



Positive solutions of nonlinear differential equations with Riemann–Stieltjes boundary conditions

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. We study the existence of positive solutions for some nonlinear second order boundary value problems with nonlocal boundary conditions. The key boundary condition considered is of the form $u(1) = \alpha[u']$, where α is a linear functional on $C[0, 1]$, that is, is given by a Riemann–Stieltjes integral $\alpha[v] = \int_0^1 v(s) dA(s)$ where A is a function of bounded variation. It is important in our case that dA is not a positive measure but can be sign changing.

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1 Introduction

There has been extensive study of nonlinear boundary value problems for ordinary differential equations (ODEs) with nonlocal boundary conditions (BCs) over the past decades. The subject actually has a long history, the study of nonlocal BCs for ODEs can be traced back to Picone [16], who considered multi-point BCs. For some historical introductions to nonlocal problems we refer the reader to the reviews [2, 15, 24]. Some important contributions, often considered mistakenly to be the first, were given by Bicadze and Samarskii, for example [1].

The interest was revived by C. P. Gupta and co-authors in a series of papers on multipoint boundary value problems (BVPs) employing degree theory, see for example [3, 4].

The study of positive solutions using topological methods was begun at the end of the 1990s.

We shall study existence of positive solutions for several simple looking problems, and show that some existing theory is applicable provided it is considered in sufficient generality. The first problem we shall study is

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad u(1) = \alpha[u'], \quad (1.1)$$

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for a suitable nonnegative function f , where α is a linear functional on $C[0,1]$, that is, is given by a Riemann–Stieltjes integral $\alpha[v] = \int_0^1 v(s)dA(s)$ where A is a function of bounded variation. The novelty of our work is having a boundary condition involving $\alpha[u']$ which is non-negative for *any solution*, a case that has been rarely studied, we shall explain below some reasons for this.

A standard methodology for non-resonant problems is to use the Green's function, which must be nonnegative and should satisfy some suitable positivity condition, to consider an integral equation acting in the positive cone $P = \{u \in C[0,1], u(t) \geq 0, t \in [0,1]\}$ of the space $C[0,1]$ of continuous functions and obtain solutions as fixed points of the integral operator, often using fixed point index theory.

To work in the space $C[0,1]$ it has been natural to consider the BC of the form $u(1) = \alpha[u]$ (not involving the derivative) and this case has been extensively studied. The usual progression has been to first study the so-called three point problem, when $\alpha[u] = \alpha u(\eta)$, with $\eta \in (0,1)$ and $\alpha \geq 0$ is suitably bounded above, then to study multipoint BCs with finite sums of such terms, then integral BCs, then Riemann–Stieltjes BCs, and to also consider more general differential equations of second and higher order. The Riemann–Stieltjes BC such as $u(1) = \alpha[u]$ is quite natural since it represents a linear functional in the space $C[0,1]$ and it includes the three-point, multipoint and integral BCs just mentioned as special cases.

Problems with BCs of Riemann–Stieltjes type have been studied by Karakostas and Tsamatos, [6, 7], and by Yang [27, 28], all with (positive) measures. Webb and Infante gave a unified approach to equations for second order equations with one or two Riemann–Stieltjes BCs of the form $u(0) = \alpha_0[u]$, $u(1) = \alpha_1[u]$ in [22] and for arbitrary order equations with many Riemann–Stieltjes BCs of this type in [23]. The new observation made by Webb and Infante was that A can be a function of bounded variation, that is dA can be a signed measure, under suitable restrictions, so it is not required that $\alpha[u] \geq 0$ for all $u \in P$ but $\alpha[u] \geq 0$ is to be satisfied by a positive solution u . In particular, for a multipoint BC such as $u(1) = \sum_{i=1}^m a_i u(\eta_i)$, some of the coefficients a_i can be negative.

