

GROWTH OF MEROMORPHIC SOLUTIONS OF GENERAL COMPLEX LINEAR DIFFERENTIAL-DIFFERENCE EQUATION

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ABSTRACT. In this paper, we investigate the relations between the growth of entire or meromorphic coefficients and the growth of meromorphic solutions of general complex linear differential-difference equation, and obtain the lower bound of the order of meromorphic solutions by comparing the (lower) orders or the (lower) types of the coefficients. Our results can be seen as generalizations for both the case of complex linear differential equation and the case of complex linear difference equation.

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1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, we assume that $f(z)$ is a meromorphic function in the whole complex plane, and use standard notations, such as $m(r, f)$, $T(r, f)$, $M(r, f)$, in the classic Nevanlinna theory [5,7,13]. And we also use the notations $\sigma(f)$ and $\mu(f)$ to denote respectively the order and the lower order of $f(z)$. Moreover, we give the definitions of the type and the lower type as follows.

Definition 1. (see[5,6]) Let $f(z)$ be a meromorphic function with the order $\sigma(f)$ ($0 < \sigma(f) < \infty$) and the lower order $\mu(f)$ ($0 < \mu(f) < \infty$), then the type and the lower type of $f(z)$ are defined respectively by

$$\tau(f) = \overline{\lim}_{r \rightarrow \infty} \frac{T(r, f)}{r^{\sigma(f)}} \quad \text{and} \quad \underline{\tau}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{T(r, f)}{r^{\mu(f)}}.$$

If $f(z)$ is also an entire function, then we denote

$$\tau_M(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\sigma(f)}} \quad \text{and} \quad \underline{\tau}_M(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\mu(f)}}.$$

Many scholars applied Nevanlinna theory and its difference analogues to study the properties of meromorphic solutions of complex linear differential equation

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \quad (1)$$

and complex linear difference equation

$$A_n(z)f(z + c_n) + \cdots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0, \quad (2)$$

and obtained fruitful results (see e.g. [1,2,8,9,11,12,14]). Interestingly, we find there are many similar properties for both the case of complex linear differential equation (1) and the case of complex linear difference equation (2). We give examples on this topic as follows.

For the case of complex linear differential equation (1), Wu and Zheng [12] weakened the normal condition (see e.g. [1,11]) that only one dominant coefficient of (1) has the (lower) order or the (lower) type strictly greater than the order or the type of other coefficients, and obtained the following Theorems 1 and 2.

Theorem 1. (see[12]) Let $A_i(z), i = 0, 1, \dots, n - 1$ be entire functions and $j \in \{1, 2, \dots, n - 1\}$. If the following three assumptions hold simultaneously:

- (1) $\max\{\mu_p(A_j), \sigma_p(A_i), i \neq 0, j\} = \sigma \leq \mu_p(A_0) < \infty, \mu_p(A_0) > 0;$
- (2) $\tau_p(A_0) > \tau_p(A_j)$, when $\mu_p(A_0) = \mu_p(A_j)$;
- (3) $\tau_p(A_0) > \max\{\tau_p(A_i) : \sigma_p(A_i) = \mu_p(A_0), i \neq 0, j\}$, when $\mu_p(A_0) = \max\{\sigma_p(A_i), i \neq 0, j\}$.

Then any solution $f(z) (\neq 0)$ of (1) satisfies $\sigma_{p+1}(f) \geq \mu_p(A_0)$.

Theorem 2. (see[12]) Let $A_i(z), i = 0, 1, \dots, n - 1$ be entire functions and $j \in \{1, 2, \dots, n - 1\}$. If the following three assumptions hold simultaneously:

- (1) $\max\{\mu_p(A_0), \sigma_p(A_i), i \neq 0, j\} = \delta \leq \mu_p(A_j) < \infty, \mu_p(A_j) > 0;$
- (2) $\tau_p(A_j) > \tau_p(A_0)$, when $\mu_p(A_j) = \mu_p(A_0)$;
- (3) $\tau_p(A_j) > \max\{\tau_p(A_i) : \sigma_p(A_i) = \mu_p(A_j), i \neq 0, j\}$, when $\mu_p(A_j) = \max\{\sigma_p(A_i), i \neq 0, j\}$.

