



Ground state solutions for asymptotically periodic fractional Choquard equations

Sitong Chen and Xianhua Tang 

School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P. R. China

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Abstract. This paper is dedicated to studying the following fractional Choquard equation

$$(-\Delta)^s u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^\mu} dy \right) Q(x)f(u), \quad u \in H^s(\mathbb{R}^N),$$

where $s \in (0, 1)$, $N \geq 3$, $\mu \in (0, N)$, $V(x)$ and $Q(x)$ are periodic or asymptotically periodic, and $F(t) = \int_0^t f(s)ds$. By combining the non-Nehari manifold approach with some new inequalities, we establish the existence of Nehari type ground state solutions for the above problem in the periodic and asymptotically periodic cases under mild assumptions on f . Our results generalize and improve the ones in [Y. H. Chen, C. G. Liu, *Nonlinearity* **29**(2016), 1827–1842] and some related literature.

Keywords: fractional Laplacian, Choquard equation, ground state solution, asymptotically periodic.

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1 Introduction

In this paper, we are concerned with the following fractional Choquard equation

$$(-\Delta)^s u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^\mu} dy \right) Q(x)f(u), \quad u \in H^s(\mathbb{R}^N), \quad (1.1)$$

where $s \in (0, 1)$, $N \geq 3$, $\mu \in (0, N)$, $F(t) = \int_0^t f(s)ds$, V, Q and f satisfy

(VQ) $V, Q \in L^\infty(\mathbb{R}^N, \mathbb{R})$, $\text{ess inf}_{x \in \mathbb{R}^N} V(x) > 0$ and $\text{ess inf}_{x \in \mathbb{R}^N} Q(x) > 0$;

(F1) $f \in C(\mathbb{R}, \mathbb{R})$, and there exist constants $C_0 > 0$ and $2 - \frac{\mu}{N} < p_1 \leq p_2 < \frac{2_s^*}{2}(2 - \frac{\mu}{N})$ such that

$$|f(t)| \leq C_0 \left(|t|^{p_1-1} + |t|^{p_2-1} \right), \quad \forall t \in \mathbb{R},$$

where $2_s^* := 2N/(N - 2s)$;

 Corresponding author. Email: tangxh@mail.csu.edu.cn

(F2) $f(t)$ is nondecreasing on \mathbb{R} ;

(F3) $\lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|} = +\infty$.

The fractional Laplacian $(-\Delta)^s$ in \mathbb{R}^N is a nonlocal pseudo-differential operator taking the form

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where $C_{N,s}$ is a normalization constant, see [3, 21]. In this paper, we consider the fractional Laplacian in the weak sense. For any $N \geq 3$ and $s \in (0, 1)$, under (VQ), the fractional Sobolev space $H^s(\mathbb{R}^N)$ can be defined as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^N) \right\}$$

endowed with scalar product and norm

$$(u, v) = \int_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv \right] dx, \quad \|u\| = \left(\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)u^2 \right] dx \right)^{1/2}.$$

From [16, Lemma 2.1], $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $2 \leq q \leq 2_s^*$ and compactly embedded into $L_{loc}^q(\mathbb{R}^N)$ for $2 \leq q < 2_s^*$. Define the energy functional $\Phi : H^s(\mathbb{R}^N) \rightarrow \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)u^2 \right] dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)F(u(x))F(u(y))}{|x - y|^\mu} dx dy. \quad (1.2)$$

As we shall see in Section 2, $\frac{2_s^*}{2} (2 - \frac{\mu}{N})$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, (VQ) and (F1) imply that $\Phi \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$. Let

$$\mathcal{N} := \{u \in H^s(\mathbb{R}^N) \setminus \{0\} : \langle \Phi'(u), u \rangle = 0\},$$

which is the Nehari manifold of Φ .

Problem (1.1) presents nonlocal characteristics in the nonlinearity as well as in the (fractional) diffusion. Such a problem has a strong physical meaning because the fractional Laplacian appears in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geographical fluid dynamics, and American options in finance, see [3, 5, 21]; and the nonlocal nonlinearities were also used to model the dynamics of pseudo-relativistic boson stars, see [15, 18].

