

A note on embedding of orthomorphisms into biorthomorphisms

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Abstract: Let A be an Archimedean f -algebra with unit. In this talk, we are deal with the embedding of space of orthomorphisms on A into biorthomorphisms on $A \times A$.

Keywords: order unit, Archimedean vector lattice, ideal, semiprime, f -algebras.

MSC2010: 46B42, 46A40, 06F25

1 Introduction

The notion of biorthomorphisms of bilinear maps was firstly introduced in [6] and studied by many authors, for example, as in [3], [4], [7]. Let X be an Archimedean vector lattice. A subset E of X is called solid if $|x| \leq |y|$, $y \in E$ imply $x \in E$. A solid linear subspace of X is said to be an order ideal. An order ideal B of X is said to be a band if $\sup A$ exists in B , whenever $\sup A$ exists in X for every subset A of B or B is an order closed ideal. A vector lattice X is called Dedekind complete if every subset of X which is bounded from above has a supremum. A linear operator T from a vector lattice X into a vector lattice Y is called a lattice homomorphism if $T(x \vee y) = Tx \vee Ty$ for every $x, y \in X$. A linear operator T from a vector lattice X into a vector lattice Y is called an order bounded if T maps order bounded sets into order bounded sets. An order bounded band preserving operator T from a vector lattice X into itself is said to be an orthomorphism. We denote the set of all orthomorphisms on X by $Orth(X)$. A bilinear map T from $X \times X$ into X is called a biorthomorphism if T is separately order bounded and separately band preserving. We denote the set of all biorthomorphisms on X by $Orth(X, X)$. $Orth(X, X)$ is a vector lattice by lattice operations defined by $(T \vee S)(x, y) = T(x, y) \vee S(x, y)$ and $(T \wedge S)(x, y) = T(x, y) \wedge S(x, y)$ for all $T, S \in Orth(X, X)$, $(x, y) \in X^+ \times X^+$, [3]. A lattice ordered algebra A is called an f -algebra if $a \wedge b = 0$, $c \in A^+$ imply $ac \wedge b = ca \wedge b = 0$. It is known that if an Archimedean f -algebra with unit, then $A = Orth(A)$, in [1]. An f -algebra A is called semiprime if the only nilpotent element is zero. Note that if an f -algebra A has unit, then A is semiprime. For the unexplained notions and terminology we refer to the standard book [1].

Definition 1 An upward directed net $a_\alpha \uparrow$ in the positive cone of an f -algebra A is called an approximate unit if $a_\alpha b \uparrow b$ for every $b \in A^+$.

Definition 2 An upward directed net $a_\alpha \uparrow$ in the positive cone of an f -algebra A is called a weak approximate unit if $f(b) = \sup_\alpha f(a_\alpha b)$ for every $f \in (A')^+$, $b \in A^+$. Here, $(A')^+$ is the positive cone of the order dual of A .

An f -algebra A is semiprime if and only if A has an approximate unit, [5]. Note that if an f -algebra A has a weak approximate unit, then A is semiprime, [5].

Definition 3 Let A be an Archimedean ,semiprime f -algebra. We say that A satisfies the Stone condition whenever $a \wedge I \in A^+$ for all $a \in A^+$. Here , we consider A as embedded in $Orth(A)$. Here , I is identity operator.

In this note , by A' , we denote the set of all order bounded linear functionals on A . By A'' , we denote the second order dual of A .By canonical emdedding, A'' has unit when A has unit. We assume that A is an f -algebra with separating order dual A' . Therefore, f -algebra A is Archimedean,associative and commutative. We establish the following bilinear mappings:

$$A \times A \longrightarrow A, (a, b) \longrightarrow ab, \quad (1)$$

$$A' \times A \longrightarrow A', (f, a) \longrightarrow fa : fa(b) = f(ab), \quad (2)$$

$$A'' \times A' \longrightarrow A', (F, f) \longrightarrow Ff : (Ff)(a) = F(fa), \quad (3)$$

$$A'' \times A'' \longrightarrow A'', (F, G) \longrightarrow FG : (FG)(f) = F(Gf), \forall a, b \in A, f \in A', F, G \in A''. \quad (4)$$

By the mapping 4, A'' is a Dedekind complete f -algebra by means of Arens multiplication,[2], [5]. Consider that $A'' = (A')'_n \oplus (A')'_s$ is an order direct sum since A'' is Dedekind complete, where $(A')'_n$ is the band of all order continuous order bidual of A , $(A')'_s$ is singular part of A'' , [1].

