



# Algebraic stability of impulsive fractional-order systems

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**Abstract.** In this paper, stability of impulsive fractional-order systems is investigated. By Lyapunov's direct method and comparison principle, results about asymptotic stability are given. To this end, comparison principles are first generalized to impulsive fractional order systems, through which a fractional inequality is derived for the linear impulsive system. Then sufficient conditions for the Mittag-Leffler stability, which is a special case of algebraic stability, of impulsive fractional-order systems are established. An example is given to show the effectiveness of the results.

**Keywords:** fractional-order system, impulsive system, Lyapunov's direct method, Mittag-Leffler stability.

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## 1 Introduction

In the past two decades, fractional-order systems have been intensively studied due to their wide applications to various fields, such as viscoelastic systems, dielectric polarization, electromagnetic waves, heat conduction, robotics, biological systems, finance, and so on, see, for example, [1, 2, 3]. As we know, practical applications heavily depend on the dynamical behavior, especially on the stability, of models. So the stability of fractional differential equations (FDEs) has become one of the most active areas of research, and has attracted increasing interests from many scientists and engineers, see, for example, [4, 5] and [6] for a survey of the stability of FDEs.

Impulsive dynamical systems, which can be viewed as a subclass of hybrid systems, have not only played an important role in modeling physical phenomena subject to abrupt changes, but also from the control point of view provided a powerful tool for stabilization and synchronization of chaotic systems [7]. For the theory of impulsive dynamical systems and its applications, refer to [8, 9] and references therein.

Recently, impulsive fractional differential equations (IFDEs) have attracted considerable interests amongst researchers since their potential applications in some modeling of dynamical systems which involve hereditary phenomena and abrupt changes. There are some valuable

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results on IFDEs, but it is worth mentioning that Fečkan et al. [10] introduced a formula for solutions of the Cauchy problem of IFDEs and gave a counterexample to show that the previous results were incorrect. The related existence, uniqueness and data dependence results were presented in [11]. In [12], some necessary and sufficient conditions of controllability and observability for the impulsive fractional linear time-invariant system have been given. A pioneering work on the Hyers–Ulam–Rassias stability for nonlinear IFDEs has been reported by Wang et al. [13]. In applications, stability is one of the main concerns of IFDEs. For example, stability and stabilization of fractional order linear systems with uncertainties was considered in [14]; the stability result of fractional order systems with noncommensurate order was given in [15]; almost sure stability of fractional order Black–Scholes model was treated in [16].

However, to the best of our knowledge, the asymptotic stability and Mittag-Leffler stability of IFDEs have not yet been established now. Note that in [17, 18, 19] results about asymptotic stability of fractional order systems have been obtained by means of Lyapunov’s direct method. Here the asymptotic stability of the impulsive models will be studied. First comparison principles of the impulsive fractional order models are established. Then by virtue of Lyapunov’s direct method and comparison principles, results about asymptotic stability are given.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preliminary results. In Section 3, the Mittag-Leffler stability and asymptotic stability of impulsive fractional order systems are investigated by Lyapunov’s direct method. In Section 4, an example is given to demonstrate the effectiveness of the main results.

## 2 Preliminaries

First, several definitions and terminologies are recalled. Generally speaking, there are three commonly used definitions of fractional derivatives, i.e., Grünwald–Letnikov fractional derivative, Riemann–Liouville fractional derivative and Caputo fractional derivative. The last one is frequently adopted by applied scientists, since it is more convenient in the setting of the initial conditions.

**Definition 2.1** ([3]). The Riemann–Liouville derivative of function  $f(t)$  with fractional order  $q \in (0, 1)$  is given by

$${}^{RL}D_{t_0}^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^t (t-s)^{-q} f(s) ds.$$

The Caputo fractional derivative of function  $f(t)$  with fractional order  $q \in (0, 1)$  is defined as:

$$D_{t_0}^q f(t) = J_{t_0}^{1-q} f'(t),$$

where  $J_{t_0}^q$  is the Riemann–Liouville integral operator of order  $q$ , which is expressed as:

$$J_{t_0}^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s) ds, \quad q > 0.$$

Here  $\Gamma(\cdot)$  is the well-known Euler Gamma function.

