



## On a Neumann boundary value problem for Ermakov–Painlevé III

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**Abstract.** A Neumann-type boundary value problem is investigated for a hybrid Ermakov–Painlevé equation. Existence properties are established and a sequence of approximate solutions is investigated. In an appendix, a novel class of coupled Hamiltonian Ermakov–Painlevé III systems is introduced and shown via a reciprocal transformation to be reducible to a canonical, integrable Ermakov–Ray–Reid system.

**Keywords:** Ermakov–Painlevé III, Neumann boundary conditions, singular ODEs, integrable Ermakov–Ray–Reid systems.

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### 1 Introduction

In recent work [22, 23, 41], prototype integrable Ermakov–Painlevé II–IV equations have been derived according to

#### Ermakov–Painlevé II

$$\ddot{\rho} + \left[ -\rho^2 + \frac{t}{2} \right] \rho = -\frac{(\alpha + \frac{1}{2})^2}{4\rho^3}, \quad (1.1)$$

#### Ermakov–Painlevé III

$$\ddot{\rho} - \left[ \frac{\dot{\rho}^2}{\rho^2} - \frac{\dot{\rho}}{\rho t} + \frac{1}{2\rho^2 t} (\alpha\rho^4 + \beta) + \frac{\gamma\rho^4}{2} \right] \rho = \frac{\delta}{2\rho^3}, \quad (1.2)$$

#### Ermakov–Painlevé IV

$$\ddot{\rho} - \left[ \frac{3}{4}\rho^4 + 2t\rho^2 + t^2 - \alpha \right] \rho = \frac{\beta}{2\rho^3}. \quad (1.3)$$

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In the above, a dot indicates a derivative with respect to the independent variable  $t$ .

It is recalled that the classical Ermakov equation with genesis in work of [13], namely

$$\ddot{\rho} + \omega(t)\rho = \frac{\alpha}{\rho^3} \quad (1.4)$$

arises, *inter alia* in the analysis of the large amplitude oscillation of thin-walled tubes of Mooney–Rivlin hyperelastic materials [30,43]. Importantly, the nonlinear superposition principle admitted by (1.4) allows the exact solution of initial value problems associated with a variety of boundary loadings [43]. In [20,21], two-component nonlinear coupled Ermakov systems were introduced according to

$$\begin{aligned} \ddot{\rho} + \omega(t)\rho &= \frac{1}{\rho^2\sigma}\Phi(\sigma/\rho), \\ \ddot{\sigma} + \omega(t)\sigma &= \frac{1}{\rho\sigma^2}\Psi(\rho/\sigma), \end{aligned} \quad (1.5)$$

and which admit a distinctive integral of motion, namely the invariant

$$\mathcal{I} = \frac{1}{2}(\rho\dot{\sigma} - \sigma\dot{\rho})^2 + \int^{\sigma/\rho} \Phi(z)dz + \int^{\rho/\sigma} \Psi(w)dw \quad (1.6)$$

together with concomitant nonlinear superposition principles. Subsequently in [34], a 2+1-dimensional extension of the Ermakov–Ray–Reid system (1.5) was constructed, while extensions to Ermakov-type systems of arbitrary order and dimension and which admit analogues of its characteristic invariant were presented in [42]. Therein, alignment of a 2+1-dimensional Ermakov system and an integrable Ernst-type system was shown to generate a novel integrable hybrid of the 2+1-dimensional solitonic sinh-Gordon system of [15,16] and a Ermakov-type system. Multi-component Ermakov systems were introduced in [39] via a symmetry reduction of a 2+1-dimensional multi-layer hydrodynamic model. Novel decomposition of classes of many-body problems into such integrable multi-component Ermakov systems have recently been obtained in [27,28]. The canonical Ermakov–Ray–Reid system (1.5) has a diverse range of physical applications, notably in nonlinear optics, hydrodynamics, gas cloud evolution theory and magneto-gasdynamics (see [29,35–38] and work cited therein).

The six classical Painlevé equations, commonly denoted by PI–PVI likewise arise in a wide range of physical contexts and play a basic role in modern soliton theory (see e.g. Conte [10]). Like Ermakov–Ray–Reid systems and their single component classical reduction (1.4) they admit nonlinear superposition principles, in the Painlevé case as generated via Bäcklund transformations. They possess linear representations and it is recalled that Ermakov–Ray–Reid systems (1.5) also possess underlying linear structure, albeit of another kind [9]. These commonalities make the analysis of hybrid integrable Ermakov–Painlevé systems of natural research interest.