Then any transcendental solution $f(z)$ of (1) satisfies $\max\{\sigma_{p+1}(f), \lambda_p(f)\} \geq \mu_p(A_j)$, and the non-transcendental solution can only be polynomials that have degree no more than $j - 1$.

For the case of complex linear difference equation (2), Luo and Zheng [9] obtained the similar results as Theorems 1 and 2, which also generalize the acknowledged good results (see e.g. [2, 8]) in the field of complex linear difference equations.

Theorem 3. (see[9]) Let $A_j(z), j = 0, 1, \dots, n$ be entire functions and $k, l (\neq k) \in \{0, 1, \dots, n\}$. If the following three assumptions hold simultaneously:

- (1) $\max\{\mu(A_k), \sigma(A_j), j \neq k, l\} = \sigma \leq \mu(A_l) < \infty, \mu(A_l) > 0;$
- (2) $\tau_M(A_l) > \tau_M(A_k),$ when $\mu(A_l) = \mu(A_k);$
- (3) $\tau_M(A_l) > \max\{\tau_M(A_j) : \sigma(A_j) = \mu(A_l), j \neq k, l\},$ when $\mu(A_l) = \max\{\sigma(A_j), j \neq k, l\}.$

Then any meromorphic solution $f(z) (\neq 0)$ of (2) satisfies $\sigma(f) \geq \mu(A_l) + 1.$

Theorem 4. (see[9]) Let $A_j(z), j = 0, 1, \dots, n$ be meromorphic functions and $k, l (\neq k) \in \{0, 1, \dots, n\}.$ If the following four assumptions hold simultaneously:

- (1) $\max\{\mu(A_k), \sigma(A_j), j \neq k, l\} = \sigma \leq \mu(A_l) < \infty, \mu(A_l) > 0;$
- (2) $\tau(A_l) > \tau(A_k),$ when $\mu(A_l) = \mu(A_k);$
- (3) $\tau(A_l) > \max\{\tau(A_j) : \sigma(A_j) = \mu(A_l), j \neq k, l\},$ when $\mu(A_l) = \max\{\sigma(A_j), j \neq k, l\};$
- (4) $\delta(\infty, A_l) = \delta > 0.$

Then any meromorphic solution $f(z) (\neq 0)$ of (2) satisfies $\sigma(f) \geq \mu(A_l) + 1.$

Remark 1. By a careful examination of the proof of Theorem 4, we note that “ $\tau(A_l)$ ” in Theorem 4 should be replaced by “ $\delta\tau(A_l)$ ”.

Inspired by the above results, we proceed to investigate the more general case than (1) and (2), that is, complex linear differential-difference equation

$$\sum_{i=0}^n \sum_{j=0}^m A_{ij}(z) f^{(j)}(z + c_i) = 0. \quad (3)$$

This is an interesting and meaningful topic. Not only for the sake that (3) is a combination of (1) and (2), and consequently inherits some common properties from (1) and (2). But also for the sake that (3) has its own application fields distinct from (1) and (2), and consequently (3) is of importance. There are already some results on this topic (see e.g. [10, 15]). As generalizations of Theorems 1-4, we obtain the following two results in the filed of complex linear differential-difference equations.

Theorem 5. Let $A_{ij}(z), i = 0, 1, \dots, n, j = 0, 1, \dots, m$ be entire functions, and $a, l \in \{0, 1, \dots, n\}, b \in \{0, 1, \dots, m\}$ such that $(a, b) \neq (l, 0).$ If the following three assumptions hold simultaneously:

- (1) $\max\{\mu(A_{ab}), \sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \sigma \leq \mu(A_{l0}) < \infty, \mu(A_{l0}) > 0;$
- (2) $\tau_M(A_{l0}) > \tau_M(A_{ab}),$ when $\mu(A_{l0}) = \mu(A_{ab});$
- (3) $\tau_M(A_{l0}) > \max\{\tau_M(A_{ij}) : \sigma(A_{ij}) = \mu(A_{l0}), (i, j) \neq (a, b), (l, 0)\},$ when $\mu(A_{l0}) = \max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\}.$

