Acta Universitatis Apulensis No. 55/2018
ISSN: 1582-5329 pp. 63-88
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2018.55.06

EXISTENCE RESULTS TO ELLIPTIC PROBLEMS WITH DIRICHLET BOUNDARY CONDITION INVOLVING A P-HARMONIC OPERATOR

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Abstract. In this note, we prove the existence of a non-trivial weak solution for a nonlinear equation involving a p-harmonic operator through a local minimization theorem, under Dirichlet boundary value conditions. In the case of terms with a sublinear growth near the origin, we ensure the existence of solutions for small positive values of the parameter. Moreover, the corresponding solutions have smaller energies as the parameter goes to zero.

2010 *Mathematics Subject Classification*: 35J35, 35J60.

Keywords: Elliptic problem, Dirichlet boundary condition, Non-trivial solution, Local minimization theorem, Lipschitz function.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded smooth open domain and let $p > 1$. In this work, we shall study the following Dirichlet problem

$$\begin{cases} \Delta\big(a(x,\ \Delta u)\big) + div\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u = \lambda f(x,u) + h(u), & \text{in } \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$, n denotes the outward unit normal to $\partial\Omega$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(f$_1$) $\qquad |f(x,t)| \leq a_1 + a_2|t|^{q-1}, \qquad \forall\ (x,t) \in \Omega \times \mathbb{R},$

for some non-negative constants a_1, a_2, where $q \in (1, p^*)$ and

$$p^* := \begin{cases} \dfrac{pN}{N-2p}, & \text{if } p < \frac{N}{2}, \\ +\infty, & \text{if } p \geq \frac{N}{2}, \end{cases}$$

$a : \Omega \times \mathbb{R} \to \mathbb{R}$, is a function such that there is $A : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, $A(x, \xi)$ is continuous in $\bar{\Omega} \times \mathbb{R}$, with continuous derivative with respect to ξ, $a = D_\xi A = A'$, with a and A having the following properties:

(a) a satisfies the growth condition: there is a constant $c_1 > 0$ such that

$$|a(x, \xi)| \leq c_1(1 + |\xi|^{p-1}), \quad \forall\, x \in \Omega,\, \xi \in \mathbb{R};$$

(b) a is monotone, i.e., $(a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2) \geq 0$ holds for a.e. $\forall\, x \in \Omega,\ \xi \in \mathbb{R}$, with equality if and only if $\xi_1 = \xi_2$,

(c) A is homogeneous of degree p, i.e., $\forall\, x \in \Omega$, $t \in [0, +\infty)$, $\xi \in \mathbb{R}$,

$$A(x, t\xi) = t^p A(x, \xi);$$

(d) A satisfies the ellipticity condition: there is a constant $c_2 > 0$ such that

$$A(x, \xi) \geq c_2|\xi|^p, \quad \forall\, x \in \Omega,\, \xi \in \mathbb{R},$$

and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function of $(p-1)$-order with the Lipschitz constant $0 < L < 1$, that is

$$|h(x) - h(y)| \leq L|x - y|^{p-1}$$

for every $x, y \in \mathbb{R}$, and $h(0) = 0$.

More precisely, employing a critical point result for differentiable functionals, we obtain some sufficient conditions to guarantee that, the problem (1) has at least one weak solution (see Theorem 2).

The operator $-\mathrm{div}(a(x, \nabla u))$, a special case of which is p-Laplacian, arises, for example, from the expression of the p-Laplacian in curvilinear coordinates. Recently, much progress has been made on the existence of solutions to the elliptic Dirichlet problems involving a general operator in divergence form, for instance, see [3, 9, 11, 17, 18, 28]. For example, De Nápoli and Mariani in [9] studied the existence of solutions to equations of p-Laplacian type. They proved the existence of at least one solution, and under further assumptions, the existence of infinitely many solutions. In order to apply mountain pass results, they introduced a notion of uniformly convex functional that generalizes the notion of uniformly convex norm. Duc and Vu in [11] studied the non-uniform case, and extended the result of [9] under the key hypothesis that the map fulfills a suitable growth condition. The authors in [28] established the existence and multiplicity of weak solutions of a problem involving a uniformly convex elliptic operator in divergence form. They discussed the existence of one nontrivial solution by the mountain pass lemma, when the nonlinearity has a $(p-1)$-superlinear growth at infinity, and two nontrivial solutions by minimization and mountain pass when the nonlinear term has a $(p-1)$-sublinear growth at infinity. Molica Bisci and Repovš in [17], exploiting variational methods, investigated the existence of three weak solutions for a class of elliptic equations involving a general operator in divergence form and with Dirichlet boundary condition. They are analyzed several special cases. In conclusion, for completeness, they presented a concrete example of an application by finding the existence of three nontrivial weak solutions for an uniformly elliptic second-order problem on a bounded Euclidean domain, while in [18] they studied a nonlinear parametric Neumann problem driven by a nonhomogeneous quasi-linear elliptic differential operator $\operatorname{div}(a(x,\nabla u))$, in which the reaction term is a nonlinearity function f which exhibits $(p-1)$-subcritical growth. By using variational methods, they proved a multiplicity result on the existence of weak solutions for such problems. Also, in [3] existence results and energy estimates of weak solutions to the following equation involving a p-harmonic operator

$$\begin{cases} \Delta\big(a(x,\Delta u)\big) = \lambda f(x,u), & \text{in } \Omega, \\ u=0, \quad \dfrac{\partial u}{\partial n}=0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where the function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfied the condition (f_1), by using Ricceri's variational principle [22], in order to prove that the problem (2) admits at least one non-trivial weak solution for a open interval involving the parameter λ, were established. In [8], Colasuonno et al. studied different and very general classes of elliptic operators in divergence form looking at the existence of multiple weak solutions. Their results represent a nice improvement, in several directions, of the results obtained by Kristály et al. in [15], in which the existence of three weak solutions to the following problem involving elliptic operators in divergence form

$$\begin{cases} -div\big(a(x,\nabla u)\big) = f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega\subseteq\mathbb{R}^N$ is a bounded domain, $N\geq 2$, while the nonlinearities $a:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ fulfill certain structural conditions, while the nonlinearity has a $(p-1)-$sublinear growth at infinity, was investigated.

The existence of multiple solutions for this type of nonlinear differential equations was studied in [10]. Many of these results are based on some three critical points theorems of Ricceri and Bonanno established in [5, 23]. In [21], Ricceri developed one of his results, [23, Theorem 1] by means of an abstract result, [24, Theorem 4].

Yang et al. in [26] studied the following singular p-Laplacian type equation

$$\begin{cases} -div\big(|x|^{-\beta}a(x,\nabla u)\big) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 \le \beta < N - p$, Ω is a smooth bounded domain in $\mathbb{R}^N$ containing the origin, f satisfies some growth and singularity conditions. Under some mild assumptions on the function a, applying the three points theorem due to Bonanno [5], authors established the existence of at least three distinct weak solutions to the above problem if f admits some hypotheses on the behavior at $u = 0$ or perturbation property. Using the minimax methods in critical point theory, Suo and Tang in [25] studied the multiplicity of solutions for degenerate semi-linear elliptic equation of the form

$$\begin{cases} -div\big(a(x)\nabla u\big) = \lambda u + f(x,u) + h(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \ge 2)$ with smooth boundary $\partial\Omega$, the function a is a nonnegative measurable weight on $\Omega, \lambda \in \mathbb{R}$, $h \in L^2(\Omega)$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the following assumption:

(e) There exist constants $c > 0$ and $q \in (1, 2)$ such that

$$|f(x,t)| \le c(1 + |t|^{q-1}),$$

for all $(x,t) \in \Omega \times \mathbb{R}$.

