin(q\tau) - 1}{A + B \cos(q\tau)} & \text{for } B + |A| < 0, \quad q = \sqrt{B^2 - A^2}, \\ \frac{1 + |A|\tau}{2|A|} & \text{for } B = |A| < 0, \\ \frac{Bq^{-1} \sinh(q\tau) - 1}{A + B \cosh(q\tau)} & \text{for } A + |B| < 0, \quad q = \sqrt{A^2 - B^2}, \end{cases}$$

where $\sinh(\cdot)$ and $\cosh(\cdot)$ are the hyperbolic sine and hyperbolic cosine functions, respectively.

3 Stability conditions

For given scalars a, b, β, γ and τ , which satisfy condition (2.5), let N^* be the unique positive root of (2.4) in the interval $[r_1, r_2]$. We denote the following positive constants

$$\delta = be^{-N^*} \quad \text{and} \quad p = \beta(1 - \gamma N^*)e^{-\gamma N^*}. \quad (3.1)$$

We have the following result.

Theorem 3.1. *Assume that the condition given in Eq. (2.5) holds. Then, the linearized equation (2.9) is AMSS if and only if the following condition holds*

$$\frac{p \cosh(\tau \sqrt{\delta^2 - p^2}) - \delta}{\frac{p}{\sqrt{\delta^2 - p^2}} \sinh(\tau \sqrt{\delta^2 - p^2}) - 1} > \sigma^2/2, \quad (3.2)$$

where δ, p are positive constants given in Eq. (3.1).

Proof. As shown in the preceding section, under condition (2.5), the positive root N^* of (2.4) exists and is unique. Moreover, we have

$$-be^{-N^*} + \beta(1 - \gamma N^*)e^{-\gamma N^*} < 0.$$

Therefore, Eq. (2.9) is AMSS if and only if (see, Lemma 2.7)

$$G^{-1} > \sigma^2/2, \quad (3.3)$$

where

$$G = \frac{2}{\pi} \int_0^\infty \frac{dt}{(p \cos \tau t - \delta)^2 + (t + p \sin \tau t)^2}. \quad (3.4)$$

Moreover, the exact value of the constant G can be calculated via elementary functions as

$$G = \frac{q + \delta + pe^{-q\tau}}{q(q + \delta - pe^{-q\tau})},$$

where $q = \sqrt{\delta^2 - p^2}$. Using the fact that $\cosh(q\tau) = \sinh(q\tau) + e^{-q\tau}$, $q^2 - \delta^2 = -p^2$, we have

$$\begin{aligned} & (q + \delta + pe^{-q\tau})(p \cosh(q\tau) - \delta) \\ &= (q + \delta + pe^{-q\tau})(p \sinh(q\tau) + pe^{-q\tau} - \delta) \\ &= p(q + \delta) \sinh(q\tau) + p^2 e^{-q\tau} (\sinh(q\tau) + e^{-q\tau}) + p q e^{-q\tau} - \delta(q + \delta) \\ &= (q + \delta)(p \sinh(q\tau) - q) + p e^{-q\tau} (q - p \sinh(q\tau)) \\ &= (q + \delta - p e^{-q\tau})(p \sinh(q\tau) - q). \end{aligned}$$

Therefore,

$$G = \frac{\frac{p}{q} \sinh(q\tau) - 1}{p \cosh(q\tau) - \delta}.$$

This, together with (3.3), leads to condition (3.2). The proof is completed. \square

Remark 3.2. In a more restrictive case, we assume that

$$2\beta(1 - \gamma N^*)e^{-\gamma N^*} < b e^{-N^*} \quad \text{i.e. } \delta > 2p, \quad (3.5)$$

then the equality (3.4) can be estimated as follows

$$\begin{aligned} G &\leq \frac{2}{\pi} \int_0^\infty \frac{dt}{(\delta^2 - 2\delta p) + (t - p)^2} \\ &= \frac{1}{\sqrt{\delta^2 - 2\delta p}} \left(1 + \frac{2}{\pi} \arctan \frac{p}{\sqrt{\delta^2 - 2\delta p}} \right). \end{aligned} \quad (3.6)$$