Then any meromorphic solution $f(z) (\neq 0)$ of (3) satisfies $\sigma(f) \geq \mu(A_{l0}) + 1.$

Theorem 6. Let $A_{ij}(z), i = 0, 1, \dots, n, j = 0, 1, \dots, m$ be meromorphic functions, and $a, l \in \{0, 1, \dots, n\}, b \in \{0, 1, \dots, m\}$ such that $(a, b) \neq (l, 0).$ If the following four assumptions hold simultaneously:

- (1) $\delta(\infty, A_{l0}) = \delta > 0$;
 (2) $\max\{\mu(A_{ab}), \sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \sigma \leq \mu(A_{l0}) < \infty, \mu(A_{l0}) > 0$;
 (3) $\delta_{\underline{\tau}}(A_{l0}) > \underline{\tau}(A_{ab})$, when $\mu(A_{l0}) = \mu(A_{ab})$;
 (4) $\delta_{\underline{\tau}}(A_{l0}) > \max\{\tau(A_{ij}) : \sigma(A_{ij}) = \mu(A_{l0}), (i, j) \neq (a, b), (l, 0)\}$, when $\mu(A_{l0}) = \max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\}$.

Then any meromorphic solution $f(z) (\neq 0)$ of (3) satisfies $\sigma(f) \geq \mu(A_{l0}) + 1$.

Remark 2. The assumption (1) in Theorem 6 can be replaced by one of $\lambda(\frac{1}{A_{l0}}) < \mu(A_{l0})$, or $N(r, A_{l0}) = o(m(r, A_{l0}))$ ($r \rightarrow \infty$), or $\mu(A_{l0}) = \liminf_{r \rightarrow \infty} \frac{\log m(r, A_{l0})}{\log r}$.

Remark 3. It is obvious that Theorems 1-4 are special cases of Theorems 5 and 6.

2. LEMMAS FOR PROOFS OF MAIN RESULTS

Lemma 7. (see[2]) Let $f(z)$ be a meromorphic function, $\eta (\neq 0)$, $\eta_1, \eta_2 (\eta_1 \neq \eta_2)$ be arbitrary complex numbers, and let $\gamma (> 1)$ and $\varepsilon (> 0)$ be given real constants. Then there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure,

(1) and a constant A depending only on γ and η , such that for all $|z| = r \notin [0, 1] \cup E$, we have

$$\left| \log \left| \frac{f(z + \eta)}{f(z)} \right| \right| \leq A \left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r} \log^\gamma r \log^+ n(\gamma r) \right);$$

(2) and if in addition that $f(z)$ has finite order σ , and such that for all $|z| = r \notin [0, 1] \cup E$, we have

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left| \frac{f(z + \eta_1)}{f(z + \eta_2)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 8. (see[4]) (1) Let $f(z)$ be a transcendental meromorphic function, $\Gamma = \{(k_1, j_1), \dots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers such that $k_i > j_i \geq 0, i = 1, \dots, q$, $\alpha (> 1)$ be a given real constant. Then there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure, and there exists a constant $B (> 0)$ that depend only on α and Γ , such that for all z satisfying $|z| \notin [0, 1] \cup E$ and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq B \left(\frac{T(\alpha r, f)}{r} \log^\alpha r \log T(\alpha r, f) \right)^{k-j};$$

(2) if $\sigma(f) = \sigma < \infty$, then for any given $\varepsilon (> 0)$, there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin [0, 1] \cup E$

and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 9. (see[3]) Let $f(z)$ be a meromorphic function, c is a non-zero complex constant. Then the following inequalities

$$(1 + o(1))T(r - |c|, f) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f)$$

hold as $r \rightarrow \infty$. Therefore, it is easy to obtain that

$$\sigma(f(z + c)) = \sigma(f), \quad \mu(f(z + c)) = \mu(f).$$

Lemma 10. (see[2]) Let η_1, η_2 be distinct complex numbers, and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$. Then for each $\varepsilon(> 0)$, we have

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

The following two lemmas are the special case $p = 1$ of the original ones in [6] and [12] respectively.