In [7] the authors in the framework of variable exponent spaces spaces, using variational methods discussed the existence of solutions for the nonlinear elliptic problem involving a $p(.)$-Laplace-type operator

$$\begin{cases} -div\big(a(x,\nabla u)\big) + |\nabla u(x)|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ u(x) = \text{constant}, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \ge 2)$, the functions $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are fulfilling appropriate conditions.

In the present paper, employing a smooth version of [6, Theorem 2.1] which is a more precise version of Ricceri's Variational Principle [22], we attempt the existence of at least one nontrivial weak solution for the problem (1) for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero, if $f(x, 0) = 0$ for a.e. $x \in \Omega$. In the case of terms with a sublinear growth near the origin, we ensure the existence of solutions for small positive values of the parameter. Moreover, the corresponding solutions have smaller energies as the parameter goes to zero, see Theorem 2. We list some consequences of the results. We illustrate the results by presenting convenient examples. Finally we present some consequences of the results, examples and a detailed discussion on systems with $p = 2$ and particular cases of the functions a and f are given.

The outline of the paper is organized as follows: In Section 2 we shall recall our main tool (Theorem 1) and some properties of variable exponent spaces and basic notations which we need in the proofs. Whereas, in Section 3 we formulate the main result and prove it, in order to discuss the existence of one weak solution for the problem (1). We also list some consequences of the main results, and present some examples to illustrate the results. Finally, in Section 4 we give an application of the main result for particular case of the function a and nonlinear term f in the case $p = 2$.

2. Preliminaries

Our main tool is a smooth version of Theorem 2.1 of [6] which is a more precise version of Ricceri's Variational Principle [22] that we recall here.

Theorem 1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for any $r > \inf_X \Phi$ let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}\big((-\infty,r)\big)} \frac{\sup_{v\in\Phi^{-1}\big((-\infty,r)\big)} \Psi(v) - \Psi(u)}{r - \Phi(u)}. \tag{4}$$

Then, for any $r > \inf_X \Phi$ and any $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional I_λ to $\Phi^{-1}\big((-\infty, r)\big)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

We refer the interested reader to the papers [4, 12, 13, 16] in which Theorem 1 has been successfully employed to the existence of at least one non-trivial solution for boundary value problems.

Now, let us denote by X the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, endowed with the norm

$$\|u\| := \left(\int_\Omega |\Delta u(x)|^p \, dx + \int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega |u(x)|^p dx \right)^{1/p}.$$

We recall that (see [27, page 1026]) if $p > N/2$, the embedding $X \hookrightarrow C^0(\bar{\Omega})$ is compact, and if $p \leq N/2$, the embedding $X \hookrightarrow L^q(\Omega)$ for all $q \in [1, p^*)$ is compact.

Hence, for the case where $p > N/2$, there exists $k > 0$ such that

$$\|u\|_\infty \leq k\|u\|, \qquad \forall\ u \in X,$$

and for the case where $p \leq N/2$, there exists $S_q > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq S_q\|u\|, \qquad \forall\ u \in X.$$

We say that a function u is a *weak solution* of problem (1), if $u \in X$ and satisfies

$$\int_\Omega \left[a(x,\ \Delta u)\Delta v + |\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv - \lambda\ f(x,u)v - h(u)v\right]\, dx = 0,$$

for every $v \in X$.

3. Main results

In this section, we state and prove the main results of this paper.

Theorem 2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition* (f_1) *holds and*

$$L < \frac{pc_3}{S_p^p}, \tag{5}$$

where $c_3 = \min\{\frac{1}{p}, c_2\}$. In addition, if $f(x,0) = 0$ for a.e. $x \in \Omega$, assume also that

(f_2) *there are a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive Lebesgue measure such that*

$$\limsup_{t \to 0^+} \frac{ess \inf_{x \in B} F(x,t)}{t^p} = +\infty,$$

and

$$\liminf_{t \to 0^+} \frac{ess \inf_{x \in D} F(x,t)}{t^p} > -\infty,$$

where

$$F(x,t) := \int_0^t f(x,\xi)\, d\xi, \ \ \forall\ (x,t) \in \Omega \times \mathbb{R}.$$

Further, assume that a and A are continuous functions and satisfy conditions (a)-(d).

Then, there exists $\lambda^\star > 0$, such that, for any $\lambda \in (0, \lambda^\star)$ problem (1) *admits at* *least one non-trivial weak solution* $u_\lambda \in X$. *Also,* $\lambda^\star = +\infty$, *provided* $q \in (1, p)$. *Moreover,*

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0,$$

and the function

$$\lambda \mapsto \int_\Omega \left(A(x,\ \Delta u_\lambda) + \frac{1}{p}|\nabla u_\lambda|^p + \frac{1}{p}|u_\lambda|^p \right) dx - \lambda \int_\Omega F(x, u_\lambda)\, dx - \int_\Omega H(u_\lambda)\, dx$$

is negative and strictly decreasing in $(0, \lambda^\star)$.

Proof. Our aim is to apply Theorem 1 to problem (1). To this end, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega A(x,\ \Delta u(x))\, dx + \frac{1}{p}\int_\Omega\ |\nabla u(x)|^p\, dx + \frac{1}{p}\int_\Omega\ |u(x)|^p\, dx - \int_\Omega\ H(u(x))\, dx,$$

where $H(t) := \int_0^t\ h(s)\, ds$, for every $t \in \mathbb{R}$, and

$$\Psi(u) := \int_\Omega F(x, u(x))\, dx,$$

for every $u \in X$, and set $I_\lambda := \Phi - \lambda\Psi$. Clearly, Φ and Ψ are well defined and continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \int_\Omega\ \left[a(x,\ \Delta u)\Delta v + |\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv - h(u)v\right]\, dx,$$

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)\, dx,$$

for every $v \in X$ (see [10, Lemma 2.2]). In the same way as in the proof [7, Lemma 5] we observe Φ is sequentially weakly lower semicontinuous. Moreover, Ψ is sequentially weakly (upper) continuous. By condition (d), for all $u \in X$, we have

$$\begin{aligned}
\Phi(u) &\geq \int_\Omega\ \left[c_2|\Delta u(x)|^p + \frac{1}{p}|\nabla u(x)|^p + \frac{1}{p}|u(x)|^p - |H(u(x))|\right]\, dx \\
&\geq c_3\|u\|^p - \int_\Omega \Big(\int_0^{u(x)}\ |h(s)|\, ds\Big)\, dx \\
&\geq c_3\|u\|^p - \int_\Omega \Big(L\int_0^{u(x)}\ |s|^{p-1}\, ds\Big)\, dx \\
&\geq c_3\|u\|^p - \frac{L}{p}\int_\Omega\ |u(x)|^p\, dx \\
&\geq (c_3 - \frac{LS_p^p}{p})\|u\|^p. \qquad (6)
\end{aligned}$$