**Remark 2.2** ([17]). If  $f(t_0) \geq 0$ , then one has  $D_{t_0}^q f(t) \leq^{RL} D_{t_0}^q f(t)$ . If  $f(t_0) > 0$ , then one has  $D_{t_0}^q f(t) <^{RL} D_{t_0}^q f(t)$ .

**Definition 2.3** ([20]). The Mittag-Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where  $\alpha > 0$  and  $z \in \mathbb{C}$ .

The two-parameter Mittag-Leffler function also appears frequently and has the form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where  $\alpha > 0, \beta > 0$  and  $z \in \mathbb{C}$ . When  $\beta = 1$ , one has  $E_\alpha(z) = E_{\alpha,1}(z)$ , further,  $E_{1,1}(z) = e^z$ .

Now consider the following impulsive fractional order system

$$\begin{cases} D_0^q x(t) = f(t, x), & t \neq t_k, \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}^+, \\ x(0) = x_0, \end{cases} \quad (2.1)$$

where  $D_0^q$  is the Caputo derivative,  $0 \equiv t_0 < t_1 < t_2 < \dots < t_k < \dots, t_k \rightarrow \infty$  as  $k \rightarrow \infty$ ,  $f: J \times PC^1 \rightarrow \mathbb{R}^n$  is Lebesgue measurable with respect to  $t$  and  $f(t, x)$  is continuous with respect to  $x$  on  $PC^1$ ,  $I_k: \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous, and  $I_k(0) = 0$ ;  $PC^1$  denotes the space of functions with piecewise continuous derivatives from  $\mathbb{R}$  to  $\mathbb{R}^n$ ;  $x_0 \in \mathbb{R}^n$ . Throughout the paper,  $\|\cdot\|$  is assumed to be a suitable complete norm in  $\mathbb{R}^n$ .

The existence and uniqueness result of system (2.1) is presented in [11]. The constant  $x_0$  is an equilibrium point of system (2.1) if  $f(t, x_0) = 0$ . Without loss of generality, assume system (2.1) admits zero solution. Stability is one of main concerns with system (2.1). Here we will investigate the Mittag-Leffler stability [17] of system (2.1), which is defined as follows.

**Definition 2.4.** The zero solution of (2.1) is said to be Mittag-Leffler stable if

$$\|x(t)\| \leq \{m(x_0)E_\alpha(-\lambda t^\alpha)\}^b, \quad t \in \mathbb{R}^+, \quad (2.2)$$

where  $\alpha \in (0, 1)$ ,  $\lambda \geq 0$ ,  $b > 0$ ,  $m(0) = 0$ ,  $m(x) \geq 0$ , and  $m(x)$  is locally Lipschitz in the domain  $B \in \mathbb{R}^n$  containing the origin with Lipschitz constant  $m_0$ .

**Definition 2.5.** The zero solution is said to be generalized Mittag-Leffler stable if

$$\|x(t)\| \leq \{m(x_0)t^{-\gamma}E_{\alpha,1-\gamma}(-\lambda t^\alpha)\}^b, \quad (2.3)$$

where  $\alpha \in (0, 1)$ ,  $-\alpha < \gamma \leq 1 - \alpha$ ,  $\lambda \geq 0$ ,  $b > 0$ ,  $m(0) = 0$ ,  $m(x) \geq 0$ , and  $m(x)$  is locally Lipschitz in the domain  $B \in \mathbb{R}^n$  containing the origin with Lipschitz constant  $m_0$ .