For existence of positive solutions the first problems studied were the following three-point problems

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \quad u(0) = 0, \quad u(1) = \alpha u(\eta), \quad (1.2)$$

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \quad u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad (1.3)$$

where f is non-negative, η can be arbitrary in $(0,1)$, and α satisfies $0 \leq \alpha < 1/\eta$ in (1.2) and $0 \leq \alpha < 1$ in (1.3). The BVP (1.2) was studied by Ma in [12] for the case when f is either sub- or super-linear (limits of $f(u)/u$ as $u \rightarrow 0+$ and as $u \rightarrow \infty$ are ∞ and 0, or 0 and ∞ , respectively) using Krasnosel'skii's compression–expansion fixed point theorem, the second BVP (1.3) was studied under more explicit conditions on the limits of $f(u)/u$ as $u \rightarrow 0+$ and as $u \rightarrow \infty$ in [18], and both (1.2), (1.3) were studied in [19] under slightly more precise conditions, in each case using fixed point index theory.

The apparently similar problem (1.1) with $u(1) = \alpha u'(\eta)$ has been neglected in the literature, the main reason is that the corresponding three-point problem with $\alpha > 0$ does not fit the usual theory. An explanation is that for (1.2) the corresponding local problem, when $\alpha = 0$, the derivative u' of a positive solution starts positive then becomes negative and so one cannot expect to have an arbitrary η in $(0,1)$ for α small. In the second problem (1.6) the derivative of a solution of the local problem is always negative so the three-point problem is allowed if $\alpha < 0$, but here we want to consider solutions with $\alpha[u']$ positive so this is different case of the problem we study.

We shall show that, by considering the Stieltjes BC $u(1) = \alpha[u']$ with a sign changing measure, the techniques developed by Webb and Infante [22,23] can be applied to this problem and existence of positive solutions which satisfy $\alpha[u'] \geq 0$ can be obtained under standard types of conditions on the nonlinearity f . Furthermore, once the Green's function of the problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad u(1) = \alpha[u']$$

is shown to have certain positivity properties then one can consider the problems

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u(0) = \beta_0[u], \quad u(1) = \alpha[u'] + \beta_1[u] \quad (1.4)$$

for Riemann–Stieltjes integrals with positive or suitably sign changing measures. The theory of [22,23] applies and the problem is reduced to the calculation of the constants related to the growth conditions on f .

We also show that the following two similar problems can be treated with the same methods

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad u'(1) = \alpha[u'], \quad (1.5)$$

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u'(0) = 0, \quad u(1) = \alpha[u']. \quad (1.6)$$

The fourth problem of this type would be

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u'(0) = 0, \quad u'(1) = \alpha[u'],$$

but this can not be treated by the methods here because the problem is always at resonance so no Green's function exists. The apparently similar problem, which models a thermostat,

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u'(0) = \alpha_0[u], \quad u'(1) + \alpha_1[u] = 0,$$

also has a resonant local BVP but the nonlocal BVP can be nonresonant; a three-point version of this type of problem was treated in [5] and the general case is discussed in detail in [20].

We now briefly discuss some previous work we have found on these problems.

The problem, of the type (1.6),

$$x''(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \quad x'(0) = 0, \quad x(1) = -\delta x'(\eta),$$

with $\delta > 0$ was studied by Sun [17] using a fixed point theorem of cone expansion and compression type. As mentioned above this does not correspond to the cases we consider but can be treated by the same methods as here by considering solutions with $\alpha[u'] \leq 0$.

Kwong and Wong [10] studied the problem

$$u''(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad u'(0) = 0, \quad u(1) = \alpha u(\eta) + \beta u'(\eta)$$

by using the shooting method not via an integral equation. They allow $f(t, u)$ to change sign but is subject to linear growth in u . Some results give existence of a nontrivial solution, other results show there is a positive solution. Wong and Kong [26] extended [10] to consider some multipoint BCs of a similar type.

Ruyun Ma, and Castaneda [13] studied a problem of the type (1.5) with BCs $u'(0) = \alpha[u']$, $u(1) = \beta[u]$ where these are multipoint BCs not the general Riemann–Stieltjes BCs, and found positive solutions for the sub- and super-linear cases using Krasnosel'skiĭ's theorem, a weaker version of using fixed point index theory.

De-Xiang Ma, Du and Ge [14] studied the same boundary conditions as Ruyun Ma but for the p -Laplace operator.

Wong [25] studied three multipoint BCs of the form

$$\begin{aligned} u(0) &= \alpha[u], & u(1) &= \beta[u], \\ u'(0) &= \alpha[u'], & u(1) &= \beta[u], \\ u(0) &= \alpha[u], & u'(1) &= \beta[u']. \end{aligned}$$

He uses various fixed point theorems including Krasnosel'skiĭ's theorem and his results use estimates involving constants similar to those we call m, M later in the paper. The BC $u(1) = \beta[u']$ is not considered, probably for the reasons we suggested above.

