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ON SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE RĂDUCANU-ORHAN DIFFERENTIAL OPERATOR

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Abstract. In this paper, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions of certain new subclasses of the bi-univalent function class Σ defined on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, which are associated with the Răducanu-Orhan differential operator. Moreover, connections to the earlier known results are indicated.

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1. Introduction

Let $\mathcal{A} = \left\{ f : \mathbb{U} \to \mathbb{C} : f \textit{ is analytic in the unit disk } \mathbb{U}, f(0) = 0, f^{'}(0) = 1 \right\}$ be the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

and the subclass of $\mathcal{A}$ consisting the univalent functions in $\mathbb{U}$ is denoted by $\mathcal{S}$. It is clear from the Koebe one quarter theorem (see [4]) that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z,\ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w,\ (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

In fact, we have:

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots , \tag{2}$$

where g be an extension of f^{-1} to $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

For more details about the bi-univalent function class Σ, see Lewin [6], Netanyahu [7], Brannan and Clunie [2], Srivastava et al. [14] etc. Also Brannan and Taha [3], (see also [15]) introduced $\mathcal{S}_\Sigma^*[\alpha]$, the class of strongly bi-starlike functions of order α where $0 < \alpha \leq 1$ and $\mathcal{S}_\Sigma^*(\beta)$, the class of bi-starlike functions of order β where $0 \leq \beta < 1$ and found the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in these subclasses. In recent investigations many researchers (viz. [5, 10, 13] etc.) introduced various subclasses of the function class Σ and obtained the non-sharp estimates on $|a_2|$ and $|a_3|$ for the functions in these subclasses.

For $f(z)$ given by (1) and $j(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product or convolution is given by

$$(f * j)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \mathbb{U}.$$

For $f \in \mathcal{A}$ and $0 \leq \mu \leq \delta$, $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$; Răducanu and Orhan [11] introduced the following differential operator:

$$D_{\delta\mu}^0 f(z) = f(z),$$

$$D_{\delta\mu}^1 f(z) = D_{\delta\mu} f(z) = \delta\mu z^2 f''(z) + (\delta - \mu) z f'(z) + (1 - \delta + \mu) f(z),$$

$$D_{\delta\mu}^n f(z) = D_{\delta\mu}\left(D_{\delta\mu}^{n-1} f(z)\right).$$

See that, for the function f given by (1), this becomes:

$$D_{\delta\mu}^n f(z) = z + \sum_{k=2}^{\infty} F_k(\delta, \mu, n) a_k z^k$$

or

$$D_{\delta\mu}^n f(z) = (f * j)(z),$$

where

$$j(z) = z + \sum_{k=2}^{\infty} F_k(\delta, \mu, n) z^k$$

and

$$F_k(\delta, \mu, n) = \left[1 + (\delta\mu k + \delta - \mu)(k - 1)\right]^n.$$

Observe that for $\mu = 0$ we get the Al-Oboudi differential operator (see [1]) and for $\mu = 0$, $\delta = 1$ we get the Sălăgean differential operator (see [12]).

The object of the present paper is to introduce the subclasses $\mathcal{B}_\Sigma^{\delta\mu}(n, \alpha, \lambda)$ and $\mathcal{H}_\Sigma^{\delta\mu}(n, \beta, \lambda)$ of the function class Σ, which are associated with the Răducanu-Orhan differential operator and to obtain estimates on $|a_2|$ and $|a_3|$ for the functions in these new subclasses using similar techniques used by Srivastava et al.[14].

We need the following lemma (see [9]) to prove our main results.

Lemma 1. *If $p(z) \in \mathcal{P}$, the Carathéodory class of analytic functions with positive real part in $\mathbb{U}$, then $|p_n| \leq 2$ for each $n \in \mathbb{N}$, where*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad (z \in \mathbb{U}).$$

2. Main Results

Definition 1. *A function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_{\Sigma}^{\delta\mu}(n, \alpha, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma, \quad \left| arg \left\{ \frac{(1-\lambda) D_{\delta\mu}^{n} f(z) + \lambda D_{\delta\mu}^{n+1} f(z)}{z} \right\} \right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U})$$

and

$$\left| arg \left\{ \frac{(1-\lambda) D_{\delta\mu}^{n} g(w) + \lambda D_{\delta\mu}^{n+1} g(w)}{w} \right\} \right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U}),$$

where the function g is given by (2).