The study of Ermakov–Painlevé equations was initiated in [22] where a symmetry reduction of a multi-component resonant Manakov system led to a novel two-component Ermakov–Painlevé II sub-system. The latter was shown to admit a key underlying Ermakov invariant which was applied to derive a single component canonical Ermakov–Painlevé II equation of the type (1.1) for an associated wave packet amplitude. In subsequent work in [6], an integrable Painlevé–Gambier equation as derived in a three-ion reduction of an m-ion electro-diffusion system in [11] was shown to be related via the electric field to the EPII equation (1.1). A connection between the latter and the classical PII equation

$$\ddot{w} = 2w^3 + tw + \alpha \quad (1.7)$$

was thereby obtained. The EPII avatar of (1.7) proves to be of importance not only because of its direct physical application but because it plays a basic role in the construction of novel hybrid integrable multi-component EPII systems which admit characteristic Ermakov invariants [40]. The application of a Bäcklund transformation to generate iteratively exact solutions of such EPII systems was presented in [40]. EPII similarity reductions have been made of the classical Korteweg capillarity system in [32] and in a cold plasma context in [33].

In [23], a Ermakov–Painlevé IV (EPIV) system was obtained through a symmetry reduction of a derivative nonlinear Schrödinger (NLS) system. The single component EPIV equation (1.3) constitutes a canonical base member of this system. Bäcklund transformations have recently been applied in [31] to generate iteratively exact solutions of these integrable EPIV systems via the linked classical PIV equation

$$\ddot{\omega} = \frac{1}{2\omega}\dot{\omega}^2 + \frac{3}{2}\omega^3 + 4t\omega^2 + 2(t^2 - \alpha)\omega + \frac{\beta}{\omega}. \quad (1.8)$$

In [26], hybrid EPII–IV systems have been set in a general context. Their admitted Ermakov invariants have been exploited to establish integrability properties.

Boundary value problems for both the classical PII and PIV equations have been treated in a series of papers [2–5, 8, 17]. Dirichlet boundary value problems have been investigated in [6, 7] respectively and existence together with uniqueness properties established. Here, a Neumann-type boundary value problem is considered for the hybrid EPIII equation (1.2) as obtained by setting  $\omega = \rho^2$  in the classical PIII equation

$$\ddot{\omega} = \frac{\dot{\omega}^2}{\omega} - \frac{\dot{\omega}}{t} + \frac{1}{t}(\alpha\omega^2 + \beta) + \gamma\omega^3 + \frac{\delta}{\omega}. \quad (1.9)$$

## 2 A class of boundary value problems

Here, the existence of positive solutions bounded over the interval  $(0, 1)$  are considered for the Ermakov–Painlevé III equation (1.2) subject to the boundary conditions

$$\lim_{t \rightarrow 0^+} t\dot{\rho}(t) = \dot{\rho}(1) = 0. \quad (2.1)$$

It is observed that, on setting  $u := t\dot{\rho}$ , (1.2) adopts the form

$$\dot{u} = \frac{u^2}{t\rho} + \frac{1}{2\rho}(\alpha\rho^4 + \beta) + \frac{t\gamma\rho^5}{2} + \frac{t\delta}{2\rho^3}. \quad (2.2)$$

Both mixed boundary and Sturm–Liouville conditions have been investigated for equations of the type

$$\dot{u} = p(t)q(t)f(t, \rho, u) \quad (2.3)$$

where  $u := p\dot{\rho}$  have previously been considered elsewhere in the literature [1, 14, 19] with standard assumptions

$$\int_0^1 p(t)q(t) dt < \infty, \quad \int_0^1 \frac{1}{p(t)} \int_0^t p(s)q(s) ds dt < \infty.$$

and  $f$  is non-singular with respect to the time variable  $t$ . Such conditions are not fulfilled for the avatar (2.2) of EPIII and, one is led to consider a Neumann boundary value problem over an interval  $[\eta, 1]$  for arbitrary  $\eta \in (0, 1)$  and obtain a solution of the original problem

by means of a sequence of approximate solutions with  $\eta \rightarrow 0$ . To this end, here we proceed under the assumptions

$$\alpha > 0, \quad \beta < 0, \quad \gamma \geq 0, \quad \delta \leq 0 \quad (2.4)$$

and will establish that the boundary value problem for Ermakov–Painlevé III with side conditions (2.1) admits at least one positive bounded solution. Thus, the existence of a positive solution of (1.2) will be proved under the Neumann-type boundary conditions

$$\dot{\rho}(\eta) = \dot{\rho}(1) = 0 \quad (2.5)$$

for arbitrary  $\eta \in (0, 1)$ . Moreover, we obtain upper and lower bounds for  $\rho(t)$  and  $u = t\dot{\rho}(t)$  that do not depend on  $\eta$ .