2 A general set-up

To avoid repetition we give a general method that applies to all the BVPs we consider and, for possible future application, can be applied to equations and BCs other than the ones we concentrate on here. The forms of the Green's function for the special BCs we consider can also be obtained by direct calculations.

In order to study the nonlocal BVPs we suppose that the corresponding local BVP has a solution operator given by the Green's function with certain good properties, that is the solutions are fixed points of the integral operator N_0 acting in the space $C[0, 1]$ endowed with the norm $\|u\| := \max_{t \in [0, 1]} |u(t)|$, where

$$N_0 u(t) := \int_0^1 G_0(t, s) f(s, u(s)) ds.$$

For simplicity we suppose that f is continuous, it is possible to consider f satisfying Carathéodory conditions but we shall not do so here. We now give a general method of finding the Green's function G for the nonlocal problem, so solutions will be fixed points of N where

$$Nu(t) := \int_0^1 G(t, s) f(s, u(s)) ds.$$

The conditions required to be satisfied by G are the following ones.

(C₁) The kernel $G \geq 0$ is measurable, and continuous in t for almost every (a.e.) s , that is, for every $\tau \in [0, 1]$ we have

$$\lim_{t \rightarrow \tau} |G(t, s) - G(\tau, s)| = 0 \quad \text{for a.e. } s \in [0, 1].$$

(C₂) There exist a subinterval $J = [t_0, t_1] \subseteq [0, 1]$, a nonnegative function $\Phi \in L^1(0, 1)$ with $\int_{t_0}^{t_1} \Phi(s) ds > 0$, and a constant $c_J \in (0, 1]$ such that

$$G(t, s) \leq \Phi(s) \quad \text{for } t \in [0, 1] \text{ and a.e. } s \in [0, 1], \quad (2.1)$$

$$G(t, s) \geq c_J \Phi(s) \quad \text{for } t \in J \text{ and a.e. } s \in [0, 1]. \quad (2.2)$$

(C₃) The partial derivative $\partial_t G(t, s)$ is continuous in t for a.e. s and there exist $\Phi_1 \in L^1$ such that $|\partial_t G(t, s)| \leq \Phi_1(s)$, a.e.

We will suppose the above conditions are satisfied by G_0 with functions Φ_0, Φ_1, c_0 and will then determine G and show that the conditions are satisfied by G .

(C_2) is a replacement for concavity properties and is often proved by showing the following condition for some function $c \in P \setminus \{0\}$,

$$c(t)\Phi(s) \leq G(t,s) \leq \Phi(s), \quad \text{for } 0 \leq t, s \leq 1. \quad (2.3)$$

If (2.3) is valid we can take any $J = [t_0, t_1] \subseteq [0, 1]$ for which $c_J := \min_{t \in [t_0, t_1]} c(t) > 0$. When $(C_1), (C_2)$ hold, N_0 maps $P = \{u \in C[0, 1], u(t) \geq 0, t \in [0, 1]\}$ into the sub-cone K where

$$K := \{u \in P : u(t) \geq c_J \|u\| \quad \text{for } t \in J\}. \quad (2.4)$$

(C_1) is assumed because, although G_0 may satisfy more, for example be continuous, this is all we can guarantee for G .

We consider the nonlocal BVP as a perturbation from the local problem and, using the methods of [22,23], we derive the Green's function for the nonlocal problem as follows. Firstly we seek solutions as fixed points of a nonlinear operator T where

$$Tu(t) := N_0u(t) + \gamma(t)\alpha[u'].$$

The function γ is a solution of the differential equation with $f(t, u)$ replaced by 0 and $\alpha[u']$ replaced by 1 in the boundary condition. In the cases we study here, $\gamma''(t) = 0$ and γ is easily found, some specific examples will be given below.

The conditions to be supposed on γ in general, are that it is twice differentiable, non-negative, and satisfies

$$0 \leq \alpha[\gamma'] < 1, \quad \text{and} \quad \gamma(t) \geq c_{\gamma, J} \|\gamma\|, \quad \text{for } t \in J = [t_0, t_1]. \quad (2.5)$$

Typically we will get $\gamma(t) \geq c_{\gamma}(t) \|\gamma\|$ with $c_{\gamma, J} = \min_{t \in J} c_{\gamma}(t) > 0$.