If $s = 1$, then (1.1) formally reduces to the following generalized Choquard equation:

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x - y|^\mu} dy \right) Q(x)f(u), \quad u \in H^1(\mathbb{R}^N), \quad (1.3)$$

which goes back to the description of the quantum theory of a polaron at rest by Pekar [30] in the case $N = 3$, $\mu = 1$ and $f(u) = u$, see [22, 31] for more details in the physical aspects. In the last decades, there have been many results on nontrivial solutions, ground state solutions and multiple solutions for (1.3), see e.g. [7, 22, 24, 26–28] for the case where $V = Q = 1$; see e.g. [1, 2, 19, 40] for V or Q is nonconstant. In particular, when $N = 3$, $Q = 1$, $\mu = 1$, $f(u) = u$ and V is a continuous periodic function, Ackermann [2] proved the existence of nontrivial solutions by reduction methods. When V and Q are asymptotically periodic, Zhang, Xu and Zhang [40] proved that (1.3) has a ground state solution, based on the generalized Nehari manifold method, developed by Szulkin and Weth in [34], if f satisfies (F1), (F3) and the following monotonicity assumption:

(Ne) $f(t)$ is increasing on \mathbb{R} .

We point out that (Ne) is very crucial in the arguments of [40]. In fact, the starting point of their approach is to show that for each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, the Nehari manifold \mathcal{N} intersects $H^1(\mathbb{R}^N)$ in exactly one point $\hat{m}(u) = t_u u$ with $t_u > 0$. The uniqueness of $\hat{m}(u)$ enables one to define a map $u \mapsto \hat{m}(u)$, which is important in the remaining arguments.

Recently, many researchers began to focus on problems like (1.1). The greatest part of the literature focuses on the study of (1.1) with $V = Q = 1$, see e.g. [17, 22, 25] for the case where $N = 3$, $s = 1/2$, $\mu = 1$ and $f(u) = u$; see [13] for the case where $N \geq 3$, $s \in (0, 1)$ and $f(u) = |u|^{p-2}u$ with $2 - \frac{\mu}{N} < p < \frac{2^*}{2}(2 - \frac{\mu}{N})$; see [33] for the case where $N \geq 3$, $s \in (0, 1)$ and f satisfies the assumption of Berestycki–Lions type [4, 29]. Since (1.1) with $V = Q = 1$ is autonomous, d’Avenia, Siciliano and Squassina [13] showed that the following two minimizing problems:

$$\inf_{u \in \mathcal{N}} \Phi(u) \quad \text{and} \quad \inf_{\substack{u \in H^s(\mathbb{R}^N) \\ \|u\|_2 = \rho}} \left[\Phi(u) - \frac{1}{2} \|u\|_2^2 \right] \quad \text{with } \rho > 0$$

are equivalent; Shen, Gao and Yang [33], inspired by Jeanjean [20] and Moroz and Van Schaftingen [29], constructed a Pohozaev–Palais–Smale sequence. With these facts in hand, they can easily prove that (1.1) has a ground state solution. However, the methods used in [13, 33] are invalid for (1.1) when V or Q is nonconstant.

When $V = 1$, $f(u) = |u|^{p-2}u$ with $2 \leq p < \frac{2^*}{2}(2 - \frac{\mu}{N})$, and $Q(x) = 1 + a(x)$ satisfies

$$(Q1) \quad a \in L^\infty(\mathbb{R}^N) \cap L^{2N/(2N-\mu-Np+2sp)}(\mathbb{R}^N) \text{ and } \lim_{|x| \rightarrow \infty} a(x) = 0;$$

$$a(x) \geq 0 \text{ and } a(x) > 0 \text{ on a positive measure set,}$$

Chen and Liu [8] proved that (1.1) has a Nehari type ground state solution by using the Nehari manifold method and comparing the critical level with the one of the problem at infinity. The main idea comes from Cerami and Vaira [6]. Note that this approach relies heavily on the special form $V = 1$ and $f(u) = |u|^{p-2}u$. Moreover the assumption $a \in L^{2N/(2N-\mu-Np+2sp)}(\mathbb{R}^N)$ also plays an important role. When $V(x)$ and $Q(x)$ are asymptotically periodic and f is continuous but not differentiable, the approach used in [8] is no longer applicable for (1.1). To the best of our knowledge, there seems to be no paper dealing with this case.

Motivated by the above works and [9, 10], in the present paper, by combining the non-Nehari manifold approach used in [35, 38, 39] with some new inequalities, we shall establish the existence of Nehari type ground state solutions for (1.1) under (F1)–(F3) in the periodic and asymptotically periodic cases.