Here, application of the method of upper and lower solutions is made as follows.

**Theorem 2.1.** *Under the conditions (2.4) on the Painlevé parameters, there exist positive constants  $\varepsilon, M, B$  and  $C$  independent of  $\eta$  such that the Ermakov–Painlevé equation (1.2) subject to the Neumann conditions (2.5) has at least one solution satisfying*

$$\varepsilon < \rho(t) < M, \quad -tC \leq t\dot{\rho}(t) \leq B, \quad \eta \leq t \leq 1.$$

*Proof.* Fix  $M, \varepsilon > 0$  such that  $M > \varepsilon$  and

$$\alpha M^4 + \beta + \frac{\delta}{M^2} > 0 > \alpha \varepsilon^4 + \beta + \gamma \varepsilon^6.$$

Thus, it is readily verified that  $(\varepsilon, M)$  is an ordered couple of a lower and upper solution for the EPIII equation (1.2) under the Neumann conditions (2.5). Next, observe that the right-hand side of (1.2) has quadratic growth with respect to  $\dot{\rho}$ . Thus, a standard Nagumo condition (see e.g. [12, 18]) holds and the existence of a solution  $\rho$  with  $\varepsilon < \rho < M$  follows. Furthermore, integration of  $\dot{u}$  over  $[\eta, 1]$  yields

$$\int_{\eta}^1 \left( \frac{t\dot{\rho}(t)^2}{\rho(t)} + \frac{\alpha\rho(t)^4 + \beta}{2\rho(t)} + \frac{t\gamma\rho(t)^5}{2} + \frac{t\delta}{2\rho(t)^3} \right) dt = u(1) - u(\eta) = 0,$$

and hence

$$\int_{\eta}^1 \frac{t\dot{\rho}(t)^2}{\rho(t)} dt \leq \int_{\eta}^1 \left| \frac{\alpha\rho(t)^4 + \beta}{2\rho(t)} + \frac{t\gamma\rho(t)^5}{2} + \frac{t\delta}{2\rho(t)^3} \right| dt.$$

Because  $\varepsilon \leq \rho \leq M$ , setting

$$\mathcal{C}(t) := \frac{\alpha\rho(t)^4 + \beta}{2\rho(t)} + \frac{t\gamma\rho(t)^5}{2} + \frac{t\delta}{2\rho(t)^3} \quad (2.6)$$

we deduce that  $|\mathcal{C}(t)| \leq C$  for some constant  $C$  depending only on  $\varepsilon$  and  $M$ , whence

$$\int_{\eta}^1 \frac{t\dot{\rho}(t)^2}{\rho(t)} dt \leq C(1 - \eta) \leq C.$$

This, in turn, implies for arbitrary  $r \in [\eta, 1]$

$$-r\dot{\rho}(r) = \int_r^1 \dot{u}(t) dt = \int_r^1 \left( \frac{t\dot{\rho}(t)^2}{\rho(t)} + \mathcal{C}(t) \right) dt$$

and consequently  $|r\dot{\rho}(r)| \leq 2C := B$ . Finally, we shall prove that  $\dot{\rho}$  is bounded from below independently of  $\eta$ . Indeed, suppose that  $\dot{\rho}$  achieves its absolute minimum value at some  $t_0 \in (\eta, 1)$  with  $\dot{\rho}(t_0) = -m < 0$ . Then  $\ddot{\rho}(t_0) = 0$  and it follows from equation (1.2) that

$$0 = \frac{m^2}{\rho(t_0)} + \frac{m}{t_0} + \frac{\alpha\rho(t_0)^4 + \beta}{2t_0\rho(t_0)} + \frac{\gamma\rho(t_0)^5}{2} + \frac{\delta}{2\rho(t_0)^3} > \frac{m}{t_0} + \frac{\mathcal{C}(t_0)}{t_0}.$$

Hence

$$\frac{m}{t_0} < \frac{C}{t_0}.$$

We conclude that  $\dot{\rho} > -C$  and the claim follows.  $\square$

### 3 A sequence of approximate solutions

Assume that (2.4) is satisfied and let  $\eta_n \searrow 0$ . From the previous section, there exist  $\varepsilon, M, B, C > 0$  and a sequence  $\{\rho_n\}$  of solutions of (1.2) satisfying (2.5) for  $\eta_n$  with  $\varepsilon < \rho_n < M$  and such that  $-tC \leq t\dot{\rho}_n(t) \leq B$  holds for  $t \in [\eta_n, 1]$ . Setting  $\rho_n(t) \equiv \rho_n(\eta_n)$  for  $0 \leq t \leq \eta_n$ , we may assume that  $\rho_n \in H^2(0, 1)$  for all  $n$ .