By (3.3) and (3.6), a sufficient condition for the AMSS of Eq. (2.9) is

$$\frac{\sqrt{\delta^2 - 2\delta p}}{1 + \frac{2}{\pi} \arctan \frac{p}{\sqrt{\delta^2 - 2\delta p}}} > \sigma^2 / 2. \quad (3.7)$$

For Nicholson-type DDEs with multiple delays

$$N'(t) = - \left(a - b e^{-N(t)} \right) + \sum_{k=1}^m \beta_k N(t - \tau_k) e^{-\gamma_k N(t - \tau_k)}, \quad (3.8)$$

condition (2.5) is extended to (see [11], Theorem 5.2)

$$\sum_{k=1}^m \beta_k \left(\frac{1}{e^{\gamma_k}} + \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma_k \ln(\frac{b}{a})}{e^{\gamma_k \ln(\frac{b}{a})}} \right\} \right) < a < b \quad (3.9)$$

and the positive root N^* of the equation

$$-a + b^{-N^*} + \left(\sum_{k=1}^m \beta_k e^{-\gamma_k N^*} \right) N^* = 0 \quad (3.10)$$

exists and is unique. By a similar process, Eq. (2.9) is now given as

$$d\tilde{x}(t) = \left[-\delta \tilde{x}(t) + \sum_{k=1}^m p_k \tilde{x}(t - \tau_k) \right] dt + \sigma \tilde{x}(t) dB(t), \quad (3.11)$$

where

$$\delta = b e^{-N^*} \quad \text{and} \quad p_k = \beta_k (1 - \gamma_k N^*) e^{-\gamma_k N^*}, \quad k = 1, 2, \dots, m. \quad (3.12)$$

Similar to Theorem 3.1, Eq. (3.11) is AMSS if and only if $G_m^{-1} > \sigma^2/2$, where

$$G_m = \frac{2}{\pi} \int_0^\infty \frac{dt}{\left(\sum_{k=1}^m p_k \cos \tau_k t - \delta\right)^2 + \left(t + \sum_{k=1}^m p_k \sin \tau_k t\right)^2}. \quad (3.13)$$

Unfortunately, the computation of exact value of G_m in (3.13) is still an unsolved problem [9]. To derive sufficient conditions, we use the estimating method as (3.7). More specifically, assume that

$$\Delta^2 = \delta^2 - 2\delta \sum_{k=1}^m p_k - 4 \sum_{1 \leq i < j \leq m} p_i p_j > 0. \quad (3.14)$$

Then, we have

$$\begin{aligned} & \left(-\delta + \sum_{k=1}^m p_k \cos \tau_k t\right)^2 + \left(t + \sum_{k=1}^m p_k \sin \tau_k t\right)^2 \\ &= t^2 + 2t \sum_{k=1}^m p_k \sin \tau_k t + \delta^2 - 2\delta \sum_{k=1}^m p_k \cos \tau_k t \\ & \quad + \left(\sum_{k=1}^m p_k \sin \tau_k t\right)^2 + \left(\sum_{k=1}^m p_k \cos \tau_k t\right)^2 \\ & \geq t^2 - 2t \sum_{k=1}^m p_k + \delta^2 - 2\delta \sum_{k=1}^m p_k + \sum_{k=1}^m p_k^2 \\ & \quad + 2 \sum_{1 \leq i < j \leq m} p_i p_j \cos(\tau_i - \tau_j)t \\ & \geq \left(t - \sum_{k=1}^m p_k\right)^2 + \Delta^2. \end{aligned}$$

Therefore,

$$G_m \leq \frac{2}{\pi} \int_0^\infty \frac{dt}{\left(t - \sum_{k=1}^m p_k\right)^2 + \Delta^2} = \frac{1}{\Delta} \left(1 + \frac{2}{\pi} \arctan \frac{\sum_{k=1}^m p_k}{\Delta}\right).$$

In summary, we have the following result.