To state our results, we first introduce a notation and some assumptions on V and Q . Let

$$\mathcal{B} = \{u \in L^\infty(\mathbb{R}^N, \mathbb{R}) : \text{meas}\{x \in \mathbb{R}^N : |u(x)| \geq \epsilon\} < \infty, \forall \epsilon > 0\}.$$

(VQ1) $V, Q \in \mathcal{C}(\mathbb{R}^N, (0, \infty))$ are 1-periodic in each x_j with $x = (x_1, x_2, \dots, x_N)$;

(VQ2) $V(x) = V_0(x) + V_1(x) > 0$, $Q(x) = Q_0(x) + Q_1(x) > 0$, $\forall x \in \mathbb{R}^N$, and

i) $V_0, Q_0 \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $V_0(x)$ and $Q_0(x)$ are 1-periodic in each x_j with $x = (x_1, x_2, \dots, x_N)$;

ii) $V_1 \in \mathcal{C}(\mathbb{R}^N, (-\infty, 0]) \cap \mathcal{B}$, $Q_1 \in \mathcal{C}(\mathbb{R}^N, [0, +\infty)) \cap \mathcal{B}$.

Now, we state our results of this paper.

Theorem 1.1. *Assume that (VQ1) and (F1)–(F3) hold. Then (1.1) has a ground state solution $\bar{u} \in H^s(\mathbb{R}^N)$ such that $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi > 0$.*

Theorem 1.2. *Assume that (VQ2) and (F1)–(F3) hold. Then (1.1) has a ground state solution $\bar{u} \in H^s(\mathbb{R}^N)$ such that $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi > 0$.*

Based on the mountain pass theorem due to Rabinowitz [32], we shall prove the above results by applying the non-Nehari manifold approach, which lies on finding a minimizing Cerami sequence for Φ outside \mathcal{N} by using the diagonal method (see Lemma 2.8), different from the Nehari manifold method and the generalized Nehari manifold method used in [8, 13, 33, 40]. To this end, we establish some new inequalities (see Lemmas 2.3 and 2.4). With these inequalities in hand, we verify the boundedness of Cerami sequences (see Lemma 2.9), and overcome the difficulties caused by the lose of \mathbb{Z}^N -translation invariance in the asymptotically periodic case.

Remark 1.3. Applying Theorem 1.2 to the equation in Chen and Liu [8], i.e. (1.1) with $V = 1$, $f(u) = |u|^{p-2}u$ and $Q(x) = 1 + a(x)$, we can weaken (Q1) to the following condition:

$$(Q2) \quad a \in L^\infty(\mathbb{R}^N), a(x) \geq 0 \text{ and } \lim_{|x| \rightarrow \infty} a(x) = 0.$$

Therefore, our results generalize and improve the existing ones in literature.

Remark 1.4. Our results are available for Choquard equation (1.3) with slight modification. From this point of view, we give an extension of the corresponding result in [40].

The paper is organized as follows. In Section 2, we give some preliminaries. We complete the proofs of Theorems 1.1 and 1.2 in Sections 3 and 4 respectively.

Throughout this paper, we denote the norm of $L^q(\mathbb{R}^N)$ by $\|u\|_q = (\int_{\mathbb{R}^N} |u|^q dx)^{1/q}$ for $q \in [2, \infty)$, $B_r(x) = \{y \in \mathbb{R}^N : |y - x| < r\}$, and positive constants possibly different in different places, by C_1, C_2, \dots

2 Preliminaries

In this section, we give some preliminaries which are crucial for proving our results. Firstly, to establish the variational setting, we present the following Hardy–Littlewood–Sobolev inequality.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality, [23]). *Let $t, r > 1$ and $0 < \mu < N$ with $1/t + 1/r + \mu/N = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$ independent of f and h such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^\mu} dx dy \leq C(t, N, \mu, r) \|f\|_t \|h\|_r.$$

Set $r = 2N/(2N - \mu)$, then $2 < rp_1 \leq rp_2 < 2_s^*$. By (VQ), (F1) and Proposition 2.1, one has

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)F(u(x))F(u(y))}{|x - y|^\mu} dx dy \\ & \leq C_1 \|F(u)\|_r^2 \leq C_2 \left(\|u\|_{rp_1}^{2p_1} + \|u\|_{rp_2}^{2p_2} \right), \quad \forall u \in H^s(\mathbb{R}^N). \end{aligned} \quad (2.1)$$