Lemma 11. (see[6]) Let $f(z)$ be an entire function with $\mu(f) < \infty$, then for any given $\varepsilon(> 0)$, there exists a subset $H \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H$, we have

$$\mu(f) = \lim_{\substack{r \rightarrow \infty \\ r \in H}} \frac{\log \log M(r, f)}{\log r} \quad \text{and} \quad M(r, f) < \exp\{r^{\mu(f)+\varepsilon}\}.$$

Lemma 12. (see[12]) Let $f(z)$ be an entire function with $0 < \mu(f) < \infty$, then for any given $\varepsilon(> 0)$, there exists a subset $H \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H$, we have

$$\tau_M(f) = \lim_{\substack{r \rightarrow \infty \\ r \in H}} \frac{\log M(r, f)}{r^{\mu(f)}} \quad \text{and} \quad M(r, f) < \exp\{(\tau_M(f) + \varepsilon)r^{\mu(f)}\}.$$

We also have the similar two lemmas for the case of meromorphic functions.

Lemma 13. (see[14]) Let $f(z)$ be a meromorphic function with $\mu(f) < \infty$. Then for any given $\varepsilon(> 0)$, there exists a subset $H \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H$, we have

$$T(r, f) < r^{\mu(f)+\varepsilon}.$$

Lemma 14. (see[9]) Let $f(z)$ be a meromorphic function with $0 < \mu(f) < \infty$. Then for any given $\varepsilon(> 0)$, there exists a subset $H \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H$, we have

$$T(r, f) < (\tau(f) + \varepsilon)r^{\mu(f)}.$$

3. PROOFS OF THEOREMS 5 AND 6

3.1 Proof of Theorem 5

Suppose that $f(z)$ is a non-zero meromorphic solution of (3). If $f(z)$ has infinite order, the result holds yet. Then we assume that $f(z)$ has finite order without loss of generality.

We divide (3) by $f(z + c_l)$ to get

$$-A_{l0}(z) = \sum_{\substack{i=0 \\ i \neq l}}^n \sum_{j=0}^m A_{ij}(z) \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \frac{f(z + c_i)}{f(z + c_l)} + \sum_{j=1}^m A_{lj}(z) \frac{f^{(j)}(z + c_l)}{f(z + c_l)}, \quad (4)$$

consequently

$$\begin{aligned} |A_{l0}(z)| \leq & \sum_{\substack{i=0 \\ i \neq l}}^n \sum_{j=0}^m |A_{ij}(z)| \left| \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \right| \left| \frac{f(z + c_i)}{f(z + c_l)} \right| \\ & + \sum_{j=1}^m |A_{lj}(z)| \left| \frac{f^{(j)}(z + c_l)}{f(z + c_l)} \right|. \end{aligned} \quad (5)$$

It follows by Lemma 7 that for any given $\varepsilon (> 0)$, there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f(z + c_i)}{f(z + c_l)} \right| \leq \exp\{r^{\sigma(f)-1+\varepsilon}\}, \quad i \neq l. \quad (6)$$

It follows by Lemmas 8 and 9 that for the above ε , there exists a subset $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_2$, we have

$$\left| \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \right| \leq r^{j(\sigma(f(z+c_i))-1+\varepsilon)} = r^{j(\sigma(f)-1+\varepsilon)}, \quad (i, j) \neq (l, 0). \quad (7)$$

In the following, we divide the proof into four cases.

Case (i) We suppose that $\sigma < \mu(A_{l0})$.

By the definitions of $\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)$, we have for the above ε and sufficiently large r ,

$$M(r, A_{ij}) \leq \exp\{r^{\sigma(A_{ij})+\varepsilon}\} \leq \exp\{r^{\sigma+\varepsilon}\}, \quad (i, j) \neq (a, b), (l, 0). \quad (8)$$