Since we want a nonnegative Green's function with suitable positivity properties we also suppose that

$$\Psi(s) := \int_0^1 \partial_t G_0(t, s) dA(t) \geq 0 \quad \text{for a.e. } s \in (0, 1). \quad (2.6)$$

The integral exists for a.e. $s \in [0, 1]$ since, by our assumptions, $\partial_t G_0$ is continuous in the t variable, and Ψ is an L^1 function because A has finite variation and $|\partial_t G_0(t, s)| \leq \Phi_1(s) \text{Var}(A)$.

Then, if u is a fixed point of T , we have

$$\begin{aligned} u(t) &= \int_0^1 G_0(t, s) f(s, u(s)) ds + \gamma(t)\alpha[u'], \\ u'(t) &= \int_0^1 \partial_t G_0(t, s) f(s, u(s)) ds + \gamma'(t)\alpha[u']. \end{aligned}$$

Applying the linear functional α to the second term gives

$$\alpha[u'] = \alpha[(N_0u)'] + \alpha[\gamma']\alpha[u'], \quad \text{hence } \alpha[u'] = \alpha[(N_0u)'] / (1 - \alpha[\gamma']),$$

where

$$\alpha[(N_0u)'] = \int_0^1 \left(\int_0^1 \partial_t G_0(t, s) dA(t) \right) f(s, u(s)) ds = \int_0^1 \Psi(s) f(s, u(s)) ds, \quad (2.7)$$

by changing the order of integration. We then have u is a fixed point of N where

$$Nu(t) := \int_0^1 G(t, s) f(s, u(s)) ds, \quad \text{with} \quad G(t, s) = G_0(t, s) + \frac{\gamma(t)\Psi(s)}{1 - \alpha[\gamma']}.$$

The Green's function G satisfies condition (C_1) and a modified (C_2) , namely

$$\begin{aligned} G(t, s) &\leq \Phi_0(s) + \frac{\|\gamma\|\Psi(s)}{(1 - \alpha[\gamma'])} := \Phi(s), \quad \text{for } t, s \in [0, 1], \\ G(t, s) &\geq c_0\Phi_0(s) + \frac{c_{\gamma, J}\|\gamma\|\Psi(s)}{(1 - \alpha[\gamma'])} \geq \min\{c_0, c_{\gamma, J}\}\Phi(s), \quad \text{for } t \in [t_0, t_1], s \in [0, 1]. \end{aligned}$$

Note that, under the above condition (C_3) ,

$$(Nu)' = \int_0^1 \partial_t G_0(t, s) f(s, u(s)) ds + \gamma'(t)\Psi(s)/(1 - \alpha[\gamma']),$$

and $(Nu)'' = -f(t, u(t))$ as G_0 is the Green's function of the local problem, so a fixed point of N is a solution of the differential equation.

3 The BCs $u(0) = 0$, $u(1) = \alpha[u']$

The local problem for this case is

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad u(1) = 0, \quad (3.1)$$

and the corresponding Green's function is well known to be

$$G_0(t, s) := \begin{cases} s(1-t), & \text{if } s \leq t, \\ t(1-s), & \text{if } s > t. \end{cases} \quad (3.2)$$

Routine calculations give $G_0(t, s) \leq \Phi_0(s) = s(1-s)$ and $G_0(t, s) \geq \min\{t, 1-t\}\Phi_0(s)$. Moreover

$$\partial_t G_0(t, s) = \begin{cases} -s, & \text{if } s \leq t, \\ 1-s, & \text{if } s > t, \end{cases} \quad (3.3)$$

so $|\partial_t G_0(t, s)| \leq \Phi_1(s) = \max\{s, 1-s\}$.

The function γ is the solution of the BVP $\gamma'' = 0$, $\gamma(0) = 0$, $\gamma(1) = 1$, that is $\gamma(t) = t$ so in (2.5) $c_\gamma(t) = t$, and we assume $0 \leq \alpha[\gamma'] < 1$ which we write as $0 \leq \alpha[\hat{1}] < 1$ where $\hat{1}$ denotes the constant function with value 1.