Furthermore, observe that  $\{t^2\dot{\rho}_n\}$  is uniformly bounded and, consequently, the sequence  $\{t^2\rho_n\}$  is bounded in  $W^{2,\infty}(0, 1)$ . Thus, passing to a subsequence if necessary, we may assume that  $\{t^2\rho_n\}$  converges in  $C^1([0, 1])$  to some mapping  $\theta(t)$ . For convenience, for  $t > 0$  we shall write  $\theta(t) = t^2\rho(t)$  and a simple computation shows that

$$\rho_n \rightarrow \rho \quad \text{in } C^1([\eta, 1])$$

for arbitrary  $\eta > 0$ .

Next, fix  $\varphi \in C_0^\infty(0, 1)$  and  $\eta \in (0, 1)$  such that  $\text{supp}(\varphi) \subset (\eta, 1)$ . Taking limits at both sides of the equality

$$-\int_\eta^1 \dot{\varphi}(t)\dot{\rho}_n(t) dt = \int_\eta^1 \varphi(t)\mathcal{E}(t, \rho_n(t), \dot{\rho}_n(t)) dt,$$

where

$$\mathcal{E}(t, \rho_n(t), \dot{\rho}_n(t)) = \frac{\dot{\rho}_n^2}{\rho_n} - \frac{\dot{\rho}_n}{t} + \frac{1}{2t\rho_n}(\alpha\rho_n^4 + \beta) + \frac{\gamma\rho_n^5}{2} + \frac{\delta}{2\rho_n^3}$$

we deduce that  $\rho$  is a weak solution (and, by standard results, classical) of (1.2) in  $(0, 1)$ . Moreover, because  $-tC \leq t\dot{\rho}_n(t) \leq B$  for all  $t$  and all  $n$ , it is deduced that  $-tC \leq t\dot{\rho}(t) \leq B$  for  $t \in (0, 1]$ . Clearly,  $\dot{\rho}(1) = 0$ , so it only remains to verify that  $t\dot{\rho}(t) \rightarrow 0$  as  $t \rightarrow 0$ . It proves convenient to write

$$\dot{u}(t) = \frac{t\dot{\rho}(t)^2}{\rho(t)} + \mathcal{C}(t),$$

where the mappings  $u(t) := t\dot{\rho}(t)$  and  $\mathcal{C}(t)$  defined as in (2.6) are bounded. By the mean value theorem, if  $\limsup_{t \rightarrow 0^+} u(t) > \liminf_{t \rightarrow 0^+} u(t)$ , then there exists a sequence  $r_n \rightarrow 0^+$  such that  $\dot{u}(r_n) \rightarrow -\infty$ , a contradiction because  $\dot{u}(t) \geq \mathcal{C}(t) \geq -C$ . We conclude that  $L := \lim_{t \rightarrow 0^+} u(t)$  exists.

Since  $-tC \leq u(t)$ , it is readily seen that  $L \geq 0$ . In order to prove that  $L \not\equiv 0$ , let us simply observe that if  $t\dot{\rho}(t) \geq c > 0$  over some interval  $(0, \eta)$  then, for  $t \in (0, \eta)$ ,

$$\rho(\eta) - \rho(t) = \int_t^\eta \dot{\rho}(s) ds \geq \int_t^\eta \frac{c}{s} ds = c \ln(\eta/t) \rightarrow +\infty$$

as  $t \rightarrow 0$ , which contradicts the fact that  $\rho$  is bounded.

Summarizing, the following existence result has been proved.