By the definition of $\mu(A_{l0})$, we have for sufficiently small ε and sufficiently large r ,

$$M(r, A_{l0}) \geq \exp\{r^{\mu(A_{l0})-\varepsilon}\}. \quad (9)$$

By the definition of $\mu(A_{ab})$ and Lemma 11, there exists a subset $H_1 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H_1$, we have

$$M(r, A_{ab}) \leq \exp\{r^{\mu(A_{ab})+\varepsilon}\}. \quad (10)$$

Then for all z satisfying $|z| = r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$ and $|A_{l_0}(z)| = M(r, A_{l_0})$, we deduce from (5)-(10) that

$$\begin{aligned} & \exp\{r^{\mu(A_{l_0})-\varepsilon}\} \\ & \leq [\exp\{r^{\sigma+2\varepsilon}\} + \exp\{r^{\mu(A_{ab})+\varepsilon}\}] \exp\{r^{\sigma(f)-1+\varepsilon}\} r^{m(\sigma(f)-1+\varepsilon)}. \end{aligned} \quad (11)$$

Now, we may choose sufficiently small ε satisfying $0 < 3\varepsilon < \mu(A_{l_0}) - \sigma$ and deduce from (11) that for $r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$,

$$\exp\{r^{\mu(A_{l_0})-2\varepsilon}\} \leq \exp\{r^{\sigma(f)-1+\varepsilon}\},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - 3\varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Case (ii) We suppose that $\max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \alpha < \mu(A_{ab}) = \mu(A_{l_0})$ and $\tau_M(A_{l_0}) > \tau_M(A_{ab})$.

By the definitions of $\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)$, we have for the above ε and sufficiently large r ,

$$M(r, A_{ij}) \leq \exp\{r^{\sigma(A_{ij})+\varepsilon}\} \leq \exp\{r^{\alpha+\varepsilon}\}, \quad (i, j) \neq (a, b), (l, 0). \quad (12)$$

By the definition of $\tau_M(A_{l_0})$, we have for sufficiently small ε and sufficiently large r ,

$$M(r, A_{l_0}) \geq \exp\{(\tau_M(A_{l_0}) - \varepsilon)r^{\mu(A_{l_0})}\}. \quad (13)$$

By the definition of $\tau_M(A_{ab})$ and Lemma 12, there exists a subset $H_2 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H_2$, we have

$$M(r, A_{ab}) \leq \exp\{(\tau_M(A_{ab}) + \varepsilon)r^{\mu(A_{ab})}\} = \exp\{(\tau_M(A_{ab}) + \varepsilon)r^{\mu(A_{l_0})}\}. \quad (14)$$

Then for all z satisfying $|z| = r \in H_2 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$ and $|A_{l_0}(z)| = M(r, A_{l_0})$, we deduce from (5)-(7), (12)-(14) that

$$\begin{aligned} \exp\{(\tau_M(A_{l_0}) - \varepsilon)r^{\mu(A_{l_0})}\} & \leq [\exp\{r^{\alpha+2\varepsilon}\} + \exp\{(\tau_M(A_{ab}) + \varepsilon)r^{\mu(A_{l_0})}\}] \\ & \exp\{r^{\sigma(f)-1+\varepsilon}\} r^{m(\sigma(f)-1+\varepsilon)}. \end{aligned} \quad (15)$$

Now, we may choose sufficiently small ε satisfying $0 < 3\varepsilon < \min\{\mu(A_{l_0}) - \alpha, \tau_M(A_{l_0}) - \tau_M(A_{ab})\}$ and deduce from (15) that for $r \in H_2 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$,

$$\exp\{(\tau_M(A_{l_0}) - \tau_M(A_{ab}) - 3\varepsilon)r^{\mu(A_{l_0})}\} \leq \exp\{r^{\sigma(f)-1+\varepsilon}\},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - \varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Case (iii) We suppose that $\mu(A_{ab}) < \max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \mu(A_{l_0})$ and $\tau_M(A_{l_0}) > \tau_1 = \max\{\tau_M(A_{ij}) : \sigma(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0)\}$.