Let $A(t) := \int_0^t dA(s)$, then also $\alpha[\hat{1}] = A(1)$. Then we can write

$$\Psi(s) = \int_0^s (1-s) dA(t) + \int_s^1 -s dA(t) = (1-s)A(s) - sA(1) + sA(s) = A(s) - sA(1),$$

so we must assume that $A(s) \geq sA(1)$. The Green's function for the nonlocal problem is therefore (after a small calculation),

$$G(t, s) = G_0(t, s) + \frac{\gamma(t)\Psi(s)}{1 - \alpha[\gamma']} = \begin{cases} s - \frac{t(s - A(s))}{1 - \alpha[\hat{1}]}, & \text{if } s \leq t, \\ t - \frac{t(s - A(s))}{1 - \alpha[\hat{1}]}, & \text{if } s > t, \end{cases} \quad (3.4)$$

From this expression it is not so obvious that $G(t, s) \geq 0$, it follows more easily without doing the extra small calculation.

We have proved the following result.

Theorem 3.1. Suppose that $\alpha[v] = \int_0^1 v(s) dA(s)$ where $0 \leq A(1) < 1$ and $A(s) \geq sA(1)$. Then positive solutions of the BVP

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad u(1) = \alpha[u']$$

are given by fixed points in the cone P of the integral operator N given by

$$Nu(t) = \int_0^1 G(t, s) f(s, u(s)) ds.$$

Under these conditions the nonlinear operator N is a compact (also called completely continuous) operator in $C[0, 1]$, that is, N is continuous and maps bounded subsets into relatively compact sets.

The theorems in [22, 23] are now directly applicable and we can give results for existence of multiple positive solutions under suitable conditions on $f(t, u)/u$ for different ranges of u . First we give some examples to show that conditions sometimes can and sometimes cannot be satisfied for some problems.

Example 3.2. The three-point problem with $\alpha[u'] = \alpha u'(\eta)$ for $\eta \in (0, 1)$ never satisfies the required conditions. For in this case

$$A(s) = \begin{cases} 0, & \text{if } s < \eta, \\ \alpha, & \text{if } s \geq \eta, \end{cases}$$

so $A(s) \geq sA(1)$ cannot be satisfied for $s < \eta$. However, the nonlocal problem with BCs $u(0) = 0, u(1) = \alpha u'(0)$ is possible for $0 \leq \alpha < 1$, as are multipoint problems with $u(1) = \alpha_0 u'(0) + \sum_{i=1}^m \alpha_i u'(\eta_i)$ under suitable restrictions, it is important to have the positive term $\alpha_0 u'(0)$. One example is $u(1) = au'(0) - bu'(\eta)$ where $a, b \geq 0$, and $0 \leq a - b < 1$. In fact for $a > 0$ it is possible to also have $b < 0$ provided that $(1 - \eta)a + \eta b \geq 0$.

Example 3.3. The BCs $u(0) = 0$ and the integral BC $u(1) = \int_0^1 \alpha u'(s) ds = \alpha(u(1) - u(0))$ is clearly not allowed if $\alpha = 1$ since the problem is then under-determined; the theory does not allow this case. But also for any $\alpha \neq 1$ this is not a nonlocal problem but is equivalent to the BC $u(1) = 0$. For the BC $u(1) = \int_0^1 \alpha u'(s) ds$ for $\alpha \neq 1$, we have $\alpha[v] = \alpha \int_0^1 v(s) ds$ hence $A(s) = \alpha s$ and the Green's function does reduce to $G_0(t, s)$ as is necessary.

Example 3.4. Let $\eta \in (0, 1)$ and a, b be positive constants and let

$$a(t) := \begin{cases} a, & 0 \leq t \leq \eta, \\ -b, & \eta < t \leq 1, \end{cases}$$

and let $\alpha[v] = \int_0^1 a(t)v(t) dt$. Then $A(s) = as$ if $s \leq \eta$ and $A(s) = a\eta - b(s - \eta)$ for $s > \eta$, so $A(1) = a\eta - b(1 - \eta) = (a + b)\eta - b$ so we require $0 \leq (a + b)\eta - b < 1$. We can easily verify that $A(s) \geq sA(1)$ is then satisfied.