**Remark 2.6.** The ordinary and generalized Mittag-Leffler functions interpolate between a purely exponential law and power-law-like behavior of phenomena. So Mittag-Leffler and generalized Mittag-Leffler stability imply asymptotic stability. When  $b = 1$ ,  $\alpha = 1$ ,  $\gamma = 0$ , they reduce to the exponential stability, commonly used in stability analysis of integer-order systems.

**Definition 2.7** ([21]). A continuous function  $\alpha: [0, t] \rightarrow [0, \infty)$  is said to belong to class- $\kappa$  if it is strictly increasing and  $\alpha(0) = 0$ .

**Definition 2.8** ([22]). The class- $\kappa$  functions  $\alpha(r)$  and  $\beta(r)$  are said to be with local growth momentum at the same level if there exist  $s_1 > 0, k_1, k_2 > 0$  such that  $k_1(s) \geq \beta(s) \geq k_2\alpha(s)$  for all  $r \in [0, s_1]$ . The class- $\kappa$  functions  $\alpha(s)$  and  $\beta(s)$  are said to be with global growth momentum at the same level if there exist  $k_1, k_2 > 0$  such that  $k_1\alpha(s) \geq \beta(s) \geq k_2\alpha(s)$  for all  $s \geq 0$ .

**Definition 2.9** ([23]). A function  $f$  is locally left Hölder continuous in  $x$  if there are nonnegative constants  $C, v, \delta$  such that  $|f(x) - f(y)| \leq C(x - y)^v$  for all  $y \in (x - \delta, x]$  in the domain of  $f$ . The constant  $v$  is called the Hölder exponent.

### 3 Main results

First, the comparison principle [24] of fractional systems is extended to the impulsive case.

**Lemma 3.1.** Let  $u(t), v(t): [0, T] \rightarrow R$  ( $T \leq +\infty$ ) be locally left Hölder continuous, and

(i)

$$D_0^q v(t) \leq f(t, v(t));$$

(ii)

$$D_0^q w(t) \geq f(t, w(t)),$$

for all  $t \neq t_k$  ( $k \in N^+$ );

(iii)

$$v(t_k^+) = (1 + d_k)v(t_k), \quad w(t_k^+) = (1 + d_k)w(t_k) \quad (k \in N^+),$$

where  $d_k \geq 0$ , and  $\prod_{k=1}^{\infty} (1 + d_k)$  converges, and let  $d = \prod_{k=1}^{\infty} (1 + d_k)$ ;  $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots, t_k \rightarrow T$ , as  $k \rightarrow +\infty$ ;

(iv)

$$f(t, x) - f(t, y) \leq \frac{L}{1 + t^q}(x - y),$$

whenever  $x \geq y$  and

$$L < \Gamma(q + 1). \tag{3.1}$$

Then

$$v(0) < w(0) \tag{3.2}$$

implies

$$v(t) \leq w(t), \quad 0 \leq t \leq T. \tag{3.3}$$

*Proof.* **Case 1.** Suppose that the inequality in (ii) is strict, then we have

$$v(t) < w(t), \quad 0 \leq t \leq T. \tag{3.4}$$

If (3.4) is not true, then because of the continuity of the function on every  $(t_n, t_{n+1}]$  ( $t_0 = 0, n \in N$ ), (iii) and (3.2), it follows that there exists a  $t_*$  such that  $0 < t_* < T$  and

$$v(t_*) = w(t_*), \quad v(t) < w(t), \quad 0 < t < t_*.$$

Then, setting  $m(t) = w(t) - v(t)$ ,  $0 \leq t \leq t_*$ , we find  $m(t) \geq 0$  for  $0 \leq t \leq t_*$  and  $m(t_*) = 0$ .