Similarly, we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)F(u(x))f(u(y))u(y)}{|x-y|^\mu} dx dy \leq C_3 \left(\|u\|_{r p_1}^{2p_1} + \|u\|_{r p_2}^{2p_2} \right), \quad \forall u \in H^s(\mathbb{R}^N). \quad (2.2)$$

By (1.2), one has

$$\Phi(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)F(u(x))F(u(y))}{|x-y|^\mu} dx dy, \quad \forall u \in H^s(\mathbb{R}^N). \quad (2.3)$$

Clearly, Φ is well defined on $H^s(\mathbb{R}^N)$. A standard argument shows that $\Phi \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and

$$\begin{aligned} \langle \Phi'(u), v \rangle &= \int_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv \right] dx \\ &\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)F(u(x))f(u(y))v(y)}{|x-y|^\mu} dx dy. \end{aligned} \quad (2.4)$$

Hence, the solutions of (1.1) are the critical points of (1.2).

Secondly, we state a version of Lions' concentration-compactness lemma for fractional Laplacian, which is an adaptation of a classical lemma of Lions [26].

Lemma 2.2 ([13, Lemma 2.3]). *Let $r > 0$ and $2 \leq \sigma < 2_s^*$. If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, and if*

$$\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^\sigma dx \rightarrow 0, \quad n \rightarrow \infty,$$

then $u_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2_s^$.*

Now, inspired by [11, 12, 36, 37], we establish some new inequalities, which are key points in the present paper.

Lemma 2.3. *Assume that (F1) and (F2) hold. Then for all $t \geq 0$ and $\tau_1, \tau_2 \in \mathbb{R}$,*

$$g(t, \tau_1, \tau_2) := F(t\tau_1)F(t\tau_2) - F(\tau_1)F(\tau_2) + \frac{1-t^2}{2} [F(\tau_1)f(\tau_2)\tau_2 + F(\tau_2)f(\tau_1)\tau_1] \geq 0. \quad (2.5)$$

Proof. It is evident that (2.5) holds for $t = 0$. Noting that $f(0) = 0$ due to (F1), it follows from (F2) that

$$f(\tau)\tau \geq F(\tau) \geq 0, \quad \forall \tau \in \mathbb{R}. \quad (2.6)$$

By (2.6), one has

$$\frac{F(\tau)}{\tau} \text{ is nondecreasing on } (-\infty, 0) \cup (0, +\infty). \quad (2.7)$$

For every $\tau_1, \tau_2 \in \mathbb{R}$, we deduce from (F2), (2.6) and (2.7) that

$$\begin{aligned} \frac{d}{dt} g(t, \tau_1, \tau_2) &= \tau_1 \tau_2 t \left[f(t\tau_2) \frac{F(t\tau_1)}{t\tau_1} - f(\tau_2) \frac{F(\tau_1)}{\tau_1} + f(t\tau_1) \frac{F(t\tau_2)}{t\tau_2} - f(\tau_1) \frac{F(\tau_2)}{\tau_2} \right] \\ &\begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases} \end{aligned}$$

which implies that $g(t, \tau_1, \tau_2) \geq g(1, \tau_1, \tau_2) = 0$ for all $t > 0$ and $\tau_1, \tau_2 \in \mathbb{R}$. \square

Lemma 2.4. *Assume that (VQ), (F1) and (F2) hold. Then*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^2}{2} \langle \Phi'(u), u \rangle, \quad \forall u \in H^s(\mathbb{R}^N), t \geq 0. \quad (2.8)$$

Proof. By (2.3), (2.4) and (2.5), one has

$$\begin{aligned} & \Phi(u) - \Phi(tu) \\ &= \frac{1-t^2}{2} \|u\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y) [F(tu(x))F(tu(y)) - F(u(x))F(u(y))]}{|x-y|^\mu} dx dy \\ &= \frac{1-t^2}{2} \langle \Phi'(u), u \rangle + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)}{|x-y|^\mu} \left[F(tu(x))F(tu(y)) - F(u(x))F(u(y)) \right. \\ &\quad \left. + \frac{1-t^2}{2} F(u(x))f(u(y))u(y) + \frac{1-t^2}{2} F(u(y))f(u(x))u(x) \right] dx dy \\ &\geq \frac{1-t^2}{2} \langle \Phi'(u), u \rangle, \quad \forall u \in H^s(\mathbb{R}^N), t \geq 0. \end{aligned}$$

