

## A RESULT BY SILVERMAN TRANSFERRED ON MULTIVALENT FUNCTIONS

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ABSTRACT. In 1999 Silverman introduced a class of univalent functions defined over the quotient of the analytical representations of convexity and starlikeness. This class was widely studied afterwards. In this paper, in a similar manner, we define a class of multivalent functions and obtain results, dual to the ones for univalent functions.

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### 1. INTRODUCTION

For  $n \in \mathbb{N}$  and  $a \in \mathbb{C}$  let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

where  $\mathcal{H}(\mathbb{D})$  is the class of all functions that are analytic in the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Also, let for a positive integer  $p$ ,  $\mathcal{A}_p$  be the subclass of  $\mathcal{H}(\mathbb{D})$  consisting of functions of the form  $f(z) = z^p + a_{p+1} z^{p+1} + \dots$  and  $\mathcal{A} \equiv \mathcal{A}_1$ , so that  $\mathcal{A}$  is the class of functions  $f$  which are analytic in  $\mathbb{D}$  with normalization  $f(0) = 0$  and  $f'(0) = 1$ . More details in [2, 3, 9].

A function  $f$  is said to be *multivalent* or *p-valent* in  $\mathbb{D}$  if it takes no value more than  $p$  times in  $\mathbb{D}$  and there is some  $\omega_0$  such that  $f(z) = \omega_0$  has exactly  $p$  solutions in  $\mathbb{D}$ , when roots are counted in accordance with their multiplicities.

The classes of *p-valently starlike functions of order  $\alpha$*  and *p-valently convex functions of order  $\alpha$*  ( $p = 1, 2, \dots$  and  $0 \leq \alpha < p$ ) are, respectively, defined by

$$\mathcal{S}_p^*(\alpha) = \left\{ f \in \mathcal{A}_p : \operatorname{Re} \left[ \frac{z f'(z)}{f(z)} \right] > \alpha, z \in \mathbb{D} \right\}$$

and

$$\mathcal{K}_p(\alpha) = \left\{ f \in \mathcal{A}_p : \operatorname{Re} \left[ 1 + \frac{z f''(z)}{f'(z)} \right] > \alpha, z \in \mathbb{D} \right\}.$$

For  $p = 1$ , we have respectively, the usual classes of starlike and convex functions of order  $\alpha$ ,  $\mathcal{S}^*(\alpha)$  and  $\mathcal{K}(\alpha)$ ; and for  $p = 2, 3, \dots$ , we have the classes of multivalent starlike and convex functions of order  $\alpha$ . In the case  $\alpha = 0$ , we will use the notations  $\mathcal{S}_p^*$  and  $\mathcal{K}_p$ ; and in the case  $\alpha = 0$  and  $p = 1$ , the notations  $\mathcal{S}^*$  and  $\mathcal{K}$ .

## 2. PRELIMINARIES

Silverman in his work [7] investigated the properties of a class of functions defined in terms of the quotient of the analytical representations of convexity and starlikeness,  $\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$ . Precisely, he considered the following classes

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, z \in \mathbb{D} \right\},$$

$0 < b \leq 1$ . This class was widely studied ever since (see [8, 6, 10, 11]).

In this paper we study an analogue expression for multivalent functions,

$$\frac{1 + zf^{(p+1)}(z)/f^{(p)}(z)}{zf^{(p)}(z)/f^{(p-1)}(z)}$$

and find sufficient conditions so that the following implication holds:

$$\left| \frac{1 + zf^{(p+1)}(z)/f^{(p)}(z)}{zf^{(p)}(z)/f^{(p-1)}(z)} - 1 \right| < \lambda \quad \Rightarrow \quad \left| \frac{zf'(z)}{f(z)} - p \right| < \mu \quad (1)$$

(both inequalities are on the whole unit disk).

For that purpose we use a method from the theory of differential subordinations to get our main result. More on this theory can be found in [3] and [1].

First we introduce the concept of subordination. Let  $f(z), g(z) \in \mathcal{A}$ . We say that  $f(z)$  is *subordinate* to  $g(z)$ , and write  $f(z) \prec g(z)$ , if there exists a function  $\omega(z)$ , analytic in the unit disc  $\mathbb{D}$ , such that  $\omega(0) = 0$ ,  $|\omega(z)| < 1$  and  $f(z) = g(\omega(z))$  for all  $z \in \mathbb{D}$ . If  $g(z)$  is univalent in  $\mathbb{D}$  then  $f(z) \prec g(z)$  if and only if  $f(0) = g(0)$  and  $f(\mathbb{D}) \subseteq g(\mathbb{D})$ .

The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [4] and [5]. In fact, if  $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ ,  $\mathbb{C}$  complex plane, is analytic in a domain  $D$ , if  $h(z)$  is univalent in  $\mathbb{D}$ , and if  $p(z)$  is analytic in  $\mathbb{D}$  with  $(p(z), zp'(z)) \in D$  when  $z \in \mathbb{D}$ , then  $p(z)$  is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \quad (2)$$

A univalent function  $q(z)$  is called a *dominant* of the differential subordination (2) if  $p(z) \prec q(z)$  for all  $p(z)$  satisfying (2). If  $\tilde{q}(z)$  is a dominant of (2) and  $\tilde{q}(z) \prec q(z)$  for all dominants of (2), then we say that  $\tilde{q}(z)$  is the *best dominant* of the differential subordination (2).