Then

$$D_0^q m(t_*) \leq^{RL} D_0^q m(t_*) \leq 0. \quad (3.5)$$

In fact,

$${}^{RL}D_0^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_0^t (t-s)^{p-1} m(s) ds, \quad (3.6)$$

where  $p = 1 - q$ . Let  $H(t) = \int_0^t (t-s)^{p-1} m(s) ds$ , take  $h > 0$ ,

$$\begin{aligned} H(t_*) - H(t_* - h) &= \int_0^{t_*-h} \left[ (t_* - s)^{p-1} - (t_* - h - s)^{p-1} \right] m(s) ds \\ &\quad + \int_{t_*-h}^{t_*} (t_* - s)^{p-1} m(s) ds = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{t_*-h} \left[ (t_* - s)^{p-1} - (t_* - h - s)^{p-1} \right] m(s) ds, \\ I_2 &= \int_{t_*-h}^{t_*} (t_* - s)^{p-1} m(s) ds. \end{aligned}$$

Since

$$\left[ (t_* - s)^{p-1} - (t_* - h - s)^{p-1} \right] < 0,$$

and  $m(s) \geq 0$  for  $0 \leq s \leq t_* - h$ , we have  $I_1 \leq 0$ .

Since  $m(t)$  is locally left Hölder continuous and  $m(t_*) = 0$ , there exists a constant  $K(t_*) > 0$  such that for  $t_* - h \leq s \leq t_*$ ,

$$m(s) \leq K(t_*) (t_* - s)^\lambda,$$

where  $\lambda > 0$  and  $\lambda + p - 1 > 0$ . We then get

$$I_2 \leq K(t_*) \int_{t_*-h}^{t_*} (t_* - s)^{p-1+\lambda} ds = \frac{K(t_*)}{p+\lambda} h^{p+\lambda}.$$

Then

$$H(t_*) - H(t_* - h) - \frac{K(t_*)}{p+\lambda} h^{p+\lambda} \leq 0,$$

for sufficiently small  $h > 0$ .

Letting  $h \rightarrow 0$ , we obtain  $H'(t_*) \leq 0$ , which implies  ${}^{RL}D_0^q m(t_*) \leq 0$ . From Remark 2.2, it follows that

$$D_0^q m(t_*) \leq^{RL} D_0^q m(t_*).$$

Then we have  $D_0^q m(t_*) \leq 0$ . Together with (i) and (ii), we have

$$f(t, w(t_*)) < D_0^q w(t_*) \leq D_0^q v(t_*) \leq f(t, v(t_*)).$$

This is a contradiction since  $v(t_*) = w(t_*)$ . Hence (3.4) is valid.

**Case 2.** Suppose that the inequality in (ii) is nonstrict. Set

$$w_\varepsilon(t) = w(t) + \prod_{k=1}^n (1 + d_k) \varepsilon (1 + t^q),$$

for small  $\varepsilon > 0$  and  $t \in (t_n, t_{n+1}]$ . Then we have

$$w_\varepsilon(t_n^+) = (1 + d_n)w_\varepsilon(t_n), \quad w_\varepsilon(0) = w(0) + \varepsilon > w(0),$$

and

$$w_\varepsilon(t) > w(t),$$

for  $t \in [0, T]$ .

Note that

$$\begin{aligned} D_0^q w_\varepsilon(t) &= D_0^q w(t) + D_0^q \left[ \prod_{k=1}^n (1 + d_k) \varepsilon (1 + t^q) \right] \\ &\geq f(t, w(t)) + \varepsilon \Gamma(q + 1) \prod_{k=1}^n (1 + d_k) \\ &\geq f(t, w_\varepsilon(t)) - \prod_{k=1}^n (1 + d_k) \varepsilon [L - \Gamma(q + 1)] \\ &> f(t, w_\varepsilon(t)). \end{aligned}$$

Here we used condition (iv), (3.1) and (3.2). Now after applying the discussions in Case 1 to  $v(t)$  and  $w_\varepsilon(t)$ , we can get  $v(t) < w_\varepsilon(t)$ ,  $0 \leq t \leq T$ . Since  $\varepsilon > 0$  is arbitrary, then (3.3) is true.  $\square$

From Lemma 3.1, the comparison principle for linear impulsive fractional systems follows immediately.