**Theorem 3.1.** *Under the conditions (2.4) on the Painlevé parameters, the Ermakov–Painlevé equation (1.2) subject to the boundary conditions (2.1) has at least one positive solution.*

## 4 An integrable Hamiltonian Ermakov–Painlevé III system

Ermakov–Ray–Reid systems (1.5) which admit a Hamiltonian have been determined in [29,35]. Therein, the requirements

$$\frac{1}{\rho^2\sigma}\Phi(\sigma/\rho) = -\frac{\partial W}{\partial\rho}, \quad \frac{1}{\sigma\rho^2}\Psi(\rho/\sigma) = -\frac{\partial W}{\partial\sigma}$$

were imposed and the parametrisation

$$\Phi = 2\left(\frac{\sigma}{\rho}\right)J(\sigma/\rho) + \left(\frac{\sigma}{\rho}\right)^2 J'(\sigma/\rho), \quad \Psi = -\left(\frac{\sigma}{\rho}\right)^2 J'(\sigma/\rho)$$

obtained. Here, the hybrid Ermakov–Painlevé III system is considered, namely

$$\begin{aligned} \ddot{\phi}_1 - \left[ \frac{\dot{\rho}^2}{\rho^2} - \frac{\dot{\rho}}{\rho t} + \frac{1}{2\rho^2 t}(\alpha\rho^4 + \beta) + \frac{\gamma\rho^4}{2} + \frac{\delta}{2\rho^4} \right] \phi_1 \\ = \frac{1}{\phi_1^2 \phi_2} \left[ 2\left(\frac{\phi_2}{\phi_1}\right)J(\phi_2/\phi_1) + \left(\frac{\phi_2}{\phi_1}\right)^2 J'(\phi_2/\phi_1) \right], \\ \ddot{\phi}_2 - \left[ \frac{\dot{\rho}^2}{\rho^2} - \frac{\dot{\rho}}{\rho t} + \frac{1}{2\rho^2 t}(\alpha\rho^4 + \beta) + \frac{\gamma\rho^4}{2} + \frac{\delta}{2\rho^4} \right] \phi_2 \\ = \frac{1}{\phi_1 \phi_2^2} \left[ -\left(\frac{\phi_2}{\phi_1}\right)^2 J'(\phi_2/\phi_1) \right] \end{aligned}$$

where  $\rho$  is governed by the EPIII equation (1.2), namely

$$\ddot{\rho} - \left[ \frac{\dot{\rho}^2}{\rho^2} - \frac{\dot{\rho}}{\rho t} + \frac{1}{2\rho^2 t}(\alpha\rho^4 + \beta) + \frac{\gamma\rho^4}{2} + \frac{\delta}{2\rho^4} \right] \rho = 0.$$

Thus,

$$\begin{aligned} \ddot{\phi}_1 \rho - \ddot{\rho} \phi_1 &= \frac{\rho}{\phi_1^2 \phi_2} \left[ 2\left(\frac{\phi_2}{\phi_1}\right)J(\phi_2/\phi_1) + \left(\frac{\phi_2}{\phi_1}\right)^2 J'(\phi_2/\phi_1) \right], \\ \ddot{\phi}_2 \rho - \ddot{\rho} \phi_2 &= \frac{\rho}{\phi_1 \phi_2^2} \left[ -\left(\frac{\phi_2}{\phi_1}\right)^2 J'(\phi_2/\phi_1) \right], \end{aligned}$$

whence, on introduction of the transformation

$$\left. \begin{aligned} c\phi_1^* &= \phi_1/\rho, & \phi_2^* &= \phi_2/\rho \\ dt^* &= \rho^{-2}dt \end{aligned} \right\} \mathbb{R}^*$$

reduction is made to the canonical Hamiltonian Ermakov–Ray–Reid system of [29,35], namely

$$\begin{aligned}\phi_{1t^{**}}^* &= \frac{1}{\phi_1^{*2}\phi_2^*} \left[ 2 \left( \frac{\phi_2^*}{\phi_1^*} \right) J(\phi_2^*/\phi_1^*) + \left( \frac{\phi_2^*}{\phi_1^*} \right)^2 J'(\phi_2^*/\phi_1^*) \right], \\ \phi_{2t^{**}}^* &= \frac{1}{\phi_1^*\phi_2^{*2}} \left[ - \left( \frac{\phi_2^*}{\phi_1^*} \right)^2 J'(\phi_2^*/\phi_1^*) \right].\end{aligned}$$

The two integrals of motion of the latter system, namely its Ermakov invariant and Hamiltonian, allow its algorithmic solution in the manner described in [29, 35]. It is noted that if the transformation  $\mathbb{R}^*$  is supplemented by the relation  $\rho^* = \rho^{-1}$  then  $\mathbb{R}^{*2} = I$  so that  $\mathbb{R}^*$  constitutes a reciprocal-type transformation. This kind of reciprocal transformation has been employed in [41] to reduce certain non-autonomous Toda–Painlevé systems to integrable canonical form. It has likewise recently been used in the exact solution of moving boundary problems of Stefan-type relevant to the analysis of seepage phenomena in heterogeneous media in soil mechanics [24,25].

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