So, from (5) and (6), we have that Φ is coercive in X and $\inf_{u\in X}\Phi(u)=0$.
Now, let $r>0$. It is easy to see that $\varphi(r)\geq 0$ for any $r>0$, where φ is defined by (4). Then, by Theorem 1,

$$\begin{aligned}&\text{for any } r>0 \text{ and any } \lambda\in\Big(0,1/\varphi(r)\Big) \text{ the restriction}\\ &\text{of } I_\lambda \text{ to } \Phi^{-1}\big((-\infty,r)\big) \text{ admits a global minimum } u_{\lambda,r},\end{aligned}\tag{7}$$

which is a critical point (namely a local minimum) of I_λ in X. Let $\lambda^\star$ be defined as follows

$$\lambda^\star:=\sup_{r>0}\frac{1}{\varphi(r)}.$$

Note that $\lambda^\star>0$, since $\varphi(r)\geq 0$ for any $r>0$. Now, fix $\bar\lambda\in(0,\lambda^\star)$. It is easy to see that

$$\text{there exists } \bar r_{\bar\lambda}>0 \text{ such that } \bar\lambda\leq 1/\varphi(\bar r_{\bar\lambda}).\tag{8}$$

Then, by (7) applied with $r=\bar r_{\bar\lambda}$, we have that for any λ such that

$$0<\lambda<\bar\lambda\leq 1/\varphi(\bar r_{\bar\lambda}),$$

the function $u_\lambda:=u_{\lambda,\bar r_{\bar\lambda}}$ is a global minimum of the functional I_λ restricted to $\Phi^{-1}\big((-\infty,\bar r_{\bar\lambda})\big)$, i.e.,

$$I_\lambda(u_\lambda)\leq I_\lambda(u) \text{ for any } u\in X \text{ such that } \Phi(u)<\bar r_{\bar\lambda}\tag{9}$$

and

$$\Phi(u_\lambda)<\bar r_{\bar\lambda},\tag{10}$$

and also u_λ is a critical point of I_λ in X and so it is a weak solution of problem (1).
Now, we show that $\lambda^\star=+\infty$, provided $q\in(1,p)$. To this end, by (f_1), one has

$$|F(x,t)|\leq a_1|t|+\frac{a_2}{q}|t|^q,\tag{11}$$

for any $(x,t)\in\Omega\times\mathbb{R}$. Also, by (6), for any $u\in X$ such that $\Phi(u)<r$, with $r>0$, we have

$$\|u\|^p<\frac{r}{c_4},$$

where $c_4:=c_3-\frac{LS_p^p}{p}>0$. So, we observe that

$$\Phi^{-1}(]-\infty,r[)=\{u\in X\ |\ \Phi(u)<r\}\subseteq\{u\in X\ |\ \|u\|^p<\frac{r}{c_4}\}.\tag{12}$$

Now, we discuss two cases.
Case 1: If $p < N/2$, from (11) and (12), for any $u \in X$ such that $\Phi(u) < r$, we obtain

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x,u(x))\,dx \\ &\le a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_{L^q(\Omega)} \\ &\le a_1 S_1\|u\| + \frac{a_2 S_q}{q}\|u\|^q \\ &< a_1 S_1\Big(\frac{r}{c_4}\Big)^{1/p} + \frac{a_2 S_q^q}{q}\Big(\frac{r}{c_4}\Big)^{q/p},\end{aligned}$$

so that

$$\sup_{u\in\Phi^{-1}\big((-\infty,r)\big)} \Psi(u) \le \frac{a_1 S_1}{c_4^{1/p}} r^{1/p} + \frac{a_2 S_q^q}{q c_4^{q/p}} r^{q/p}$$

for any $r > 0$. Now, by definition of φ, for any $r > 0$ we have

$$\varphi(r) \le \frac{\sup_{u\in\Phi^{-1}\big((-\infty,r)\big)} \Psi(u)}{r} \le \frac{a_1 S_1}{c_4^{1/p}} r^{1/p-1} + \frac{a_2 S_q^q}{q c_4^{q/p}} r^{q/p-1},$$

since $\Phi(0) = \Psi(0) = 0$. Namely,

$$\frac{1}{\varphi(r)} \ge \frac{q c_4^{q/p}}{a_1 S_1 q c_4^{(q-1)/p} r^{(1-p)/p} + a_2 S_q^q r^{(q-p)/p}},$$

so that

$$\lambda^\star = \sup_{r>0}\frac{1}{\varphi(r)} \ge \sup_{r>0}\frac{q c_4^{q/p}}{a_1 S_1 q c_4^{(q-1)/p} r^{(1-p)/p} + a_2 S_q^q r^{(q-p)/p}} = +\infty,$$

provided $q \in (1,p)$. Hence, $\lambda^\star = +\infty$ if $q \in (1,p)$.

Case 2: If $p \ge N/2$, from (11), for any $u \in X$ such that $\Phi(u) < r$, we obtain

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x,u(x))\,dx \\ &\le meas(\Omega)\left(a_1\|u\|_\infty + \frac{a_2}{q}\|u\|^q_\infty\right) \\ &\le meas(\Omega)\left(a_1 k\|u\| + \frac{a_2 k^q}{q}\|u\|^q\right) \\ &< meas(\Omega)\left(a_1 k\Big(\frac{r}{c_4}\Big)^{1/p} + \frac{a_2 k^q}{q}\Big(\frac{r}{c_4}\Big)^{q/p}\right),\end{aligned}$$

so that

$$\sup_{u\in\Phi^{-1}\big((-\infty,r)\big)} \Psi(u) \leq meas(\Omega)\left(\frac{a_1 k}{c_4^{1/p}} r^{1/p} + \frac{a_2 k^q}{q c_4^{q/p}} r^{q/p}\right)$$

for any $r > 0$. Now, by definition of φ, for any $r > 0$ we have

$$\varphi(r) \leq \frac{\sup_{u\in\Phi^{-1}\big((-\infty,r)\big)} \Psi(u)}{r} \leq meas(\Omega)\left(\frac{a_1 k}{c_4^{1/p}} r^{1/p-1} + \frac{a_2 k^q}{q c_4^{q/p}} r^{q/p-1}\right).$$

Namely,

$$\lambda^\star = \sup_{r>0} \frac{1}{\varphi(r)} \geq \sup_{r>0} \frac{q c_4^{q/p}}{meas(\Omega)\left(a_1 k q c_4^{(q-1)/p} r^{(1-p)/p} + a_2 k^q r^{(q-p)/p}\right)} = +\infty,$$

provided $q \in (1, p)$. Hence, we obtain again $\lambda^\star = +\infty$ if $q \in (1, p)$. Now, we have to show that for any $\lambda \in (0, \lambda^\star)$ the solution u_λ is not trivial. If $f(\cdot, 0) \neq 0$, we have $u_\lambda \not\equiv 0$ in X, since the trivial function does not solve problem (1). Let us consider the case when $f(\cdot, 0) = 0$ and let us fix $\bar{\lambda} \in (0, \lambda^\star)$ and $\lambda \in (0, \bar{\lambda})$. Finally, let u_λ be as in (9) and (10). We will prove that $u_\lambda \not\equiv 0$ in X. To this end, let us show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{13}$$