This shows that (2.8) holds. \square

Corollary 2.5. *Assume that (VQ), (F1) and (F2) hold. Then*

$$\Phi(u) = \max_{t \geq 0} \Phi(tu), \quad \forall u \in \mathcal{N}. \quad (2.9)$$

Lemma 2.6. *Assume that (VQ) and (F1)–(F3) hold. Then, for any $u \in H^s(\mathbb{R}^N) \setminus \{0\}$, there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$.*

Proof. By (2.1), (2.3) and the Sobolev embedding theorem, one has

$$\Phi(u) \geq \frac{1}{2} \|u\|^2 - \frac{C_4}{2} (\|u\|^{2p_1} + \|u\|^{2p_2}), \quad \forall u \in H^s(\mathbb{R}^N),$$

which, together with $2 < 2(2 - \mu/N) < 2p_1 \leq 2p_2 \leq 2_2^*(2 - \mu/N)$, implies that there exists $\rho_0 > 0$ such that

$$\delta_0 := \inf_{\|u\|=\rho_0} \Phi(u) > 0. \quad (2.10)$$

For any fixed $u \in H^s(\mathbb{R}^N) \setminus \{0\}$, we define a function $\psi(t) = \Phi(tu)$ on $[0, \infty)$. Clearly, by (2.3) and (2.4), one has

$$\psi'(t) = 0 \Leftrightarrow \langle \Phi'(tu), tu \rangle = 0 \Leftrightarrow tu \in \mathcal{N}. \quad (2.11)$$

Using (F3), (2.3) and (2.10), it is easy to verify that $\psi(t) > 0$ for small $t > 0$ and $\psi(t) < 0$ for large t . Therefore, $\max_{t \geq 0} \psi(t)$ is achieved at some $t_u > 0$ so that $\psi'(t_u) = 0$. This, together with (2.11), shows that $t_u u \in \mathcal{N}$. \square

Lemma 2.7. *Assume that (VQ) and (F1)–(F3) hold. Then*

$$\inf_{u \in \mathcal{N}} \Phi(u) := m = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} \Phi(tu) > 0.$$

Proof. Corollary 2.5 and Lemma 2.6 imply that $m = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} \Phi(tu)$. Moreover, from (2.10) and Corollary 2.5, we conclude that $m \geq \inf_{\|u\|=\rho_0} \Phi(u) > 0$. \square

In the following, based on the mountain pass theorem due to P. H. Rabinowitz [32] in 1992, we will find a minimizing Cerami sequence for Φ outside \mathcal{N} by the diagonal method, this idea goes back to [35, 38], which is essential in the proofs of Theorems 1.1 and 1.2.

Lemma 2.8. *Assume that (VQ) and (F1)–(F3) hold. Then there exist a constant $c_* \in (0, m]$ and a sequence $\{u_n\} \subset H^s(\mathbb{R}^N)$ satisfying*

$$\Phi(u_n) \rightarrow c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad (2.12)$$

Proof. In view of the definition of m , we choose $v_k \in \mathcal{N}$ such that

$$m \leq \Phi(v_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \quad (2.13)$$

By (F3) and (2.10), we have $\Phi(t_k v_k) < 0$ for some $t_k > \rho_0 / \|v_k\|$, and $\Phi(u) \geq \delta_0 > 0$ for all $u \in S_{\rho_0} := \{u \in H^s(\mathbb{R}^N) : \|u\| = \rho_0\}$. Applying the Mountain pass lemma to $\{S_{\rho_0}, t_k v_k\}$, there exists a sequence $\{u_{k,n}\}_{n \in \mathbb{N}} \subset H^s(\mathbb{R}^N)$ satisfying

$$\Phi(u_{k,n}) \rightarrow c_k, \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}, \quad (2.14)$$

where $c_k \in [\delta_0, \sup_{t \geq 0} \Phi(t v_k)]$. By virtue of Corollary 2.5, one has $\Phi(v_k) = \sup_{t \geq 0} \Phi(t v_k)$. Thus, from (2.13) and (2.14), one has

$$\Phi(u_{k,n}) \rightarrow c_k \in \left[\delta_0, m + \frac{1}{k} \right), \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}. \quad (2.15)$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi(u_{k,n_k}) \in \left[\delta_0, m + \frac{1}{k} \right), \quad \|\Phi'(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}. \quad (2.16)$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\Phi(u_n) \rightarrow c_* \in [\delta_0, m], \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad \square$$

Lemma 2.9. *Assume that (VQ) and (F1)–(F3) hold. Then any sequence $\{u_n\} \subset H^s(\mathbb{R}^N)$ satisfying*

$$\Phi(u_n) \rightarrow c \geq 0, \quad \langle \Phi'(u_n), u_n \rangle \rightarrow 0 \quad (2.17)$$

is bounded.