**Lemma 3.2.** Let  $u(t), v(t): [0, T] \rightarrow R$  ( $T \leq +\infty$ ) be locally left Hölder continuous, and

(i)

$$D_0^q v(t) \leq -\lambda v;$$

(ii)

$$D_0^q w(t) \geq -\lambda w,$$

for all  $t \neq t_k$  ( $k \in N^+$ );

(iii)

$$v(t_k^+) = (1 + d_k)v(t_k), w(t_k^+) = (1 + d_k)w(t_k) \quad (k \in N^+),$$

where  $d_k \geq 0$ , and  $\prod_{k=1}^\infty (1 + d_k)$  converges, and let  $d = \prod_{k=1}^\infty (1 + d_k)$ .

Then

$$v(0) < w(0)$$

implies

$$v(t) \leq w(t), \quad 0 \leq t \leq T.$$

Now consider the following one dimensional linear impulsive fractional system

$$\begin{cases} D_0^q u(t) = -\lambda u, & t \neq t_k \\ u(0) = u_0, \\ \Delta u(t_k) := u(t_k^+) - u(t_k) = d_k u(t_k), & k \in N^+, \end{cases} \quad (3.7)$$

where  $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$ ,  $t_k \rightarrow T$ , as  $k \rightarrow +\infty$ ;  $\lambda > 0$ ,  $d_k > 0$ ,  $u_0$  are real constants;  $\prod_{k=1}^{+\infty} (1 + d_k)$  converges, and let  $d = \prod_{k=1}^{+\infty} (1 + d_k)$ .

**Case 1.**  $u_0 > 0$ .

Let  $u(t)$  be the solution of system (3.7). From Lemma 3.1,  $u(t) \geq 0$ . For  $t \in [0, t_1)$ , by the formula of the solution of linear fractional equations [3], one can have

$$u(t) = u_0 E_q(-\lambda t^q). \quad (3.8)$$

Then we have

$$u(t_1^+) = (1 + d_1)u(t_1^-) = (1 + d_1)u_0 E_q(-\lambda t_1^q). \quad (3.9)$$

Define  $\tilde{u}_1(t) = [(1 + d_1)u_0 + \varepsilon]E_q(-\lambda t^q)$ ,  $\varepsilon > 0$ ,  $t \in [0, t_2)$ , then  $u(t_1^+) \leq \tilde{u}_1(t_1)$  and

$$\begin{cases} D_0^q \tilde{u}_1(t) = -\lambda \tilde{u}_1, \\ \tilde{u}_1(0) = (1 + d_1)u_0 + \varepsilon. \end{cases} \quad (3.10)$$

For  $t \in (t_1, t_2]$ , it will be proved that

$$u(t) \leq \tilde{u}_1(t). \quad (3.11)$$

In fact, if (3.11) is not true, then there exist  $t_* \in (t_1, t_2]$ , such that,

$$u(t_*) = \tilde{u}_1(t_*), \quad (3.12)$$

and

$$u(t) \leq \tilde{u}_1(t),$$

for  $t \in [0, t_*)$ .

Denote

$$m(t) = \tilde{u}_1(t) - u(t),$$

then  $m(t) \geq 0$  for  $[0, t_*)$ .