Theorem 2. *If the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma}^{\delta\mu}(n, \alpha, \lambda)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\begin{array}{c} 2\alpha[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)] - \\ (\alpha-1)[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2 \end{array}}} \tag{3}$$

and

$$|a_3| \leq \frac{4\alpha^2}{[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2} + \frac{2\alpha}{[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}. \tag{4}$$

Proof. Definition 1 implies that we can write:

$$\frac{(1-\lambda) D_{\delta\mu}^{n} f(z) + \lambda D_{\delta\mu}^{n+1} f(z)}{z} = [s(z)]^{\alpha} \tag{5}$$

and

$$\frac{(1-\lambda) D_{\delta\mu}^{n} g(w) + \lambda D_{\delta\mu}^{n+1} g(w)}{w} = [t(w)]^{\alpha}, \tag{6}$$

where $s(z), t(w) \in \mathcal{P}$ such that:

$$s(z) = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \ (z \in \mathbb{U}) \tag{7}$$

and

$$t(w) = 1 + t_1 w + t_2 w^2 + t_3 w^3 + \cdots, \ (w \in \mathbb{U}). \tag{8}$$

Clearly, we have:

$$[s(z)]^\alpha = 1 + \alpha s_1 z + \left[\alpha s_2 + \frac{\alpha(\alpha-1)}{2} s_1^2\right] z^2 + \cdots$$

and

$$[t(w)]^\alpha = 1 + \alpha t_1 w + \left[\alpha t_2 + \frac{\alpha(\alpha-1)}{2} t_1^2\right] w^2 + \cdots.$$

Also, using (1) and (2), we get:

$$\frac{(1-\lambda)D^n_{\delta\mu} f(z) + \lambda D^{n+1}_{\delta\mu} f(z)}{z} = 1 + [1 + (2\delta\mu + \delta - \mu)]^n [1 + \lambda(2\delta\mu + \delta - \mu)] a_2 z + [1 + 2(3\delta\mu + \delta - \mu)]^n [1 + 2\lambda(3\delta\mu + \delta - \mu)] a_3 z^2 + \cdots \tag{9}$$

and

$$\frac{(1-\lambda)D^n_{\delta\mu} g(w) + \lambda D^{n+1}_{\delta\mu} g(w)}{w} = 1 - [1 + (2\delta\mu + \delta - \mu)]^n [1 + \lambda(2\delta\mu + \delta - \mu)] a_2 w + [1 + 2(3\delta\mu + \delta - \mu)]^n [1 + 2\lambda(3\delta\mu + \delta - \mu)](2a_2^2 - a_3) w^2 + \cdots. \tag{10}$$

Now, equating the coefficients in (5) and (6), we obtain:

$$[1 + (2\delta\mu + \delta - \mu)]^n [1 + \lambda(2\delta\mu + \delta - \mu)] a_2 = \alpha s_1, \tag{11}$$

$$[1 + 2(3\delta\mu + \delta - \mu)]^n [1 + 2\lambda(3\delta\mu + \delta - \mu)] a_3 = \alpha s_2 + \frac{\alpha(\alpha-1)}{2} s_1^2, \tag{12}$$

$$-[1 + (2\delta\mu + \delta - \mu)]^n [1 + \lambda(2\delta\mu + \delta - \mu)] a_2 = \alpha t_1, \tag{13}$$

$$[1 + 2(3\delta\mu + \delta - \mu)]^n [1 + 2\lambda(3\delta\mu + \delta - \mu)](2a_2^2 - a_3) = \alpha t_2 + \frac{\alpha(\alpha-1)}{2} t_1^2. \tag{14}$$

Using (11) and (13), we get:

$$s_1 = -t_1 \tag{15}$$

and

$$2[1 + (2\delta\mu + \delta - \mu)]^{2n} [1 + \lambda(2\delta\mu + \delta - \mu)]^2 a_2^2 = \alpha^2 (s_1^2 + t_1^2). \tag{16}$$

Adding (12) in (14) and then using (16), we obtain:

$$a_2^2 = \frac{\alpha^2(s_2+t_2)}{\left\{\begin{array}{l}2\alpha[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]- \\ (\alpha-1)[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2\end{array}\right\}}.$$

Now, by using Lemma 1, this gives:

$$\left|a_2^2\right| \leq \frac{4\alpha^2}{\left\{\begin{array}{l}2\alpha[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]- \\ (\alpha-1)[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2\end{array}\right\}},$$

which proves the result (3). Next, for the estimate on $|a_3|$, subtracting (14) from (12) in light of (15), we get:

$$a_3 - a_2^2 = \frac{\alpha(s_2-t_2)}{2[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}.$$

This by using (16), becomes:

$$a_3 = \frac{\alpha^2(s_1^2+t_1^2)}{2[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2} + \frac{\alpha(s_2-t_2)}{2[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}.$$

Finally, by using Lemma 1, we get:

$$|a_3| \leq \frac{4\alpha^2}{[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2} + \frac{2\alpha}{[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]},$$

which is the desired result (4). This completes the proof of Theorem 2.