In fact, this problem actually reduces to a simpler problem. For, since $u(0) = 0$,

$$\alpha[u'] = a \int_0^\eta u'(s) ds - b \int_\eta^1 u'(s) ds = au(\eta) - b(u(1) - u(\eta)) = (a + b)u(\eta) - bu(1).$$

Therefore this becomes the usual three point problem with BCs $u(0) = 0, u(1) = \frac{a+b}{1+b}u(\eta)$, and, as is well known, the necessary condition for positive solutions to exist is $0 \leq \frac{a+b}{1+b}\eta < 1$ which is the same as $0 \leq (a + b)\eta - b < 1$ found above, so our condition is sharp.

Example 3.5. Let $\alpha[v] := \int_0^1 a(t)v(t) dt$ where

$$a(t) = \begin{cases} 1, & \text{for } t \leq 1/2, \\ t/2, & \text{for } t > 1/2. \end{cases}$$

Then we have

$$A(s) = \begin{cases} s, & \text{for } s \leq 1/2, \\ 1/2 + (s^2 - 1/4)/4, & \text{for } s > 1/2. \end{cases}$$

We obtain $A(1) = 11/16 < 1$ and it is readily verified that $A(s) \geq sA(1)$.

4 The BCs $u(0) = 0$, $u'(1) = \alpha[u']$

We now discuss the BVP (1.5). Let H denote the Heaviside function,

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

The Green's function of the corresponding local problem can be written

$$G_0(t, s) := t - (t - s)H(t - s). \quad (4.1)$$

Hence we obtain $\Phi_0(s) = s$, $c_0(t) = t$, $\gamma(t) = t$. We proceed as before and obtain

$$\Psi(s) = \int_0^1 \partial_t G_0(t, s) dA(t) = \int_0^s dA(t) = A(s). \quad (4.2)$$

Theorem 4.1. Suppose that $\alpha[v] = \int_0^1 v(s) dA(s)$ where $A(s) \geq 0$ and $A(1) < 1$. Then positive solutions of the BVP

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad u'(1) = \alpha[u']$$

are given by fixed points in the cone P of the compact integral operator N given by

$$Nu(t) = \int_0^1 G(t, s) f(s, u(s)) ds$$

where $G(t, s) = G_0(t, s) + \frac{tA(s)}{1-A(1)}$, and G satisfies the conditions $(C_1), (C_2)$.

Remark 4.2. Clearly it is much simpler to give examples of measures satisfying the required conditions $A(s) \geq 0$ and $A(1) < 1$ than it is for (1.6). For this problem the three-point BVP with $u'(1) = \alpha u'(\eta)$ is allowed if $0 \leq \alpha < 1$.

If $\alpha > 1$ the Green's function for this three-point problem is not nonnegative so the theory we have used does not apply. However it is possible for positive solutions to exist for certain f and η , as in the following example; this type of problem requires some other theory.

Example 4.3. The BVP

$$u'' + 2 = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad u'(1) = \alpha u'(\eta)$$

has solution $u(t) = \frac{(2\alpha\eta-2)}{(\alpha-1)}t - t^2$. If $\alpha > \alpha\eta > 1$ then $u(t) \geq 0$ on $[0, 1]$ if $\alpha(2\eta - 1) \geq 1$. This is impossible for every $\eta \leq 1/2$ but is possible for $\eta > 1/2$. For example, for $\eta = 3/4, \alpha = 2$ the solution is $u(t) = t - t^2$.

5 The BCs $u'(0) = 0$, $u(1) = \alpha[u']$

This problem is similar to the problem in Section 3. The local Green's function is

$$G_0(t, s) = 1 - s - (t - s)H(t - s), \quad \text{with } \Phi_0(s) = 1 - s, \quad c_0(t) = 1 - t.$$

We have $\gamma(t) = 1$ with $c_\gamma(t) = 1$ and $\Psi(s) = \int_s^1 (-1) dA(s) = A(s) - A(1)$. The Green's function is $G(t, s) = G_0(t, s) + A(s) - A(1)$.

Since $\gamma' = 0$ we only require $A(s) \geq A(1)$ for all $s \in [0, 1]$. In particular strictly positive measures are not allowed. A theorem very similar to Theorem 3.1 is valid, we do not give the obvious statement.

6 Existence and nonexistence theorems

We state some existence results which are special cases of results from [23] and are proved using some results on the fixed point index.