For proving implications in this paper we will use the following lemma from the theory of differential subordinations. She give efficient tool for obtaining sufficient conditions when certain differential inequality holds.

**Lemma 1** (Theorem 2.3h(i), p.34, [3]). *Let  $\Omega \subset \mathbb{C}$  and suppose that the function  $\psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$  satisfies  $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$  for all  $K \geq Mn$ ,  $\theta \in \mathbb{R}$  and  $z \in \mathbb{D}$ . If  $p \in \mathcal{H}[a, n]$  and  $\psi(p(z), zp'(z); z) \in \Omega$  for all  $z \in \mathbb{D}$ , then  $|p(z)| < M$ ,  $z \in \mathbb{D}$ .*

### 3. MAIN RESULTS

In the beginning we will study implication:

$$\left| \frac{1 + zf^{(p+1)}(z)/f^{(p)}(z)}{zf^{(p)}(z)/f^{(p-1)}(z)} - 1 \right| < \lambda \quad \Rightarrow \quad \left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1 \right| < \lambda_1 \quad (3)$$

(both inequalities are on the whole unit disk), that will later lead to the implication (1).

**Theorem 2.** *Let  $p$  be a positive integer,  $p \geq 2$  and  $\lambda_1 > 0$ . Also, let  $f \in \mathcal{A}_p$ . If  $\lambda \leq \lambda(\lambda_1) \equiv \frac{\lambda_1}{(\lambda_1+1)^2}$ , then implication (3) holds.*

*Proof.* In the view of Lemma 1, let put  $M = \lambda_1$  and let define function  $p(z) = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1$ . Further, for the function  $\psi(r, s; z) = \frac{s}{(r+1)^2}$ , such that  $p(z) \in \mathcal{H}[0, 1]$  ( $n = 1$  in the Lemma 1) we have

$$\psi(p(z), zp'(z); z) = \left| \frac{zp'(z)}{(p(z) + 1)^2} \right| = \left| \frac{1 + zf^{(p+1)}(z)/f^{(p)}(z)}{zf^{(p)}(z)/f^{(p-1)}(z)} - 1 \right| < \lambda(\lambda_1)$$

for all  $z \in \mathbb{D}$  and

$$\begin{aligned} \psi(Me^{i\theta}, Ke^{i\theta}; z) &= \left| \frac{Ke^{i\theta}}{(Me^{i\theta} + 1)^2} \right| = \frac{K|e^{i\theta}|}{|Me^{i\theta} + 1|^2} = \\ &= \frac{K}{|Me^{i\theta} + 1|^2} \geq \frac{M}{|Me^{i\theta} + 1|^2} \geq \frac{M}{M^2 + 2M + 1} = \\ &= \frac{M}{(M + 1)^2} = \frac{\lambda_1}{(\lambda_1 + 1)^2} \geq \lambda \end{aligned}$$

for all  $z \in \mathbb{D}$ . So,  $|p(z)| < M$  for all  $z \in \mathbb{D}$ .

In a similar way we prove the following result.

**Theorem 3.** *Let  $p$  and  $k$  are integers such that  $p \geq 2$  and  $0 \leq k \leq p - 2$ . Also let  $f \in \mathcal{A}_p$ ,  $\lambda_{k+1} > 0$  and*

$$\lambda_k \leq \lambda_k(\lambda_{k+1}) \equiv \frac{\lambda_{k+1}}{\lambda_{k+1} + k + 2} + \lambda_{k+1}.$$

If

$$\left| \frac{zf^{(p-k)}(z)}{f^{(p-k-1)}(z)} - (k+1) \right| < \lambda_k(\lambda_{k+1}) \quad (z \in \mathbb{D}),$$

then

$$\left| \frac{zf^{(p-k-1)}(z)}{f^{(p-k-2)}(z)} - (k+2) \right| < \lambda_{k+1} \quad (z \in \mathbb{D}).$$

*Proof.* Let put  $M = \lambda_{k+1}$  and let define functions

$$p(z) = \frac{zf^{(p-k-1)}(z)}{f^{(p-k-2)}(z)} - k - 2$$

and

$$\psi(r, s; z) = \frac{s}{r + k + 2} + r,$$

such that  $p(z) \in \mathcal{H}[0, 1]$  ( $n = 1$  in the Lemma 1) and

$$\begin{aligned} \psi(p(z), zp'(z); z) &= \left| \frac{zp'(z)}{p(z) + k + 2} + p(z) \right| \\ &= \left| \frac{zf^{(p-k)}(z)}{f^{(p-k-1)}(z)} - k - 1 - p(z) + p(z) \right| \\ &= \left| \frac{zf^{(p-k)}(z)}{f^{(p-k-1)}(z)} - k - 1 \right| < \lambda_k(\lambda_{k+1}) \end{aligned}$$