For this, first note that, by (c), we have

$$A(x, t\xi) = t^p A(x, \xi),$$

for all $x \in \Omega$, $t \in [0, +\infty)$ and $\xi \in \mathbb{R}$. Thus, for all $t \in [0, 1]$ and $u \in X\backslash\{0\}$, we have

$$\begin{aligned} 0 < \Phi(t^{\frac{1}{p}} u) &\leq |\Phi(t^{\frac{1}{p}} u)| \\ &\leq \int_\Omega |A(x, \Delta(t^{\frac{1}{p}} u(x)))|\, dx + \frac{1}{p} t \int_\Omega |\nabla u(x)|^p\, dx \\ &+ \frac{1}{p} t \int_\Omega |u(x)|^p\, dx + \int_\Omega \Big(\int_0^{t^{\frac{1}{p}} u(x)} |h(s)|\, ds\Big)\, dx \\ &\leq t\Big(\int_\Omega \left[A(x, \Delta u(x)) + \frac{1}{p}|\nabla u(x)|^p + \frac{1}{p}(1 + L)\ |u(x)|^p\right] dx\Big) \end{aligned} \tag{14}$$

Due to (f_2), we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and a constant $\kappa \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{ess \inf_{x\in B} F(x, \xi_n)}{\xi_n^p} = +\infty,$$

and

$$ess \inf_{x\in D} F(x,\xi_n) \geq \kappa \xi_n^p,$$

for n sufficiently large. Now, fix a set $C \subset B$ of positive measure and a function $v \in X$ such that:

(i) $v(x) \in [0,1]$, for every $x \in \bar{\Omega}$;

(ii) $v(x) = 1$, for every $x \in C$;

(iii) $v(x) = 0$, for every $x \in \Omega \setminus D$.

Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta meas(C) + \kappa \displaystyle\int_{D\setminus C} v(x)\,dx}{\int_\Omega A(x,\Delta v(x))\,dx + \frac{1}{p}\int_\Omega \ |\nabla v(x)|^p\,dx + \frac{1}{p}(1+L)\int_\Omega |v(x)|^p\,dx}.$$

Then, there is $\nu \in \mathbb{N}$ such that $\xi_n < 1$ and

$$ess \inf_{x\in B} F(x,\xi_n) \geq \eta \xi_n^p,$$

for every $n > \nu$. Finally, let $w_n := \xi_n^{\frac{1}{p}} v$ for every $n \in \mathbb{N}$. It is easy to see that $w_n \in X$ for any $n \in \mathbb{N}$. Now, for every $n > \nu$, bearing in mind the properties of the function v ($0 \leq w_n(x) < \sigma$ for n sufficiently large and some $\sigma > 0$) and considering in fact that $\xi_n^{\frac{1}{p}} \in [0,1]$ for every $n \in \mathbb{N}$ from (14), one has

$$\begin{aligned}
\frac{\Psi(w_n)}{\Phi(w_n)} &= \frac{\displaystyle\int_C F(x,\xi_n^{\frac{1}{p}})\,dx + \int_{D\setminus C} F(x,\xi_n^{\frac{1}{p}} v(x))\,dx}{\Phi(\xi_n^{\frac{1}{p}} v)} \\
&\geq \frac{\displaystyle\int_C \eta\xi_n dx + \int_{D\setminus C} \kappa\xi_n v^p(x)\,dx}{\Phi(\xi_n^{\frac{1}{p}} v)} \\
&\geq \frac{\eta\xi_n meas(C) + \kappa\xi_n \displaystyle\int_{D\setminus C} v^p(x)\,dx}{\xi_n\left(\int_\Omega \ A(x,\Delta v(x))\,dx + \frac{1}{p}\int_\Omega \ |\nabla v(x)|^p\,dx + \frac{1}{p}(1+L)\int_\Omega \ |v(x)|^p\,dx\right)} \\
&\geq \frac{\eta meas(C) + \kappa \displaystyle\int_{D\setminus C} (v(x))^p\,dx}{\int_\Omega \ A(x,\Delta v(x))\,dx + \frac{1}{p}\int_\Omega \ |\nabla v(x)|^p\,dx + \frac{1}{p}(1+L)\int_\Omega \ |v(x)|^p\,dx} > M.
\end{aligned}$$

Since M could be arbitrarily large, it follows that

$$\lim_{n\to\infty} \frac{\Psi(w_n)}{\Phi(w_n)} = +\infty,$$

from which (13) clearly follows. Hence, there exists a sequence $\{w_n\} \subset X$ strongly converging to zero, such that, for every n sufficiently large, $w_n \in \Phi^{-1}\big((-\infty, \bar{r}_{\bar{\lambda}})\big)$, and

$$I_\lambda(w_n) := \Phi(w_n) - \lambda\Psi(w_n) < 0. \tag{15}$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}\big((-\infty, \bar{r}_{\bar{\lambda}})\big)$ (see (9)), by (15) we conclude that

$$I_\lambda(u_\lambda) \leq I_\lambda(w_n) < 0 = I_\lambda(0), \tag{16}$$

so that $u_\lambda \not\equiv 0$ in X. Thus, u_λ is a nontrivial weak solution of problem (1). Moreover, from (16) we can easily see that the map

$$(0, \lambda^\star) \ni \lambda \mapsto I_\lambda(u_\lambda) \text{ is negative.} \tag{17}$$

Now, we claim that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Indeed, let again $\bar{\lambda} \in (0, \lambda^\star)$ and $\lambda \in (0, \bar{\lambda})$. Bearing in mind (6) and the fact that $\Phi(u_\lambda) < \bar{r}_{\bar{\lambda}}$ for any $\lambda \in (0, \bar{\lambda})$ (see (10)), one has that

$$c_4\|u_\lambda\|^p \leq \Phi(u_\lambda) < \bar{r}_{\bar{\lambda}},$$

that is,

$$\|u_\lambda\|^p < \frac{\bar{r}_{\bar{\lambda}}}{c_4}.$$

Again, we consider two cases.

Case 1: If $p < N/2$, we have

$$\begin{aligned} \left|\int_\Omega f(x, u_\lambda(x))u_\lambda(x)\,dx\right| &\leq a_1\|u_\lambda\|_{L^1(\Omega)} + a_2\|u_\lambda\|^q_{L^q(\Omega)} \\ &\leq a_1S_1\|u_\lambda\| + a_2S_q^q\|u_\lambda\|^q \\ &< a_1S_1\Big(\frac{\bar{r}_{\bar{\lambda}}}{c_4}\Big)^{1/p} + a_2S_q^q\Big(\frac{\bar{r}_{\bar{\lambda}}}{c_4}\Big)^{q/p} =: M_{\bar{r}_{\bar{\lambda}}}, \end{aligned} \tag{18}$$

for every $\lambda \in (0, \bar{\lambda})$.