If A has unit, then $A'' = (A')'_n$, [5]. If A has a weak approximate unit with Stone condition, then $(A')'_n$ is semiprime, [5].In this case, A'' is semiprime. We suppose that A is an Archimedean semiprime f -algebra. Then, the mapping $\rho : Orth(A) \longrightarrow Orth(A, A)$ is defined by $\rho(\pi)(x, y) = \pi(xy) = \pi(x)y$ for all $x, y \in A, \pi \in Orth(A)$. ρ is an injective lattice homomorphism, but it is not surjective. The following is a typical example of non surjectivity of ρ .

Example 1 Let $A = C[0, 1]$ with the f -algebra multiplicaion defined by $(f.g)(x) = xf(x)g(x)$ for $f, g \in C[0, 1]$ and $x \in [0, 1]$. Here, $f.g \in Orth(A, A)$ but $f.g$ does not belong to $Orth(A)$, [3].

The question is that when $\rho(Orth(A))$ is an order ideal in $Orth(A, A)$.

It is known that if A is a Dedekind complete semiprime f -algebra , then image of $Orth(A)$ under ρ is an order ideal,[4], [3].We extend this result to the order bidual of A . For this problem, we have two cases.In the first case, we assume that an f -algebra A has unit. In the second case, we suppose that f -algebra A has a weak approximate unit with Stone condition. For the first case, we consider the following mappings: $\rho : A = Orth(A) \rightarrow Orth(A, A)$, $k : Orth(A, A) \rightarrow Orth(A'', A'')$, $j : Orth(A) \rightarrow Orth(A'') = A''$ which are defined by $\rho(\pi)(x, y) = \pi(xy) = \pi(x)y, k(T)=T'''$ (triadjoint of T in [7]), $j(\pi) = \pi''$. And the mapping $\varrho : Orth(A'') \rightarrow Orth(A'', A'')$ is defined by $\varrho(\pi)(F, G) = \pi(F)G$.It is well-known from [5] that A'' is a Dedekind complete f -algebra with unit (so it is semiprime), the image of $Orth(A'')$ under ϱ is an order ideal in $Orth(A'', A'')$. Secondly, we suppose that A has a weak approximate unit with Stone condition.Then, A'' is a Dedekind complete semiprime f -algebra.We observe that the mappings $\sigma : A \rightarrow Orth(A)$ defined by $\sigma_a(b) = ab$ for every $a, b \in A$,canonical embedding $i : A \rightarrow A''$ defined by $i(a) = a'' : a''(f) = f(a)$ and $\phi : A'' \rightarrow Orth(A'')$ defined by $\phi_F(G) = FG$ for every $F, G \in A''$. Here, the mappings σ and ϕ are one to one algebra and lattice homomorphisms. So, the conclusion follows as in the first case.Then, our claim is also true. Hence, we can give the main results in this presentation.

Theorem 1 Let A be an Archimedean f -algebra with unit and suppose that a mapping $\varrho : Orth(A'') \rightarrow Orth(A'', A'')$ is defined by $\varrho(\pi)(F, G) = \pi(F)G = \pi(F)G$ for every $\pi \in Orth(A'')$, $F, G \in A''$. Then, the image of $Orth(A'')$ under ϱ is an order ideal in $Orth(A'', A'')$.

Theorem 2 Let A be an Archimedean f -algebra having weak approximate unit satisfying Stone condition. Suppose that a mapping $\varrho : Orth(A'') \rightarrow Orth(A'', A'')$ is defined by $\varrho(\pi)(F, G) = \pi(F)G = \pi(F)G$ for every $\pi \in Orth(A'')$, $F, G \in A''$. Then, the image of $Orth(A'')$ under ϱ is an order ideal in $Orth(A'', A'')$.

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