Proposition 3.3. Consider model (3.8) and assume that the derived conditions in Eqs. (3.9) and (3.14) are fulfilled, where δ and p_k , $k = 1, 2, \dots, m$, are positive constants defined in (3.12). Then, the linearized equation (3.11) is AMSS if the following condition holds

$$\frac{\sqrt{\delta^2 - 2\delta \sum_{k=1}^m p_k - 4 \sum_{1 \leq i < j \leq m} p_i p_j}}{1 + \frac{2}{\pi} \arctan \frac{\sum_{k=1}^m p_k}{\sqrt{\delta^2 - 2\delta \sum_{k=1}^m p_k - 4 \sum_{1 \leq i < j \leq m} p_i p_j}}} > \frac{\sigma^2}{2}. \quad (3.15)$$

Remark 3.4. Clearly, conditions (3.2), (3.7) and (3.15) hold for sufficiently small σ . In other words, the positive equilibrium N^* of model (2.1) or (3.8) is stable in probability under small stochastic perturbations. In this regard, the result of Proposition 3.3 in this paper extends that of Theorem 5.2 in [11].

4 Simulations

Consider model (2.1) with $\beta = 1$. It can be seen that condition (2.5) holds if and only if

$$\frac{1}{\gamma e} + \max \left\{ \frac{1}{e^2}, \frac{1 - \ln \kappa}{\kappa} \right\} < a < b, \quad (4.1)$$

where $\kappa = \left(\frac{b}{a}\right)^\gamma$. Since the equation $\frac{1 - \ln \kappa}{\kappa} = \frac{1}{e^2}$ has a unique positive root $\kappa_* \simeq 2.0576$, condition (4.1) holds if and only if

$$a > \begin{cases} \frac{1}{\gamma e} + \frac{1 - \ln \kappa}{\kappa} & \text{if } \kappa \in (1, \kappa_*) \\ \frac{1}{\gamma e} + \frac{1}{e^2} & \text{if } \kappa \geq \kappa_*. \end{cases} \quad (4.2)$$

For $\gamma = 0.5$, $\kappa = 1.1$, $a = 1.6$ and $b = 1.936$, Eq. (2.4) has a unique positive root $N^* = 0.4399$. Then, we have $\delta = 1.247$ and $p = 0.626$. With the delay $\tau = 2$, by condition (3.2), the linearized equation (2.9) is AMSS if and only if $\sigma^2 < 2.0266$. Simulation results given in Figure 4.1 are taken with $\sigma = 1.42$ and various initial conditions. It can be seen that all sample trajectories converge to N^* , which supports the conclusion.

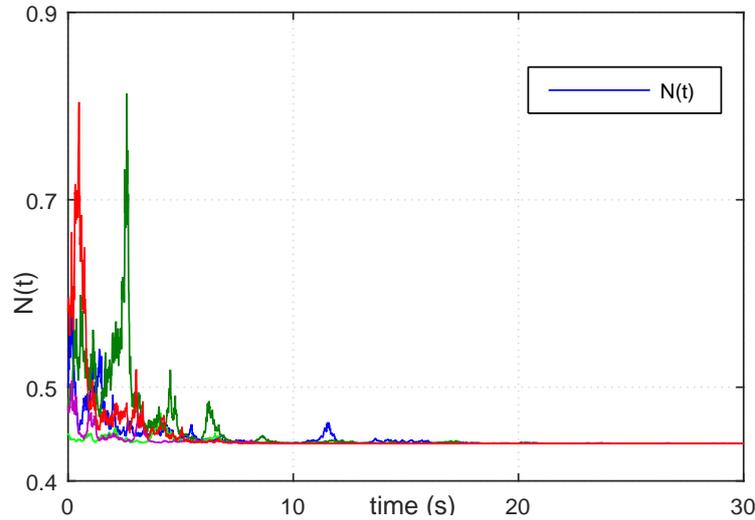


Figure 4.1: Sample trajectories of $N(t)$

5 Conclusions

In this paper, a stochastic Nicholson-type blowflies model with nonlinear density-dependent mortality rate has been investigated. Sufficient conditions have been derived to ensure the existence of a unique positive equilibrium which is stable in probability subject to stochastic perturbations of the white noise type. Numerical simulations have been given to illustrate the effectiveness of the derived stability conditions.

Acknowledgements

The authors would like to thank the editor(s) and anonymous reviewers for their constructive comments and suggestions which help to improve the present paper.