Consider the problem

$$u(t) = Nu(t) = \int_0^1 G(t, s)f(u(s)) ds, \quad t \in [0, 1], \quad (6.1)$$

where G satisfies $(C_1), (C_2)$, where f is continuous and nonnegative and for simplicity we suppose f depends only on u ; for more complicated cases see for example [23]. Let L be the associated linear operator

$$Lu(t) = \int_0^1 G(t, s)u(s) ds, \quad t \in [0, 1]. \quad (6.2)$$

Let K denote the cone $K = \{u \in P : u(t) \geq c_J \|u\| \text{ for } t \in J = [t_0, t_1]\}$. By (C_2) it is easy to show that N, L map P into K . Furthermore

$$L\hat{1}(t) = \int_0^1 G(t, s) ds \geq \int_{t_0}^{t_1} c_J \Phi(s) ds \hat{1}.$$

This proves that the spectral radius $r(L)$ of L satisfies $r(L) \geq \int_{t_0}^{t_1} c_J \Phi(s) ds > 0$. By the Krein–Rutman theorem, $r(L)$ is an eigenvalue of L with eigenvector in P and hence also in K , usually called the principal eigenvalue. We let $\mu_1 := 1/r(L)$, the principal characteristic value of L .

We will use the following notations.

$$\begin{aligned} f^0 &:= \limsup_{u \rightarrow 0} \frac{f(u)}{u}; & f^\infty &:= \limsup_{u \rightarrow \infty} \frac{f(u)}{u}, \\ f_0 &:= \liminf_{u \rightarrow 0} \frac{f(u)}{u}; & f_\infty &:= \liminf_{u \rightarrow \infty} \frac{f(u)}{u}. \end{aligned}$$

We first state a sharp result for the existence of one positive solution.

Theorem 6.1. *Assume (C_1) holds and that (C_2) holds for an arbitrary $J = [t_0, t_1] \subset (0, 1)$. Then equation (6.1) has a positive solution $u \in K$ if one of the following conditions holds.*

$$(S_1) \quad 0 \leq f^0 < \mu_1 \text{ and } \mu_1 < f_\infty \leq \infty.$$

$$(S_2) \quad 0 \leq f^\infty < \mu_1 \text{ and } \mu_1 < f_0 \leq \infty.$$

The reason this is sharp is because ‘crossing the eigenvalue’ is necessary for positive solutions to exist, namely we have the following nonexistence result.

Theorem 6.2 ([21, Theorem 4.1]). *Assume (C_1) holds and that (C_2) holds for an arbitrary $J = [t_0, t_1] \subset (0, 1)$. Let $\mu_1 = 1/r(L)$ be the principal characteristic value of L . Consider the following conditions:*

- (i) $f(u) \leq au$, for all $u > 0$, where $a < \mu_1$;
- (ii) $f(u) \geq bu$, for all $u > 0$, where $b > \mu_1$.

If (i) or (ii) holds then the equation $u = Nu$ has no solution in $P \setminus \{0\}$.

For existence of two positive solutions we need some further notations.

We write $f^{0,r} := \sup_{\{0 \leq u \leq r\}} f(u)/r$, $f_{r,r/c} := \inf_{\{r \leq u \leq r/c\}} f(u)/r$.

We use the following constants.

$$m := \left(\sup_{t \in [0,1]} \int_0^1 G(t,s) ds \right)^{-1}, \quad M := \left(\inf_{t \in [t_0, t_1]} \int_{t_0}^{t_1} G(t,s) ds \right)^{-1}. \quad (6.3)$$

Theorem 6.3. *Assume (C_1) holds and that (C_2) holds for an arbitrary $J = [t_0, t_1] \subset (0, 1)$. Then equation (6.1) has two positive solutions in K if one of the following conditions holds.*

- (D₁) $0 \leq f^0 < \mu_1$, $f_{\rho, \rho/c} > M$ for some $\rho > 0$, and $0 \leq f^\infty < \mu_1$.
- (D₂) $\mu_1 < f_0 \leq \infty$, $f^{0,\rho} < m$ for some $\rho > 0$, and $\mu_1 < f_\infty \leq \infty$.

Remark 6.4. There are theorems for three or more solutions, we do not state them here but refer to [23] for details.