Since  $m(0) > 0$ , from Remark 2.2 we have

$$D_0^q m(t) < {}^{RL}D_0^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{p-1} m(s) ds,$$

for  $t \in [0, t_2)$ , where  $p = 1 - q$ . Denote  $H(t) = \int_0^t (t-s)^{p-1} m(s) ds$ . Then for small  $h > 0$ ,

$$\begin{aligned} H(t_*) - H(t_* - h) &= \int_0^{t_*-h} [(t_* - s)^{p-1} - (t_* - h - s)^{p-1}] m(s) ds \\ &\quad + \int_{t_*-h}^{t_*} (t_* - s)^{p-1} m(s) ds = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{t_*-h} [(t_* - s)^{p-1} - (t_* - h - s)^{p-1}] m(s) ds, \\ I_2 &= \int_{t_*-h}^{t_*} (t_* - s)^{p-1} m(s) ds = I_1 + I_2. \end{aligned}$$

Note that

$$(t_* - s)^{p-1} - (t_* - h - s)^{p-1} \leq 0, \quad m(s) \geq 0,$$

for  $s \in [0, t_* - h]$ , so  $I_1 \leq 0$ .

Since  $m(t)$  is locally left Hölder continuous and  $m(t_*) = 0$ , there exists a constant  $K(t_*) > 0$ , such that for  $t_* - h \leq s \leq t_*$ ,

$$m(s) \leq K(t_*)(t_* - s)^\lambda,$$

where  $\lambda > 0$  and  $\lambda + p - 1 > 0$ . Then one gets

$$I_2 \leq K(t_*) \int_{t_*-h}^{t_*} (t_* - s)^{\lambda+p-1} ds = \frac{K(t_*)}{(p+\lambda)} h^{p+\lambda}.$$

Then

$$H(t_*) - H(t_* - h) - \frac{K(t_*)}{(p+\lambda)} h^{p+\lambda} \leq 0,$$

for sufficiently small  $h > 0$ .

Letting  $h \rightarrow 0$ , one has

$$H'(t_*) \leq 0,$$

which implies  ${}^{RL}D_0^q m(t_*) \leq 0$ .

Then we have

$$D_0^q m(t_*) < {}^{RL}D_0^q m(t_*) \leq 0,$$

which gives

$$-\lambda \tilde{u}_1(t_*) = D_0^q \tilde{u}_1(t_*) < D_0^q u(t_*) = -\lambda u(t_*).$$

This contradicts with (3.12). Then (3.11) is valid. Since  $\varepsilon > 0$  is arbitrary, then we have

$$u(t) \leq (1 + d_1)u_0 E_q(-\lambda t^q),$$

for  $t \in (t_1, t_2]$ .

Inductively, we can easily deduce that the solution  $u(t)$  of system (3.7) satisfy

$$u(t) \leq u_0 \prod_{k=1}^n (1 + d_k) E_q(-\lambda t^q), \quad t \in (t_n, t_{n+1}]. \quad (3.13)$$

That is,

$$0 \leq u(t) \leq u_0 d E_q(-\lambda t^q), \quad t \geq 0. \quad (3.14)$$

**Case 2.**  $u_0 < 0$ .

Let  $v(t) = -u(t)$ , then we have

$$\begin{cases} D_0^q v(t) = -\lambda v, & t \neq t_k, \\ v(0) = -u_0 > 0, \\ \Delta v(t_k) := v(t_k^+) - v(t_k) = d_k v(t_k), & k \in \mathbb{N}^+. \end{cases} \quad (3.15)$$

From the analysis in Case 1, we have

$$0 \leq v(t) \leq -u_0 d E_q(-\lambda t^q), \quad t \geq 0, \quad (3.16)$$

which gives

$$0 \geq u(t) \geq u_0 d E_q(-\lambda t^q).$$

Based on the discussions in both Case 1 and Case 2, one arrives at

$$|u(t)| \leq |u_0| d E_q(-\lambda t^q), \quad (3.17)$$

which implies the following theorem.