Proof. To prove the boundedness of $\{\|u_n\|\}$, arguing by contradiction, suppose that $\|u_n\| \rightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n|^2 dx = 0,$$

then by Lemma 2.2, one has $v_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2_s^*$, and so $\|v_n\|_{rp_1} \rightarrow 0$ and $\|v_n\|_{rp_2} \rightarrow 0$ due to $2 < rp_1 \leq rp_2 < 2_s^*$. Let $t_n = 2\sqrt{c+1}/\|u_n\|$, it follows from (2.1), (2.3), (2.8) and (2.17) that

$$\begin{aligned} c + o(1) &= \Phi(u_n) \geq \Phi(t_n u_n) + \frac{1 - t_n^2}{2} \langle \Phi'(u_n), u_n \rangle \\ &= \Phi(2\sqrt{c+1}v_n) + o(1) \\ &\geq 2(c+1) - C_5 \left(\|v_n\|_{rp_1}^{2p_1} + \|v_n\|_{rp_2}^{2p_2} \right) = 2(c+1) + o(1), \end{aligned} \quad (2.18)$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |v_n|^2 dx > \frac{\delta}{2}. \quad (2.19)$$

Let $\tilde{v}_n(x) = v_n(x + k_n)$, then passing to a subsequence, we have $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^s(\mathbb{R}^N)$, $\tilde{v}_n \rightarrow \tilde{v}$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for $2 \leq p < 2_s^*$, and $\tilde{v}_n \rightarrow \tilde{v}$ a.e. in \mathbb{R}^N . Thus, (2.19) implies that $\tilde{v} \neq 0$. Denote $\tilde{u}_n(x) = u_n(x + k_n)$, then $|\tilde{u}_n| = |\tilde{v}_n| \|u_n\|$ and $\tilde{u}_n / \|u_n\| \rightarrow \tilde{v}$ a.e. in \mathbb{R}^N . For $x \in \{y \in \mathbb{R}^N : \tilde{v}(y) \neq 0\}$, we have $\lim_{n \rightarrow \infty} |\tilde{u}_n(x)| = \infty$. Thus, it follows from (VQ), (F3), (2.3), (2.17) and Fatou's lemma that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{c + o(1)}{\|u_n\|^2} = \lim_{n \rightarrow \infty} \frac{\Phi(u_n)}{\|u_n\|^2} \\ &= \frac{1}{2} - \frac{1}{2} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)}{|x-y|^\mu} \frac{F(u_n(x))F(u_n(y))}{\|u_n\|^2} dx dy \\ &\leq \frac{1}{2} - \frac{1}{2} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x+k_n)Q(y+k_n)}{|x-y|^\mu} \frac{F(\tilde{u}_n(x))}{|\tilde{u}_n(x)|} |\tilde{v}_n(x)| \frac{F(\tilde{u}_n(y))}{|\tilde{u}_n(y)|} |\tilde{v}_n(y)| dx dy \\ &\leq \frac{1}{2} - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \liminf_{n \rightarrow \infty} \left[\frac{Q(x+k_n)Q(y+k_n)}{|x-y|^\mu} \frac{F(\tilde{u}_n(x))}{|\tilde{u}_n(x)|} |\tilde{v}_n(x)| \frac{F(\tilde{u}_n(y))}{|\tilde{u}_n(y)|} |\tilde{v}_n(y)| \right] dx dy \\ &= -\infty. \end{aligned} \quad (2.20)$$

This contradiction shows that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. \square