for all  $z \in \mathbb{D}$ . In order to prove the theorem it is enough to show that  $\psi(Me^{i\theta}, Ke^{i\theta}; z) \geq \lambda_k(\lambda_{k+1})$ . Indeed, since  $k + 2 \geq M$  we have

$$\begin{aligned} \psi(Me^{i\theta}, Ke^{i\theta}; z) &= \left| \frac{Ke^{i\theta}}{Me^{i\theta} + k + 2} + Me^{i\theta} \right| = \left| \frac{K}{Me^{i\theta} + k + 2} + M \right| \\ &\geq \frac{K}{M + k + 2} + M \geq \frac{M}{M + k + 2} + M \\ &= \frac{\lambda_{k+1}}{\lambda_{k+1} + k + 2} + \lambda_{k+1} = \lambda_k(\lambda_{k+1}) \end{aligned}$$

for all  $z \in \mathbb{D}$ .

Since the functions  $\lambda(\lambda_1) = \frac{\lambda_1}{(\lambda_1+1)^2}$  and  $\lambda_k(\lambda_{k+1}) = \frac{\lambda_{k+1}}{\lambda_{k+1}+k+2} + \lambda_{k+1}$  are strictly monotone on the interval  $(0, 1)$  and  $(0, +\infty)$ , respectively, we realize that there exist their inverse  $\lambda_1(\lambda) = \frac{1-2\lambda+\sqrt{1-4\lambda}}{2\lambda}$  and

$$\lambda_{k+1}(\lambda_k) = \frac{1}{2} \left( \lambda_k - k - 3 + \sqrt{(\lambda_k + k + 3)^2 - 4\lambda_k} \right).$$

So, Theorem 2 and Theorem 3 can be rewritten in the following, equivalent form.

**Theorem 4.** *Let  $p$  be a positive integer,  $p \geq 2$  and  $0 < \lambda \leq 1/4$ . Also, let  $f \in \mathcal{A}_p$ . If*

$$\left| \frac{1 + z f^{(p+1)}(z) / f^{(p)}(z)}{z f^{(p)}(z) / f^{(p-1)}(z)} - 1 \right| < \lambda \quad (z \in \mathbb{D}),$$

then

$$\left| \frac{z f^{(p)}(z)}{f^{(p-1)}(z)} - 1 \right| < \frac{1 - 2\lambda + \sqrt{1 - 4\lambda}}{2\lambda} \quad (z \in \mathbb{D}).$$

**Theorem 5.** *Let  $p$  and  $k$  are integers such that  $p \geq 2$  and  $0 \leq k \leq p - 2$ . Also let  $f \in \mathcal{A}_p$  and  $0 < \lambda_k < 1$ . If*

$$\left| \frac{z f^{(p-k)}(z)}{f^{(p-k-1)}(z)} - (k + 1) \right| < \lambda_k \quad (z \in \mathbb{D}),$$

then

$$\begin{aligned} \left| \frac{z f^{(p-k-1)}(z)}{f^{(p-k-2)}(z)} - (k + 2) \right| &< \lambda_{k+1} \\ &\equiv \frac{1}{2} \left[ \lambda_k - k - 3 + \sqrt{(\lambda_k + k + 3)^2 - 4\lambda_k} \right] \quad (z \in \mathbb{D}). \end{aligned}$$

We reach the final goal of this paper by combining the two previous theorems. Theorem 5 can be applied recursively because  $\frac{1}{2} \left[ \lambda_k - k - 3 + \sqrt{(\lambda_k + k + 3)^2 - 4\lambda_k} \right]$  is in the interval  $(0, 1)$  when  $\lambda_k \in (0, 1)$ .

**Theorem 6.** Let  $p$  be a positive integer,  $p \geq 2$  and  $0 < \lambda \leq 1/4$ . Also, let  $\lambda_1 = \frac{1-2\lambda+\sqrt{1-4\lambda}}{2\lambda}$  and for  $k = 1, 2, \dots$ ,

$$\lambda_{k+1} = \frac{1}{2} \left[ \lambda_k - k - 3 + \sqrt{(\lambda_k + k + 3)^2 - 4\lambda_k} \right].$$

If  $f \in \mathcal{A}_p$  and

$$\left| \frac{1 + zf^{(p+1)}(z)/f^{(p)}(z)}{zf^{(p)}(z)/f^{(p-1)}(z)} - 1 \right| < \lambda \quad (z \in \mathbb{D}),$$

then

$$\left| \frac{zf'(z)}{f(z)} - p \right| < \lambda_{p-1} \quad (z \in \mathbb{D}).$$

For  $p = 2$  and  $\lambda = 1/4$  we have  $\lambda_1 = 1$  and receive the following corollary.

**Corollary 7.** If  $f \in \mathcal{A}_2$  and

$$\left| \frac{1 + zf'''(z)/f''(z)}{zf''(z)/f'(z)} - 1 \right| < \frac{1}{4} \quad (z \in \mathbb{D}),$$

then

$$\left| \frac{zf'(z)}{f(z)} - 2 \right| < 1 \quad (z \in \mathbb{D}).$$

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