Case 2: If $p \geq N/2$, we have

$$\begin{aligned} \left|\int_\Omega f(x,u_\lambda(x))u_\lambda(x)\,dx\right| &\leq meas(\Omega)\,(a_1\|u_\lambda\|_\infty + a_2\|u_\lambda\|_\infty^q) \\ &\leq meas(\Omega)\,(a_1k\|u_\lambda\| + a_2k^q\|u_\lambda\|^q) \qquad (19)\\ &< meas(\Omega)\left(a_1k\Big(\frac{\bar r_{\bar\lambda}}{c_4}\Big)^{1/p} + a_2k^q\Big(\frac{\bar r_{\bar\lambda}}{c_4}\Big)^{q/p}\right) =: N_{\bar r_{\bar\lambda}}, \end{aligned}$$

for every $\lambda \in (0,\bar\lambda)$. Since u_λ is a critical point of I_λ, then $I'_\lambda(u_\lambda)(v) = 0$, for any $v \in X$ and every $\lambda \in (0,\bar\lambda)$. In particular, $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is

$$\Phi'(u_\lambda)(u_\lambda) = \lambda\int_\Omega f(x,u_\lambda(x))u_\lambda(x)\,dx, \qquad (20)$$

for every $\lambda \in (0,\bar\lambda)$. On the other hand, since A is homogeneous, we have

$$\begin{aligned} \eta a(x,t\eta) &= \frac{\partial}{\partial t}\Big(A(x,t\eta)\Big) \\ &= \frac{\partial}{\partial t}\Big(t^pA(x,\eta)\Big) \\ &= pt^{p-1}A(x,\eta). \end{aligned}$$

So, $\eta\, a(x,\eta) = pA(x,\eta) \geq A(x,\eta)$, for every $x \in \Omega$ and each $\eta \in \mathbb{R}$. Then,

$$a(x,\xi)\ \xi \geq A(x,\xi) \geq c_2|\xi|^p, \qquad (21)$$

for all $\xi \in \mathbb{R}$. Then, from (20) and (21), it follows that

$$0 \leq c_2\|u_\lambda\|^p \leq \Phi'(u_\lambda)(u_\lambda) = \lambda\int_\Omega f(x,u_\lambda(x))u_\lambda(x)\,dx,$$

for any $\lambda \in (0,\bar\lambda)$. Taking into account (18) or (19) and letting $\lambda \to 0^+$, we get

$$\lim_{\lambda\to 0^+}\|u_\lambda\| = 0,$$

as claimed. Finally, we show that the map

$$\lambda \mapsto I_\lambda(u_\lambda) \text{ is strictly decreasing in } (0,\lambda^\star).$$

Indeed, we observe that for any $u \in X$, one has

$$I_\lambda(u) = \lambda\left(\frac{\Phi(u)}{\lambda} - \Psi(u)\right). \qquad (22)$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \bar{\lambda} < \lambda^\star$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi^{-1}\big((-\infty, \bar{r}_{\bar{\lambda}})\big)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} := \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i})\right) = \inf_{v \in \Phi^{-1}\left((-\infty, \bar{r}_{\bar{\lambda}})\right)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\right),$$

for every $i = 1, 2$. Clearly, (17) together (22) and the positivity of λ imply that

$$m_{\lambda_i} < 0, \quad \text{for } i = 1, 2. \tag{23}$$

Moreover,

$$m_{\lambda_2} \leq m_{\lambda_1}, \tag{24}$$

thanks to $0 < \lambda_1 < \lambda_2$. Then, by (22)-(24) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \bar{\lambda})$. The arbitrariness of $\bar{\lambda} < \lambda^\star$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^\star)$. Thus, the proof is complete.

Now we give some remarks on our results.

Remark 1. In Theorem 2, employing Ricceri's variational principle we searched for a critical point of the functional I_λ naturally associated with the the problem (1). We note that, in general, the functional I_λ can be unbounded from the below in X. Indeed, for example, when $f(x,t) = 1 + |t|^{\zeta-2}t$ for $(x,t) \in \Omega \times \mathbb{R}$ with $\zeta > p > 2$, for any fixed $u \in X \setminus \{0\}$ and $\tau \in \mathbb{R}$, we obtain

$$\begin{aligned}
I_\lambda(\tau u) &= \Phi(\tau u) - \lambda \int_\Omega F(x, \tau u)\, dx \\
&\leq \tau^p \left(\int_\Omega \left[A(x, \Delta u) + \frac{1}{p}|\nabla u|^p + \frac{1}{p}(1+L)|u|^p\right] dx\right) \\
&- \lambda\tau\|u\|_{L^1} - \lambda\frac{\tau^\zeta}{\zeta}\|u\|_{L^\zeta}^\zeta \\
&\leq (1 + \frac{L}{p})\tau^p\|u\|^p - \lambda\tau\|u\|_{L^1} - \lambda\frac{\tau^\zeta}{\zeta}\|u\|_{L^\zeta}^\zeta \\
&\to -\infty
\end{aligned}$$

as $\tau \to +\infty$. Hence, using direct minimization is not possible to find critical points of the functional I_λ.

Remark 2. If function f is non-negative then the solution ensured in Theorem 2 is non-negative. Indeed, let u_* be a non-trivial weak solution of the problem (1), then u_* is non-negative. Arguing by a contradiction, suppose that the set $\mathcal{A} = \{x \in \Omega \ ; \ u_*(x) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(x) = \min\{u_*(x), 0\}$. Using this fact that u_* also is a solution of (1), so for every $\bar{v} \in X$ one has

$$\int_\Omega \left[a(x, \Delta u_*)\Delta \bar{v} + |\nabla u_*|^{p-2}\nabla u_* \nabla \bar{v} + |u_*|^{p-2}u_*\bar{v} - h(u_*)\bar{v}\right] dx \\ - \lambda \int_\Omega f(x, u_*)\bar{v}\, dx = 0$$

and by choosing $\bar{v} = u_*$ and since f is non-negative, we have

$$0 \leq \left(\min\{pc_2, 1\} - LS_p^p\right) \|u_*\|_{\mathcal{A}}^p \\ \int_{\mathcal{A}} [a(x, \Delta u_*)\Delta u_* + |\nabla u_*|^p + |u_*|^p - h(u_*)u_*]\, dx \\ = \lambda \int_{\mathcal{A}} f(x, u_*(x))u_*(x)\, dx \leq 0$$

since $L < \frac{pc_3}{S_p^p}$, where $c_3 = \min\{\frac{1}{p}, c_2\}$, we have $\|u_*\|_{\mathcal{A}}^p \leq 0$ which contradicts fact that u_* is a non-trivial solution. Hence, u_* is positive.