References

- [1] E. BERETTA, V. KOLMANOVSKII, L. SHAIKHET, Stability of epidemic model with time delays influenced by stochastic perturbations, *Math. Comput. Simulat.* **45**(1998), No. 3–4, 269–277. [https://doi.org/10.1016/S0378-4754\(97\)00106-7](https://doi.org/10.1016/S0378-4754(97)00106-7); Zbl 1017.92504
- [2] L. BEREZANSKY, E. BRAVERMAN, L. IDELS, Nicholson’s blowflies differential equations revisited: Main results and open problems, *Appl. Math. Model.* **34**(2010), No. 6, 1405–1417. <https://doi.org/10.1016/j.apm.2009.08.027>; Zbl 1193.34149
- [3] L. EDELSTEIN-KESHET, *Mathematical models in biology*, SIAM, Philadelphia, 2004. <https://doi.org/10.1137/1.9780898719147>; Zbl 0674.92001
- [4] T. ERNEUX, *Applied delay differential equations*, Springer-Verlag, New York, 2009. <https://doi.org/10.1007/978-0-387-74372-1>; Zbl 1201.34002
- [5] W. S. C. GURNEY, S. P. BLYTHE, R. M. NISBET, Nicholson’s blowflies revisited, *Nature* **287**(1980), 17–21. <https://doi.org/10.1038/287017a0>
- [6] X. LI, G. LIU, Analysis of stochastic Nicholson-type delay system with patch structure, *Appl. Math. Lett.* **96**(2019), 223–229. <https://doi.org/10.1016/j.aml.2019.05.016>; Zbl 07111465
- [7] B. LIU, Global exponential stability of positive periodic solutions for a delayed Nicholson’s blowflies model, *J. Math. Anal. Appl.* **412**(2015), No. 1, 212–221. <https://doi.org/10.1016/j.jmaa.2013.10.049>; Zbl 1308.34096
- [8] R. M. MAY, *Stability and complexity in model ecosystems*, Princeton University Press, 1973.
- [9] L. SHAIKHET, *Lyapunov functional and stability of stochastic functional differential equations*, Springer, New York, 2013. <https://doi.org/10.1007/978-3-319-00101-2>; Zbl 1277.34003
- [10] L. SHAIKHET, Stability of equilibrium states of a nonlinear delay differential equation with stochastic perturbations, *Int. J. Robust Nonlinear Control* **27**(2017), No. 6, 915–924. <https://doi.org/10.1002/rnc.3605>; Zbl 1369.93702
- [11] D. T. SON, L. V. HIEN, T. T. ANH, Global attractivity of positive periodic solution of a delayed Nicholson model with nonlinear density-dependent mortality term, *Electron J. Qual. Theory Differ. Equ.* **2019**, No. 8, 1–21. <https://doi.org/10.14232/ejqtde.2019.1.8>; Zbl 1424.34281
- [12] L. WANG, Almost periodic solution for Nicholson’s blowflies model with patch structure and linear harvesting terms, *Appl. Math. Model.* **37**(2013), No. 4, 2153–2165. <https://doi.org/10.1016/j.apm.2012.05.009>; Zbl 1349.34288
- [13] W. WANG, L. WANG, W. CHEN, Stochastic Nicholson’s blowflies delayed differential equations, *Appl. Math. Lett.* **87**(2019), 20–26. <https://doi.org/10.1016/j.aml.2018.07.020>; Zbl 1408.34067
- [14] W. WANG, F. LIU, W. CHEN, Exponential stability of pseudo almost periodic delayed Nicholson-type system with patch structure, *Math. Meth. Appl. Sci.* **42**(2019), No. 2, 592–604. <https://doi.org/10.1002/mma.5364>; Zbl 1174.76310

- [15] Y. XU, New stability theorem for periodic Nicholson's model with mortality term, *Appl. Math. Lett.* **94**(2019), 59–65. <https://doi.org/10.1016/j.aml.2019.02.021>; Zbl 1412.34232
- [16] L. YAO, Dynamics of Nicholson's blowflies models with a nonlinear density-dependent mortality, *Appl. Math. Model.* **64**(2018), 185–195. <https://doi.org/10.1016/j.apm.2018.07.007>; Zbl 07183285
- [17] Y. ZHU, K. WANG, Y. REN, Y. D. ZHUANG, Stochastic Nicholson's blowflies delay differential equation with regime switching, *Appl. Math. Lett.* **94**(2019), 187–195. <https://doi.org/10.1016/j.aml.2019.03.003>; Zbl 07048204