7 Examples

We give a few simple examples to illustrate our results and we give some explicit constants.

Example 7.1. We consider the BVP (see Example 3.2 above)

$$u''(t) + \lambda f(u(t)) = 0, \quad u(0) = 0, \quad u(1) = \alpha u'(0). \quad (7.1)$$

The restriction on α is $0 \leq \alpha < 1$. For the existence of one positive solution we need to calculate μ_1 . In this case the eigenfunction is of the form $\sin(\omega t)$ and $\mu_1 = \omega^2$ where ω is the smallest positive root of the equation $\sin(\omega) = \alpha \omega$.

For $\alpha = 0$, the local problem, $\mu_1 = \pi^2$ as is well known.

For $\alpha = 1/4$, we calculate $\mu_1 \approx 6.1235$.

For $\alpha = 1/2$, we calculate $\mu_1 \approx 3.5929$.

For $\alpha = 3/4$, we calculate $\mu_1 \approx 1.6274$.

Now take as example $f(u) = u(au + b)/(1 + u)$, where $a > b > 0$. Then we have $f(u)/u = (au + b)/(1 + u)$ and $f^0 = b$, $f_\infty = a$, so for every λ such that $\lambda b < \mu_1 < \lambda a$ there is at least one (strictly) positive solution. That is, for $\lambda \in (\mu_1/b, \mu_1/a)$ there is a positive solution by Theorem 6.1 and for $\lambda < \mu_1/b$ and for $\lambda > \mu_1/a$ no positive solution exists by Theorem 6.2 (of course there is the zero solution here).

Take $a = 1, b = 1/2$, the intervals for which a positive solution exists for the above values of α are (approximately): for $\alpha = 1/4$, $\lambda \in (6.1235, 12.2471)$; for $\alpha = 1/2$, $\lambda \in (3.5929, 7.1858)$; for $\alpha = 3/4$, $\lambda \in (1.6274, 3.25481)$. There are no positive solutions for λ outside these intervals.

Example 7.2. For the same BVP as in example 7.1 we let $f(u) = 2$. The $f_0 = \infty$ and $f^\infty = 0$ so this problem has a positive solution for every $\lambda > 0$ provided that $0 \leq \alpha < 1$. In fact, the solution can be found easily here and is $u(t) = \lambda t(1/(1-\alpha) - t)$ which shows that the condition $0 \leq \alpha < 1$ is sharp.

Example 7.3. We consider the same BVP as in example 7.1 but with $f(u) = u^2 + u^{1/2}$.

Now we have $f_0 = \infty$ and $f_\infty = \infty$ so we can obtain 2 positive solutions for every positive λ satisfying $\lambda f^{0,\rho} < m$ using (D_2) of Theorem 6.3, where for $\rho > 0$,

$$f^{0,\rho} = \max_{0 \leq u \leq \rho} f(u)/\rho = f(\rho)/\rho = \rho^{3/2} + \rho^{-1/2},$$

since f is increasing.

The Green's function is

$$G_0(t,s) + \gamma(t)\Psi(s)/(1-\alpha) = t(1-s) - (t-s)H(t-s) + t\alpha(1-s)/(1-\alpha).$$

As an example we take $\alpha = 1/2$ and calculate m where $1/m = \max_{t \in [0,1]} \int_0^1 G(t,s) ds$, we find $m = 2$. We choose $\rho = 0.63$ (this is approximately $2^{1/3}/2$ where the minimum of $f(\rho)/\rho$ occurs). Then there are two positive solution for λ satisfying $1.8899\lambda < 2$ that is $0 < \lambda < 1.058$ (approx.).

It is possible to give examples of integral BCs for this differential equation and also for the other problems. The problem is now reduced to calculation of the constants that occur.

Remark 7.4. The expressions for the Green's functions for the problems we have studied enable us to use the theory of [22, 23] to study more complicated BCs with added nonlocal terms of the type $\beta[u]$. For example from our knowledge of the problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \quad u(0) = 0, \quad u(1) = \alpha[u'],$$

we can then study the problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \quad u(0) = \beta_0[u], \quad u(1) = \alpha[u'] + \beta_1[u],$$

where $\beta_i[u] = \int_0^1 u(s) dB_i(s)$ where dB_i can be sign changing measures. We omit further details here.

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