Remark 3. It is worth to mention that Theorem 2 is a bifurcation result in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in X \times (0, +\infty) \ : \ u_\lambda \text{ is a non-trivial weak solution of (1)}\}$$

in $X \times \mathbb{R}$. Indeed, by Theorem 2 we have that

$$\|u_\lambda\| \to 0 \text{ as } \lambda \to 0.$$

Hence, there exist two sequences $\{u_j\}$ in X and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that $\lambda_j \to 0^+$ and $\|u_j\| \to 0$, as $j \to +\infty$. Moreover, we emphasis that due to the fact that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, the solutions u_{λ_1} and u_{λ_2} given by Theorem 2 are different.

Several special cases of Theorem 2 read as follows.

Corollary 3. *Let $g : \bar{\Omega} \to \mathbb{R}$ be a function that $g \in L^\infty(\bar{\Omega})$ with*

$$ess \inf_{x \in \bar{\Omega}} g(x) > 0.$$

Further, let $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$|f(t)| \leq c_0 |t|^{q-1}, \tag{25}$$

for every $t \in \mathbb{R}$ *and some positive constant* c_0*, where* $q \in (1, p^\star)$*. In addition, assume that*

$$\lim_{t \to 0^+} \frac{\int_0^t f(s)\, ds}{t^p} = +\infty,$$

and suppose also that the condition (5) *holds, and* a, A *are continuous functions and satisfy the conditions* $(a) - (d)$*. Then, there exists* $\lambda^\star > 0$*, such that, for any* $\lambda \in (0, \lambda^\star)$ *problem*

$$\begin{cases} \Delta\big(a(x,\ \Delta u)\big) + div\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u = \lambda\, g(x) f(u) + h(u), & in\ \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 0, & on\ \partial\Omega, \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in X$*. Also,* $\lambda^\star = +\infty$*, provided* $q \in (1, p)$*. Moreover,*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$$

and the function

$$\lambda \longmapsto \int_\Omega \left[A(x, \Delta u_\lambda) + \frac{1}{p}|\nabla u_\lambda|^p + \frac{1}{p}|u_\lambda|^p - H(u_\lambda(x) - \lambda\ g(x)\Big(\int_0^{u_\lambda(x)} f(s)\, ds \Big) \right] dx$$

is negative and strictly decreasing in $(0, \lambda^\star)$*.*

Proof. It is enough for the proof to put

$$f_0(x, t) = g(x) f(t)$$

for every $(x, t) \in \Omega \times \mathbb{R}$. Clearly, all of assumptions of Theorem 2 are satisfying. Thus, by using Theorem 2 for the function f_0 the proof is complete.

Corollary 4. *Let* $g : \bar{\Omega} \to \mathbb{R}$ *be a function that* $g \in L^\infty(\bar{\Omega})$ *with*

$$ess \inf_{x \in \bar{\Omega}} g(x) > 0.$$

Further, let $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$\lim_{|t| \to +\infty} \frac{f(t)}{|t|^{q-1}} = 0, \tag{26}$$

where $q \in (1, p^\star)$ *and the condition* (5) *holds. In addition, if* $f(0) = 0$, *assume that*

$$\lim_{t \to 0^+} \frac{\int_0^t f(s)\,ds}{t^p} = +\infty.$$

Then, there exists $\lambda^\star > 0$, *such that, for any* $\lambda \in (0, \lambda^\star)$ *the following p-biharmonic problem*

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) + div(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = \lambda\, g(x) f(u) + h(u), & in\ \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 0, & on\ \partial\Omega, \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in X$. *Also,* $\lambda^\star = +\infty$, *provided* $q \in (1, p)$. *Moreover,*

$$\lim_{\lambda \to 0^+} \int_\Omega \Big(|\Delta u_\lambda(x)|^p + |\nabla u_\lambda(x)|^p + |u_\lambda(x)|^p \Big)\, dx = 0$$

and the function

$$\lambda \longmapsto \frac{1}{p} \int_\Omega \Big(|\Delta u_\lambda(x)|^p + |\nabla u_\lambda(x)|^p + |u_\lambda(x)|^p \Big)\, dx$$

$$- \int_\Omega \Big(\int_0^{u_\lambda(x)} h(\xi)\, d\xi \Big)\, dx - \lambda \int_\Omega g(x) \left(\int_0^{u_\lambda(x)} f(s)\, ds \right) dx$$

is negative and strictly decreasing in $(0, \lambda^\star)$.

Proof. It is enough to put

$$a(x, \xi) := \xi\ |\xi|^{p-2}$$

for every $(x, \xi) \in \Omega \times \mathbb{R}$. From (26), the condition (25) is satisfying. So, the conclusion follows by applying Corollary 3.

We end this section by giving the following examples to illustrate our results.

Example 1. *Consider the following problem*

$$\begin{cases} \Delta\big(a(x,\ \Delta u)\big) + div\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u, = \lambda f(x, u) + \mu_0 \tan^{-1}(u), & in\ \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 0, & on\ \partial\Omega. \end{cases} \tag{27}$$

Now, by applying Theorem 2 with choosing $\mu_0 \le \frac{p\, c_3}{S_p^p}$ and

$$f(x, t) := \alpha(x)\, |t|^{r-p} t + \sigma(x)\, |t|^{s-p} t, \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

where $r \in (1, p-2)$, $2p < s$ with $s-p+2 < p^*$ and $\alpha, \beta : \overline{\Omega} \to \mathbb{R}$ are two continuous functions with $\alpha(x)$ negative, the problem (27) has at least one non-trivial weak solution for any $\lambda \in (0, \lambda^*)$, for some $\lambda^* > 0$. Note that

$$\begin{aligned} \lim_{t\to 0^+} \frac{\int_0^t f(x,\xi)\,d\xi}{t^p} &= \lim_{t\to 0^+} \frac{\alpha(x)}{r-p+2} t^{r-2p+2} + \lim_{t\to 0^+} \frac{\sigma(x)}{s-p+2} t^{s-2p+2} \\ &= \lim_{t\to 0^+} \frac{\alpha(x)}{r-p+2} t^{r-2p+2}. \end{aligned}$$

If $r \in (1, p-2)$, then $r-p+2 < 0$ and $r-2p+2 < 0$, hence

$$\lim_{t\to 0^+} \frac{\alpha(x)}{r-p+2} t^{r-2p+2} = +\infty.$$

Also

$$\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{q-1}} = 0,$$

if $s-p+2 < q < p^*$.

Example 2. *Now, we consider the following problem*

$$\begin{cases} \Delta\big(a(x,\ \Delta u)\big) + div\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u = \lambda f(u) + h(u), & in\ \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 0, & on\ \partial\Omega, \end{cases} \tag{28}$$

where $f(t) = qt^{q-1}$ and $h(t) = \nu \sin(t)$ for every $t \in \mathbb{R}$ where $\nu < \min\{1, p\ c_2\}$ and $2 < q < p$. By the definition of the function f we obtain that $F(t) = t^q$. Clearly, we have $|f(t)| \le q|t|^{q-1}$, for all $t \in \mathbb{R}$, and since $q < p$

$$\lim_{t\to 0^+} \frac{F(t)}{t^p} = \lim_{t\to 0^+} \frac{t^q}{t^p} = +\infty.$$

Then all conditions in Corollary 3 are satisfied. So, conclusions of Corollary 3 are applicable for the problem 28.