**Theorem 3.3.** *The one dimensional linear impulsive fractional-order system (3.7) is Mittag-Leffler stable.*

**Theorem 3.4.** *Suppose  $\prod_{k=1}^{\infty}(1 + d_k)$  converges,  $d_k > 0$ , and let  $d = \prod_{k=1}^{\infty}(1 + d_k)$ ,  $\alpha_1, \alpha_2, a, b$  and  $\beta$  are positive constants. Let  $V(t, x)$  be a locally left Hölder continuous function. If the following conditions are satisfied*

$$(i) \quad \alpha_1 \|x\|^a \leq V(t, x(t)) \leq \alpha_2 \|x\|^{ab}, \quad (3.18)$$

for all  $t \geq 0$ ;

$$(ii) \quad D_0^\gamma V(t, x(t)) \leq -\alpha_3 \|x\|^{ab}, \quad (3.19)$$

for all  $t \geq 0$  and  $t \neq t_k, k \in N^+, \gamma \in (0, 1]$ ;

$$(iii) \quad \Delta V(t, x(t)) := V(t^+, x(t^+)) - V(t, x(t)) = d_k(V(t, x(t))) \quad (3.20)$$

for  $t = t_k, k \in N^+$ ,

then system (2.1) is Mittag-Leffler stable.

*Proof.* Given any  $x_0 \in R^n$ , (3.18), (3.19), (3.20) imply that

$$\begin{cases} D_0^\gamma V(t, x(t)) \leq -\frac{\alpha_3}{\alpha_2} V(t, x(t)), & t \neq t_k, \\ \Delta V(t_k, x(t_k)) = d_k(V(t_k, x(t_k))), & k \in N^+. \end{cases} \quad (3.21)$$

From Lemma 3.2 and (3.17), we have

$$V(t, x(t)) \leq V(0, x(0)) \prod_{k=1}^{+\infty} (1 + d_k) E_\gamma \left( -\frac{\alpha_3}{\alpha_2} t^\gamma \right), \quad (3.22)$$

for  $t \geq 0$ .

From (3.18), one has

$$\begin{aligned} \alpha_1 \|x\|^a &\leq V(t, x(t)) \leq V(0, x(0)) \prod_{k=1}^{+\infty} (1 + d_k) E_\gamma \left( -\frac{\alpha_3}{\alpha_2} t^\gamma \right) \\ &\leq \alpha_2 \|x_0\|^{ab} \prod_{k=1}^{+\infty} (1 + d_k) E_\gamma \left( -\frac{\alpha_3}{\alpha_2} t^\gamma \right), \end{aligned} \quad (3.23)$$

that is,

$$\begin{aligned} \|x(t)\| &\leq \|x_0\|^b \left[ \frac{\alpha_2}{\alpha_1} \prod_{k=1}^{+\infty} (1 + d_k) E_\gamma \left( -\frac{\alpha_3}{\alpha_2} t^\gamma \right) \right]^{\frac{1}{a}} \\ &\leq \|x_0\|^b \left[ \frac{\alpha_2}{\alpha_1} d E_\gamma \left( -\frac{\alpha_3}{\alpha_2} t^\gamma \right) \right]^{\frac{1}{a}}, \end{aligned} \quad (3.24)$$

which implies the Mittag-Leffler stability of system (2.1).  $\square$

**Theorem 3.5.** *Suppose  $d_k \geq 0$  and  $\prod_{k=1}^{\infty}(1 + d_k)$  converges. Let  $d = \prod_{k=1}^{\infty}(1 + d_k)$  and  $V(t, x)$  be a locally left Hölder continuous function. Assume*

(i) there exist class- $\kappa$  functions  $\alpha_i$ ,  $i = 1, 2, 3$ , having global growth momentum at the same level and satisfying

$$\alpha_1(\|x\|) \leq V(t, x(t)) \leq \alpha_2(\|x\|), \quad (3.25)$$

for all  $t \geq 0$ ;

(ii)

$$D_0^\gamma V(t, x(t)) \leq -\alpha_3(\|x\|), \quad (3.26)$$

for all  $t \geq 0$  and  $t \neq t_k$ ,  $\gamma \in (0, 1]$ ;

(iii)

$$\Delta V(t, x(t)) = d_k(V(t, x(t))) \quad (3.27)$$

for  $t = t_k$ ,  $k \in \mathbb{N}^+$ ;

(iv) there exists  $a > 0$  such that  $\alpha_1(r)$  and  $r^a$  have global growth momentum at the same level.