3 The periodic case

In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of Lemmas 2.8 and 2.9, there exists a bounded sequence $\{u_n\} \subset H^s(\mathbb{R}^N)$ such that (2.12) holds. If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx = 0,$$

then by Lemma 2.2, one has $u_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2_s^*$, and so $\|u_n\|_{rp_1} \rightarrow 0$ and $\|u_n\|_{rp_2} \rightarrow 0$ due to $2 < rp_1 \leq rp_2 < 2_s^*$. Hence, it follows from (2.1), (2.2), (2.3), (2.4) and (2.12) that

$$\begin{aligned} c^* + o(1) &= \Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)}{|x-y|^\mu} F(u_n(x)) [f(u_n(y))u_n(y) - F(u_n(y))] dx dy \\ &\leq C_6 \left(\|u_n\|_{rp_1}^{2p_1} + \|u_n\|_{rp_2}^{2p_2} \right) = o(1). \end{aligned} \quad (3.1)$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^2 dx > \frac{\delta}{2}. \quad (3.2)$$

Let $\bar{u}_n(x) = u_n(x + k_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |\bar{u}_n|^2 dx > \frac{\delta}{2}. \quad (3.3)$$

Since $V(x)$ and $Q(x)$ are periodic on x , we have

$$\Phi(\bar{u}_n) \rightarrow c_* \in (0, m], \quad \|\Phi'(\bar{u}_n)\|(1 + \|\bar{u}_n\|) \rightarrow 0. \quad (3.4)$$

Passing to a subsequence, we have $\bar{u}_n \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$, $\bar{u}_n \rightarrow \bar{u}$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for $2 \leq p < 2^*$, and $\bar{u}_n \rightarrow \bar{u}$ a.e. in \mathbb{R}^N . Thus, (3.3) implies that $\bar{u} \neq 0$. It is easy to verify that $\Phi'(\bar{u}) = 0$. Since $f(t) = 0$ for all $t \leq 0$, as in [8], by minor modification of [13, Theorem 3.2] and using the maximum principle for fractional Laplacian in [14], we have $\bar{u} > 0$. This shows that $\bar{u} \in \mathcal{N}$ is a solution of (1.1) and so $\Phi(\bar{u}) \geq m$. From (2.3), (2.4), (2.6), (3.4) and Fatou's lemma, we have

$$\begin{aligned} m \geq c_* &= \lim_{n \rightarrow \infty} \left[\Phi(\bar{u}_n) - \frac{1}{2} \langle \Phi'(\bar{u}_n), \bar{u}_n \rangle \right] \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)}{|x-y|^\mu} F(\bar{u}_n(x)) [f(\bar{u}_n(y))\bar{u}_n(y) - F(\bar{u}_n(y))] dx dy \right\} \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(x)Q(y)}{|x-y|^\mu} F(\bar{u}(x)) [f(\bar{u}(y))\bar{u}(y) - F(\bar{u}(y))] dx dy \\ &= \Phi(\bar{u}) - \frac{1}{2} \langle \Phi'(\bar{u}), \bar{u} \rangle = \Phi(\bar{u}). \end{aligned}$$

This shows that $\Phi(\bar{u}) \leq m$ and so $\Phi(\bar{u}) = m = \inf_{\mathcal{N}} \Phi > 0$. \square

4 The asymptotically periodic case

In this section, we have $V(x) = V_0(x) + V_1(x)$ and $Q(x) = Q_0(x) + Q_1(x)$. Define functional $\Phi_0 : H^s(\mathbb{R}^N) \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} \Phi_0(u) &= \frac{1}{2} \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u|^2 + V_0(x)u^2 \right] dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q_0(x)Q_0(y)F(u(x))F(u(y))}{|x-y|^\mu} dx dy. \end{aligned} \quad (4.1)$$

By (VQ2), (F1) and Proposition 2.1, we have $\Phi_0 \in \mathcal{C}^1(H^s(\mathbb{R}^N), \mathbb{R})$ and

$$\begin{aligned} \langle \Phi'_0(u), v \rangle &= \int_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V_0(x)uv \right] dx \\ &\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q_0(x)Q_0(y)F(u(x))}{|x-y|^\mu} f(u(y))v(y) dx dy. \end{aligned} \quad (4.2)$$

Similar to the proof of [40, Lemma 4.3], we can obtain the following lemma.