Example 3. *Let $p > 2$ and $2 < q < p < p^* = \frac{pN}{N-2p}$. Put*

$$f(t) = \begin{cases} 0 & t \le 0 \\ \sin t & t > 0. \end{cases}$$

Thus, one has

$$f(0) = 0 \quad and \quad \lim_{|t|\to+\infty} \frac{f(t)}{|t|^{q-1}} = 0.$$

On the other hand, since $p > 2$ (or $p-1 > 1$), we obtain that

$$\lim_{t\to 0^+} \frac{\int_0^t f(s)\,ds}{t^p} = \lim_{t\to 0^+} \frac{\sin t}{pt^{p-1}} = +\infty.$$

So, all the assumptions Corollary 4 is satisfying. Then, Corollary 4 is applicable for the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) + div(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = \lambda\, f(u) + \mu_1 \tan^{-1}(\frac{u}{3}), & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\mu_1 < 3$. Recall that for $a(x,\xi) = |\xi|^{p-2}$, we have $A(x,\xi) = \frac{|\xi|^p}{p}$, for all $(x,\xi) \in \Omega \times \mathbb{R}$ and $c_3 = \frac{1}{p}$.

4. A particular case of the problem

In this section, let $p = 2$ and the function $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined as

$$a(x,\xi) := |x|^{-2\alpha}\,\xi$$

and

$$A(x,\xi) := \int_0^\xi a(x,s)\,ds,$$

for every $(x,\xi) \in \Omega \times \mathbb{R}$, where $0 \le \alpha < \frac{N-2}{2}$. We consider the following nonlinear system:

$$\begin{cases} \Delta(|x|^{-2\alpha}\Delta u(x)) + div\big(\nabla u(x)\big) + u(x) = \lambda|x|^{-2\beta} f(u(x)) + h(u(x)), & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{29}$$

where λ is a real positive parameter and Ω is a bounded domain similar to Section 1, and α, β are real numbers such that

$$0 \le \alpha < \frac{N-2}{2} \qquad \text{and} \qquad \alpha \le \beta < \alpha + 1,$$

and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following subcritical growth condition

$$|f(t)| \le a_1 + a_2|t|^{r-1} \quad \text{for all } t \in \mathbb{R},$$

for some nonnegative constants a_1, a_2 and $r \in]1, r_\star[$, where

$$r_\star := 2\min\{\frac{(N-2\beta)}{N-2(\alpha+1)}, \frac{N}{N-2}\},$$

and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitzian constant $0 < L < 1$, that is

$$|h(t_1) - h(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$ and $h(0) = 0$. Since Ω is bounded, we have

(a) $|a(x,\xi)| = |x|^{-2\alpha}|\xi| \leq c_1(1+|\xi|), \quad \forall\, (x,\xi) \in \Omega \times \mathbb{R}$, (If Ω does not contain 0)

for some positive constant c_1. Now, put $g(s) := A(x,s)$ for each $x \in \Omega$ and $s \in \mathbb{R}$. Then, we obtain that the function g is homogeneous of degree 2.

(c) there is a constant $c_2 > 0$ such that

$$\begin{aligned} A(x,\xi) = \int_0^\xi a(x,s)\,ds &= \frac{|x|^{-2\alpha}}{2}\xi^2 \\ &\geq \frac{\left[\sup_{z\in\Omega}\Big(d(z,0)\Big)\right]^{-2\alpha}}{2}\xi^2 \\ &= c_2|\xi|^2. \end{aligned}$$

Also a is monotone. Thus, all the conditions $(a)-(d)$ in the Section 1 are satisfied for the function a.

With the previous notations, the main result of this section reads as follows:

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following growth condition*

$$|f(t)| \leq a_3 + a_4|t|^{r-1} \quad \text{for all } t \in \mathbb{R} \tag{30}$$

for some nonnegative constants a_3, a_4 and $r \in]1, r_\star[$, where $r_\star$ has been given above, α and β are real numbers such that $0 \leq \alpha < \frac{N-2}{2}$ and $\alpha \leq \beta < \alpha + 1$. In addition, if $f(0) = 0$, assume also that

$$\limsup_{t\to 0^+} \frac{\int_0^t f(s)\,ds}{t^2} = +\infty \quad \text{and} \quad \liminf_{t\to 0^+} \frac{\int_0^t f(s)\,ds}{t^2} > -\infty. \tag{31}$$

Moreover, suppose that

$$L < \frac{2c_3}{S_2^2}, \tag{32}$$

where $c_3 = \min\{\frac{1}{2}, c_2\}$ and $c_2 = \frac{\left[\sup_{z\in\Omega}\Big(d(z,0)\Big)\right]^{-2\alpha}}{2}$. Then, there exists $\lambda^\star > 0$, such that, for any $\lambda \in (0, \lambda^\star)$ the problem (29) admits at least one non-trivial weak solution $u_\lambda \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Also, $\lambda^\star = +\infty$, provided $q \in (1,2)$. Moreover,

$$\lim_{\lambda\to 0^+} \int_\Omega \Big(|\Delta u_\lambda(x)|^2 + |\nabla u_\lambda(x)|^2 + |u_\lambda(x)|^2\Big)\,dx = 0,$$

and the function

$$\lambda \mapsto \frac{1}{2} \int_\Omega \left(|x|^{-2\alpha} |\Delta u_\lambda(x)|^2 + |\nabla u_\lambda(x)|^2 + |u_\lambda(x)|^2 \right) dx$$
$$- \int_\Omega \left(\int_0^{u_\lambda(x)} h(\xi)\, d\xi \right) dx - \lambda \int_\Omega |x|^{-2\alpha} \left(\int_0^{u_\lambda(x)} f(\xi)\, d\xi \right) dx$$

is negative and strictly decreasing in $(0, \lambda^\star)$.

Proof. By applying Theorem 2, put $f(x,t) := |x|^{-2\beta} f(t)$ and $a(x,t) := |x|^{-2\alpha} t$, for each $x \in \Omega$ and $t \in \mathbb{R}$. So, $f(x,0) = |x|^{-2\beta} f(0) = 0$. Clearly, $r_\star \le \frac{2N}{N-2} = \frac{pN}{N-2} \le p^\star$ for $p = 2$. Therefore, from condition (30), assumption (f_1) holds. Condition (31) shows that (f_2) of Theorem 2 holds. From what we said before this Theorem for the function $a : \Omega \times \mathbb{R} \to \mathbb{R}$ defined as $a(x,t) := |x|^{-2\alpha}\, t$ and from (32), we observe that all of assumptions of Theorem 2 are satisfying. Thus, Theorem 2 ensures the conclusion.

We give here the following special case of our result in this section; see also Remark 6.