By the definitions of $\sigma(A_{ij})$ and $\tau_M(A_{ij})$, $(i, j) \neq (a, b), (l, 0)$, we have for sufficiently small ε and sufficiently large r ,

$$M(r, A_{ij}) \leq \begin{cases} \exp\{r^{\mu(A_{l_0})-\varepsilon}\}, & \text{if } \sigma(A_{ij}) < \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0); \\ \exp\{(\tau_1 + \varepsilon)r^{\mu(A_{l_0})}\}, & \text{if } \sigma(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0). \end{cases} \quad (16)$$

Then for all z satisfying $|z| = r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$ and $|A_{l_0}(z)| = M(r, A_{l_0})$, we deduce from (5)-(7), (10), (13) and (16) that

$$\begin{aligned} & \exp\{(\tau_M(A_{l_0}) - \varepsilon)r^{\mu(A_{l_0})}\} \\ & \leq [\exp\{r^{\mu(A_{l_0})-\frac{\varepsilon}{2}}\} + \exp\{r^{\mu(A_{ab})+\varepsilon}\} + \exp\{(\tau_1 + 2\varepsilon)r^{\mu(A_{l_0})}\}] \\ & \quad \exp\{r^{\sigma(f)-1+\varepsilon}\}r^{m(\sigma(f)-1+\varepsilon)}. \end{aligned} \quad (17)$$

Now, we may choose sufficiently small ε satisfying $0 < 4\varepsilon < \min\{\mu(A_{l_0}) - \mu(A_{ab}), \tau_M(A_{l_0}) - \tau_1\}$ and deduce from (17) that for $r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$,

$$\exp\{(\tau_M(A_{l_0}) - \tau_1 - 4\varepsilon)r^{\mu(A_{l_0})}\} \leq \exp\{r^{\sigma(f)-1+\varepsilon}\},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - \varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Case (iv) We suppose that $\max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \mu(A_{ab}) = \mu(A_{l_0})$ and $\tau_M(A_{l_0}) > \tau_2 = \max\{\tau_M(A_{ab}), \tau_M(A_{ij}) : \sigma(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0)\}$.

Then for all z satisfying $|z| = r \in H_2 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$ and $|A_{l_0}(z)| = M(r, A_{l_0})$, we deduce from (5)-(7), (13), (14) and (16) that

$$\exp\{(\tau_M(A_{l_0}) - \varepsilon)r^{\mu(A_{l_0})}\} \leq [\exp\{(\tau_2 + 2\varepsilon)r^{\mu(A_{l_0})}\} + \exp\{r^{\mu(A_{l_0})-\frac{\varepsilon}{2}}\}] \exp\{r^{\sigma(f)-1+\varepsilon}\}r^{m(\sigma(f)-1+\varepsilon)}. \quad (18)$$

Now, we may choose sufficiently small ε satisfying $0 < 4\varepsilon < \tau_M(A_{l_0}) - \tau_2$ and deduce from (18) that for $r \in H_2 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow \infty$,

$$\exp\{(\tau_M(A_{l_0}) - \tau_2 - 4\varepsilon)r^{\mu(A_{l_0})}\} \leq \exp\{r^{\sigma(f)-1+\varepsilon}\},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - \varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Therefore, the proof of Theorem 5 is complete.

3.2 Proof of Theorem 6

Suppose that $f(z)$ is a non-zero meromorphic solution of (3). If $f(z)$ has infinite order, the result holds yet. Then we assume that $f(z)$ has finite order without loss of generality.

It follows by (4) that

$$\begin{aligned}
 m(r, A_{l0}) \leq & \sum_{\substack{i=0 \\ i \neq l}}^n \sum_{j=0}^m m(r, A_{ij}) + \sum_{j=1}^m m(r, A_{lj}) + \sum_{\substack{i=0 \\ i \neq l}}^n m\left(r, \frac{f(z+c_i)}{f(z+c_l)}\right) \\
 & + \sum_{j=1}^m \sum_{i=0}^n m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) + O(1).
 \end{aligned} \tag{19}$$

It follows by Lemmas 9 and 10 and the lemma of logarithmic derivative that for any given $\varepsilon(> 0)$, we have

$$\begin{aligned}
 m\left(r, \frac{f(z+c_i)}{f(z+c_l)}\right) &= O(r^{\sigma(f)-1+\varepsilon}), \quad i \neq l, \\
 m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) &= O(\log r), \quad j \neq 0.
 \end{aligned} \tag{20}$$

In the following, we divide the proof into four cases.

Case (i) We suppose that $\sigma < \mu(A_{l0})$.