Then system (2.1) is Mittag-Leffler stable.

*Proof.* It follows from Conditions (i) and (ii) that there exists  $k_1 > 0$  such that

$$\begin{aligned} D_0^\gamma V(t, x(t)) &\leq -\alpha_3(\|x\|) \\ &\leq -k_1 \alpha_2(\|x\|) \leq -k_1 V(t, x(t)). \end{aligned} \quad (3.28)$$

Using (3.25), (3.28) and Lemma 3.2, we obtain

$$\alpha_1(\|x\|) \leq V(t, x(t)) \leq V(0) dE_\gamma(-k_1 t^\gamma). \quad (3.29)$$

In addition, using Condition (iv), one gets

$$(k_2 \|x\|)^a \leq \alpha_1(\|x\|), \quad (3.30)$$

where  $k_2 > 0$ .

Substituting (3.30) into (3.29), we finally get

$$\|x(t)\| \leq \left\{ \frac{V(0)}{k_2^a} dE_\gamma(-k_1 t^\gamma) \right\}^{1/a}, \quad (3.31)$$

which implies that system (2.1) is Mittag-Leffler stable.  $\square$

## 4 An illustrative example

For the impulsive fractional-order system

$$\begin{cases} D^q |x_1(t)| = -2|x_1(t)| - 3|x_2(t)| - 4|x_3(t)|, & t \neq t_k, \\ D^q |x_2(t)| = -2\sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}, & t \neq t_k, \\ D^q |x_3(t)| = -6\sqrt{x_1^4(t) + x_2^4(t) + x_3^4(t)}, & t \neq t_k, \\ \Delta x_1(t) = d_k x_1(t^+), & t = t_k, \\ \Delta x_1(t) = d_k x_1(t^+), & t = t_k, \\ \Delta x_1(t) = d_k x_1(t^+), & t = t_k, \end{cases} \quad (4.1)$$

where  $x(t) \equiv (x_1(t), x_2(t), x_3(t)) \in R^3$ ,  $d_k \geq 0$ , and  $\prod_{k=1}^{\infty}(1 + d_k)$  converges and let  $d = \prod_{k=1}^{\infty}(1 + d_k)$ .

Consider the Lyapunov function candidate  $V(t, x(t)) = |x_1| + |x_2| + |x_3|$ , then

$$\begin{cases} D_0^q V \leq -6\|x\|, & t \neq t_k, \\ V(t, x(t)) = (1 + d_k)V(t^+, x(t^+)), & t = t_k. \end{cases} \quad (4.2)$$

Let  $\gamma = q$ ,  $a = b = 1$ ,  $\alpha_2 = 1$ ,  $\alpha_2 = \sqrt{3}$ ,  $\alpha_3 = 6$ , then

$$\alpha_1\|x\| \leq V(t, x(t)) \leq \alpha_2\|x\|.$$

From Theorem 3.4, it gives

$$\begin{aligned} \|x(t)\| &\leq \|x(0)\| \sqrt{3} \prod_{k=1}^{+\infty} d_k E_q(-2\sqrt{3}t^q) \\ &\leq \|x(0)\| \sqrt{3} d E_q(-2\sqrt{3}t^q), \end{aligned} \quad (4.3)$$

which means system (4.1) is Mittag-Leffler stable.

## 5 Conclusions

Impulsive fractional order systems, which appear in several areas of science and engineering, involve hereditary phenomena and abrupt changes. The combined use of the fractional derivative and impulsive system may lead to a better description of systems in applications. By comparison principles and Lyapunov's direction method, results about the Mittag-Leffler stability of such systems are obtained, in the presence of Caputo fractional derivative.

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