Theorem 6. *Let* $0 \le \beta < 1$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $f(0) = 0$ *and*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty, \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(t)}{|t|^s} = 0,$$

for some $0 \le s < \frac{(N+2(1-2\beta))}{N-2}$, *and assume that the condition* (32) *also holds. Then, there exists* $\lambda^\star > 0$ *such that for all* $\lambda \in]0, \lambda^\star[$, *the following problem*

$$\begin{cases} \Delta(\Delta u) + \Delta u + u = \lambda |x|^{-2\beta} f(u) + h(u), & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

admits at least one nontrivial weak solution $u_\lambda \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. *Also,* $\lambda^\star = +\infty$, *provided* $q \in (1,2)$. *Moreover, we have*

$$\lim_{\lambda \to 0^+} \int_\Omega \left(|\Delta u_\lambda(x)|^2 + |\nabla u_\lambda(x)|^2 + |u_\lambda(x)|^2 \right) dx = 0,$$

and the mapping

$$\lambda \mapsto \frac{1}{2} \int_\Omega \left(|\Delta u_\lambda(x)|^2 + |\nabla u_\lambda(x)|^2 + |u_\lambda(x)|^2 \right) dx$$
$$- \int_\Omega H(u_\lambda(x))\, dx - \lambda \int_\Omega |x|^{-2\beta} \left(\int_0^{u_\lambda(x)} f(\xi)\, d\xi \right) dx$$

is negative and strictly decreasing in $]0, \lambda^\star[$.

Proof. Clearly, for $\alpha = 0$, the conclusion follows from Theorem 5.

Remark 4. *(see [16, Remark 4.1]) By simple direct computations, since*

$$0 \leq \alpha < \frac{N-2}{2} \quad and \quad \alpha \leq \beta < \alpha + 1,$$

it follows that

$$2 < \frac{2(N-2\beta)}{N-2(\alpha+1)}.$$

Further, we observe that if, instead of $0 \leq \alpha < \frac{N-2}{2}$, *we require the more condition* $0 \leq \alpha < \frac{\beta(N-2)}{N}$, *one has*

$$\frac{2(N-2\beta)}{N-2(\alpha+1)} \in]2, 2^\star[.$$

Thus, in this special case, the above relation yields

$$r_\star = \frac{2(N-2\beta)}{N-2(\alpha+1)}.$$

Taking Remark 4 into account, we have the following results from Theorem 5 motivated by [16, Corollary 4.5].

Corollary 7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying the condition* (30), *for* $r \in]2, r_\star[$, *where* $r_\star = \frac{2(N-2\beta)}{N-2(\alpha+1)}$. *Further, assume that* $f(0) = 0$ *and*

$$\lim_{t \to 0^+} \frac{\int_0^t f(s)\,ds}{t^2} = +\infty, \tag{33}$$

and condition (32) *holds. Then, there exists* $\lambda^\star > 0$ *such that for any* $\lambda \in]0, \lambda^\star[$, *the problem* (29) *admits at least one nontrivial weak solution* $u_\lambda \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. *Also,* $\lambda^\star = +\infty$, *provided* $q \in (1, 2)$. *Moreover,*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0,$$

and the mapping

$$\begin{aligned}\lambda \mapsto \quad & \frac{1}{2} \int_\Omega \left(|x|^{-2\alpha} |\Delta u_\lambda(x)|^2 + |\nabla u_\lambda(x)|^2 + |u_\lambda(x)|^2 \right) dx \\ & - \int_\Omega H(u_\lambda(x))\,dx - \lambda \int_\Omega |x|^{-2b}\ F(u_\lambda(x))\,dx\end{aligned}$$

is negative and strictly decreasing in $]0,\lambda^\star[$, *where* H *is the primitive of the nonlinearity function* h, *i.e.,*

$$H(t) := \int_0^t h(s)\,ds, \qquad \textit{for each } t \in \mathbb{R}.$$

The next example we choose a f vanishing at zero with the construction partially motivated by [16, Example 4.6]. The existence of one nontrivial solution for the problem involving f is achieved employing Corollary 7.

Example 4. *Considere the problem* (29) *with* $f(x) := |x|^{\gamma-2}x + |x|^{s-2}x$, *for any* $x \in \mathbb{R}$, $1<\gamma<2$ *and* $2<s<r_\star$, *and* $h:\mathbb{R}\to\mathbb{R}$ *is a Lipschitz continuous function of* 1*-order with the Lipschitzian constant* $0<L<1$, *a.e.,*

$$|h(x)-h(y)| \le L\,|x-y|,$$

for every $x,y\in\mathbb{R}$, *and* $h(0)=0$ *(for example, we can assume that the function* h *be defined as* $h(x)=\mu\tan^{-1}x$, *for each* $x\in\mathbb{R}$, *where* μ *is a positive constant which is satisfying in the condition* (32)*). In fact,* $f(0)=0$ *and it is easy to verify that*

$$|f(t)| \le \frac{s-\gamma}{s-1} + \frac{s+\gamma}{s-1}|t|^{s-2}, \qquad \textit{for all } t\in\mathbb{R}.$$

On the other hand, we observe that

$$\begin{aligned}
\lim_{t\to 0^+}\frac{\int_0^t f(s)\,ds}{t^2} &= \lim_{t\to 0^+}\frac{\int_0^t \left(|x|^{\gamma-2}x+|x|^{s-2}x\right)dx}{t^2}\\
&= \lim_{t\to 0^+}\frac{\int_0^t \left(x^{\gamma-2}+x^{s-2}\right)dx}{t^2}\\
&= \lim_{t\to 0^+}\frac{\left(\frac{1}{\gamma-1}t^{\gamma-1}+\frac{1}{s-1}t^{s-1}\right)}{t^2}\\
&= +\infty.
\end{aligned}$$

Thus clearly, all the assumptions of Corollary 7 are fulfilled, ant it is applicable for the problem (29)*.*

Remark 5. *(see [16, Remark 4.3.]) Since* $f(0)=0$ *in the Theorem 5,* $\lambda=0$ *is a bifurcation point for problem* (29)*, in the sense that the point* $(0,0)$ *belongs to the closure of the set*

$$\Sigma := \{(u,\lambda)\in (W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))\times]0,+\infty[:\ u \textit{ is a weak solution of } (29),\ u\neq 0\}$$

in the space $\left(W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)\right)\times\mathbb{R}$.

Remark 6. *(see [16, Remark 4.4]) Theorem 6 easily follows from Theorem 5 taking into account that the following s-sublinear assumption at infinity*

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|^s} = 0,$$

in which $s \in [0, s_\star)$, *where*

$$s_\star := \min\Big\{\frac{N+2(\alpha-2\beta+1)}{N-2(\alpha+1)}, \frac{N+2}{N-2}\Big\},$$

implies the growth condition (30)*. In addition, if*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty,$$

also condition (31) *holds true.*

Remark 7. *(see [16, Remark 4.8]) In conclusion, we also note that a related bifurcation result (in respect to Theorem 5) for perturbed elliptic problems and involving the bi-harmonic operator* $\Delta(\Delta u) := div\Big(\nabla(\Delta u)\Big)$. *More precisely, if* f *is satisfying in the condition* (30)*, under our assumptions at zero, problem*

$$\begin{cases} \Delta(\Delta u) + \Delta u + u = \lambda f(u) + h(u), & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

admits at least one nontrivial weak solution and the other conclusions of Theorem 5 hold true.

Acknowledgements. We would like to show our great thanks to Professor Jifeng Chu for his valuable suggestions and comments, which improved the former version of this paper and made us to rewrite the paper in a more clear way.

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