Lemma 4.1. *Assume that (VQ2) and (F1) hold. If $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V_1(x)u_n^2 dx &= 0, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V_1(x)u_n v dx = 0, \quad \forall v \in H^s(\mathbb{R}^N), \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[Q(x)Q(y) - Q_0(x)Q_0(y)] F(u_n(x))F(u_n(y))}{|x-y|^\mu} dx dy &= 0 \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[Q(x)Q(y) - Q_0(x)Q_0(y)] F(u_n(x))}{|x-y|^\mu} f(u_n(y))v(y) dx dy = 0, \quad \forall v \in H^s(\mathbb{R}^N).$$

Proof of Theorem 1.2. If $V_1(x) \equiv 0$ and $Q_1(x) \equiv 0$, then Theorem 1.2 is contained in Theorem 1.1. So we can assume that $V_1(x) \leq 0$ and $Q_1(x) \geq 0$ but $|V_1(x)| + |Q_1(x)| \not\equiv 0$. In view of Lemmas 2.8 and 2.9, there exists a bounded sequence $\{u_n\} \subset H^s(\mathbb{R}^N)$ such that (2.12) holds. Passing to a subsequence, we may assume that $u_n \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$, $u_n \rightarrow \bar{u}$ in $L_{\text{loc}}^p(\mathbb{R}^N)$ for $2 \leq p < 2_s^*$ and $u_n \rightarrow \bar{u}$ a.e. in \mathbb{R}^N . Next, we prove that $\bar{u} \neq 0$.

Arguing by contradiction, suppose that $\bar{u} = 0$. Then $u_n \rightarrow 0$ in $H^s(\mathbb{R}^N)$, $u_n \rightarrow 0$ in $L_{\text{loc}}^p(\mathbb{R}^N)$ for $2 \leq p < 2_s^*$ and $u_n \rightarrow 0$ a.e. in \mathbb{R}^N . From (1.2), (2.4), (2.12), (4.1), (4.2) and Lemma 4.1, we deduce

$$\Phi_0(u_n) \rightarrow c^* \in (0, m], \quad \|\Phi_0'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad (4.3)$$

Analogous to the proof of (3.2), there exists $k_n \in \mathbb{Z}^N$, going if necessary to a subsequence, such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^2 dx > \frac{\delta}{2} > 0.$$

Define $v_n(x) = u_n(x + k_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |v_n|^2 dx > \frac{\delta}{2}. \quad (4.4)$$

Since $V_0(x)$ and $Q_0(x)$ are periodic in x , it follows from (4.3) that

$$\Phi_0(v_n) \rightarrow c^* \in (0, m], \quad \|\Phi_0'(v_n)\|(1 + \|v_n\|) \rightarrow 0. \quad (4.5)$$

Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in $H^s(\mathbb{R}^N)$, $v_n \rightarrow \bar{v}$ in $L_{\text{loc}}^p(\mathbb{R}^N)$ for $2 \leq p < 2_s^*$ and $v_n \rightarrow \bar{v}$ a.e. in \mathbb{R}^N . Thus, (4.4) implies that $\bar{v} \neq 0$. Arguing as in Theorem 1.1, we can prove that $\Phi_0'(\bar{v}) = 0$, $\Phi_0(\bar{v}) \leq c^*$ and $\bar{v} > 0$. In view of Lemma 2.6, there exists $\bar{t} > 0$ such that $\bar{t}\bar{v} \in \mathcal{N}$ and so $\Phi(\bar{t}\bar{v}) \geq m$. Noting that the conclusion of Lemma 2.4 holds for Φ_0 , from (VQ2), (1.2), (4.1), (4.2) and (4.5), we derive

$$\begin{aligned} m &\geq c^* \geq \Phi_0(\bar{v}) \\ &\geq \Phi_0(\bar{t}\bar{v}) - \frac{1 - \bar{t}^2}{2} \langle \Phi_0'(\bar{v}), \bar{v} \rangle \\ &= \Phi(\bar{t}\bar{v}) - \frac{\bar{t}^2}{2} \int_{\mathbb{R}^N} V_1(x) \bar{v}^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[Q(x)Q(y) - Q_0(x)Q_0(y)]}{|x - y|^\mu} F(\bar{t}\bar{v}(x)) F(\bar{t}\bar{v}(y)) dx dy \\ &> m. \end{aligned} \quad (4.6)$$

This contradiction shows that $\bar{u} \neq 0$. In the same way as the last part of the proof of Theorem 1.1, we can prove that $\bar{u} \in H^s(\mathbb{R}^N)$ is a ground state solution for (1.1) with $\Phi(\bar{u}) = m = \inf_{\mathcal{N}} \Phi > 0$. \square

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