By the definitions of $\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)$, we have for the above ε and sufficiently large r ,

$$m(r, A_{ij}) \leq T(r, A_{ij}) \leq r^{\sigma(A_{ij})+\varepsilon}, \quad (i, j) \neq (a, b), (l, 0). \tag{21}$$

By the definition of $\mu(A_{l0})$ and the assumption $\delta(\infty, A_{l0}) = \delta > 0$, we have for sufficiently small ε and sufficiently large r ,

$$m(r, A_{l0}) \geq \frac{\delta}{2} T(r, A_{l0}) \geq \frac{\delta}{2} r^{\mu(A_{l0})-\frac{\varepsilon}{2}} \geq r^{\mu(A_{l0})-\varepsilon}. \tag{22}$$

By the definition of $\mu(A_{ab})$ and Lemma 13, there exists a subset $H_1 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H_1$, we have

$$m(r, A_{ab}) \leq T(r, A_{ab}) \leq r^{\mu(A_{ab})+\varepsilon}. \tag{23}$$

Then it follows by (19)-(23) that for sufficiently large $r \in H_1$, we have

$$\begin{aligned}
 r^{\mu(A_{l0})-\varepsilon} &\leq r^{\mu(A_{ab})+\varepsilon} + O(r^{\sigma+\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r) \\
 &\leq r^{\sigma+2\varepsilon} + r^{\sigma(f)-1+2\varepsilon}.
 \end{aligned} \tag{24}$$

Now, we may choose sufficiently small ε satisfying $0 < 3\varepsilon < \mu(A_{l0}) - \sigma$ and deduce from (24) that for sufficiently large $r \in H_1$,

$$r^{\mu(A_{l0})-2\varepsilon} \leq r^{\sigma(f)-1+2\varepsilon},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - 4\varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Case (ii) We suppose that $\max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \alpha < \mu(A_{ab}) = \mu(A_{l_0})$ and $\delta_{\mathcal{T}}(A_{l_0}) > \mathcal{T}(A_{ab})$.

By the definition of $\mathcal{T}(A_{l_0})$ and the assumption $\delta(\infty, A_{l_0}) = \delta > 0$, we have for sufficiently small ε and sufficiently large r ,

$$\begin{aligned} m(r, A_{l_0}) &\geq (\delta - \varepsilon)T(r, A_{l_0}) \geq (\delta - \varepsilon)(\mathcal{T}(A_{l_0}) - \varepsilon)r^{\mu(A_{l_0})} \\ &\geq (\delta_{\mathcal{T}}(A_{l_0}) - (\mathcal{T}(A_{l_0}) + 1)\varepsilon)r^{\mu(A_{l_0})}. \end{aligned} \quad (25)$$

By the definition of $\mathcal{T}(A_{ab})$ and Lemma 14, there exists a subset $H_2 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in H_2$, we have

$$m(r, A_{ab}) \leq T(r, A_{ab}) \leq (\mathcal{T}(A_{ab}) + \varepsilon)r^{\mu(A_{ab})} = (\mathcal{T}(A_{ab}) + \varepsilon)r^{\mu(A_{l_0})}. \quad (26)$$

Then it follows by (19)-(21), (25) and (26) that for sufficiently large $r \in H_2$, we have

$$\begin{aligned} &(\delta_{\mathcal{T}}(A_{l_0}) - (\mathcal{T}(A_{l_0}) + 1)\varepsilon)r^{\mu(A_{l_0})} \\ &\leq (\mathcal{T}(A_{ab}) + \varepsilon)r^{\mu(A_{l_0})} + O(r^{\alpha+\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r) \\ &\leq (\mathcal{T}(A_{ab}) + \varepsilon)r^{\mu(A_{l_0})} + r^{\alpha+2\varepsilon} + r^{\sigma(f)-1+2\varepsilon}. \end{aligned} \quad (27)$$

Now, we may choose sufficiently small ε satisfying $0 < (\mathcal{T}(A_{l_0}) + 3)\varepsilon < \min\{\mu(A_{l_0}) - \alpha, \delta_{\mathcal{T}}(A_{l_0}) - \mathcal{T}(A_{ab})\}$ and deduce from (27) that for sufficiently large $r \in H_2$,

$$(\delta_{\mathcal{T}}(A_{l_0}) - \mathcal{T}(A_{ab}) - (\mathcal{T}(A_{l_0}) + 3)\varepsilon)r^{\mu(A_{l_0})} \leq r^{\sigma(f)-1+2\varepsilon},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - 2\varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Case (iii) We suppose that $\mu(A_{ab}) < \max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \mu(A_{l_0})$ and $\delta_{\mathcal{T}}(A_{l_0}) > \tau_1 = \max\{\tau(A_{ij}) : \sigma(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0)\}$.

By the definitions of $\sigma(A_{ij})$ and $\tau(A_{ij})$, $(i, j) \neq (a, b), (l, 0)$, we have for sufficiently small ε and sufficiently large r ,

$$T(r, A_{ij}) \leq \begin{cases} r^{\mu(A_{l_0})-\varepsilon}, & \text{if } \sigma(A_{ij}) < \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0); \\ (\tau_1 + \varepsilon)r^{\mu(A_{l_0})}, & \text{if } \sigma(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0). \end{cases} \quad (28)$$

Then it follows by (19), (20), (23), (25) and (28) that for sufficiently large $r \in H_1$, we have

$$\begin{aligned} &(\delta_{\mathcal{T}}(A_{l_0}) - (\mathcal{T}(A_{l_0}) + 1)\varepsilon)r^{\mu(A_{l_0})} \\ &\leq r^{\mu(A_{ab})+\varepsilon} + O((\tau_1 + \varepsilon)r^{\mu(A_{l_0})}) + O(r^{\mu(A_{l_0})-\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r) \\ &\leq r^{\mu(A_{ab})+\varepsilon} + (\tau_1 + 2\varepsilon)r^{\mu(A_{l_0})} + r^{\sigma(f)-1+2\varepsilon}. \end{aligned} \quad (29)$$

Now, we may choose sufficiently small ε satisfying $0 < (\underline{\tau}(A_{l_0})+3)\varepsilon < \min\{\mu(A_{l_0})-\mu(A_{ab}), \delta\underline{\tau}(A_{l_0}) - \tau_1\}$ and deduce from (29) that for sufficiently large $r \in H_1$,

$$(\delta\underline{\tau}(A_{l_0}) - \tau_1 - (\underline{\tau}(A_{l_0}) + 3)\varepsilon)r^{\mu(A_{l_0})} \leq r^{\sigma(f)-1+2\varepsilon},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - 2\varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Case (iv) We suppose that $\max\{\sigma(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \mu(A_{ab}) = \mu(A_{l_0})$ and $\delta\underline{\tau}(A_{l_0}) > \tau_2 = \max\{\underline{\tau}(A_{ab}), \tau(A_{ij}) : \sigma(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0)\}$.

Then it follows by (19), (20), (25), (26) and (28) that for sufficiently large $r \in H_2$, we have

$$\begin{aligned} & (\delta\underline{\tau}(A_{l_0}) - (\underline{\tau}(A_{l_0}) + 1)\varepsilon)r^{\mu(A_{l_0})} \\ & \leq O((\tau_2 + \varepsilon)r^{\mu(A_{l_0})}) + O(r^{\mu(A_{l_0})-\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r) \quad (30) \\ & \leq (\tau_2 + 2\varepsilon)r^{\mu(A_{l_0})} + r^{\sigma(f)-1+2\varepsilon}. \end{aligned}$$

Now, we may choose sufficiently small ε satisfying $0 < (\underline{\tau}(A_{l_0})+3)\varepsilon < \delta\underline{\tau}(A_{l_0}) - \tau_2$ and deduce from (30) that for sufficiently large $r \in H_2$,

$$(\delta\underline{\tau}(A_{l_0}) - \tau_2 - (\underline{\tau}(A_{l_0}) + 3)\varepsilon)r^{\mu(A_{l_0})} \leq r^{\sigma(f)-1+2\varepsilon},$$

that is, $\sigma(f) \geq \mu(A_{l_0}) + 1 - 2\varepsilon$. Since ε is arbitrary, we have $\sigma(f) \geq \mu(A_{l_0}) + 1$.

Therefore, the proof of Theorem 6 is complete.

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