Definition 2. *A function $f(z)$ given by (1) is said to be in the class $\mathcal{H}_{\Sigma}^{\delta\mu}(n,\beta,\lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma, \quad \Re\left\{\frac{(1-\lambda)D_{\delta\mu}^n f(z) + \lambda D_{\delta\mu}^{n+1} f(z)}{z}\right\} > \beta$$

$$(0 \leq \beta < 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U})$$

and

$$\Re\left\{ \frac{(1-\lambda)D^n_{\delta\mu}g(w) + \lambda D^{n+1}_{\delta\mu}g(w)}{w} \right\} > \beta$$

$$(0 \leq \beta < 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U}),$$

where the function g is given by (2).

Note that in Definition 1 and Definition 2, by putting $\mu = 0$ we obtain the classes $\mathcal{B}_\Sigma(\delta, n, \alpha, \lambda)$ and $\mathcal{H}_\Sigma(\delta, n, \beta, \lambda)$ introduced by Patil and Naik [8]; by putting $\mu = 0, \delta = 1$ we obtain the classes $\mathcal{B}_\Sigma(n, \alpha, \lambda)$ and $\mathcal{H}_\Sigma(n, \beta, \lambda)$ introduced by Porwal and Darus [10]; by putting $\mu = 0, \delta = 1, n = 0$ we obtain the classes $\mathcal{B}_\Sigma(\alpha, \lambda)$ and $\mathcal{H}_\Sigma(\beta, \lambda)$ introduced by Frasin and Aouf [5] and by putting $\mu = 0, \delta = 1, n = 0, \lambda = 1$ we obtain the classes $\mathcal{H}^\alpha_\Sigma$ and $\mathcal{H}_\Sigma(\beta)$ introduced by Srivastava et al. [14].

Theorem 3. *If the function $f(z)$ given by (1) be in the class $\mathcal{H}^{\delta\mu}_\Sigma(n, \beta, \lambda)$, then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}} \tag{17}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2} + \frac{2(1-\beta)}{[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}. \tag{18}$$

Proof. Definition 2 implies that there exists $s(z), t(w) \in \mathcal{P}$ such that:

$$\frac{(1-\lambda)D^n_{\delta\mu}f(z) + \lambda D^{n+1}_{\delta\mu}f(z)}{z} = \beta + (1-\beta)\, s(z) \tag{19}$$

and

$$\frac{(1-\lambda)D^n_{\delta\mu}g(w) + \lambda D^{n+1}_{\delta\mu}g(w)}{w} = \beta + (1-\beta)\, t(w), \tag{20}$$

where $s(z)$ and $t(w)$ are given by (7) and (8) respectively.
See that we have equations (9), (10) and also:

$$\beta + (1-\beta)\, s(z) = 1 + (1-\beta)s_1 z + (1-\beta)s_2 z^2 + \cdots$$

and

$$\beta + (1-\beta)\, t(w) = 1 + (1-\beta)t_1 w + (1-\beta)t_2 w^2 + \cdots.$$

Now, equating the coefficients in (19) and (20), we obtain:

$$[1+(2\delta\mu+\delta-\mu)]^n[1+\lambda(2\delta\mu+\delta-\mu)]a_2=(1-\beta)s_1, \tag{21}$$

$$[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]a_3=(1-\beta)s_2, \tag{22}$$

$$-[1+(2\delta\mu+\delta-\mu)]^n[1+\lambda(2\delta\mu+\delta-\mu)]a_2=(1-\beta)t_1, \tag{23}$$

$$[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)](2a_2^2-a_3)=(1-\beta)t_2. \tag{24}$$

Using (21) and (23), we obtain:

$$s_1=-t_1$$

and

$$2[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2a_2^2=(1-\beta)^2(s_1^2+t_1^2). \tag{25}$$

Adding (22) in (24), we obtain:

$$2[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]a_2^2=(1-\beta)(s_2+t_2)$$

or

$$a_2^2=\frac{(1-\beta)(s_2+t_2)}{2[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}.$$

This by using Lemma 1, gives:

$$\left|a_2^2\right|\leq\frac{2(1-\beta)}{[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]},$$

which gives the desired result (17). Next, subtracting (24) from (22), we obtain:

$$2[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)](a_3-a_2^2)=(1-\beta)(s_2-t_2)$$

or

$$a_3=a_2^2+\frac{(1-\beta)(s_2-t_2)}{2[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}.$$

Using (25), this becomes:

$$a_3=\frac{(1-\beta)^2(s_1^2+t_1^2)}{2[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2}+\frac{(1-\beta)(s_2-t_2)}{2[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]}.$$

This by using Lemma 1, yields:

$$|a_3|\leq\frac{4(1-\beta)^2}{[1+(2\delta\mu+\delta-\mu)]^{2n}[1+\lambda(2\delta\mu+\delta-\mu)]^2}+\frac{2(1-\beta)}{[1+2(3\delta\mu+\delta-\mu)]^n[1+2\lambda(3\delta\mu+\delta-\mu)]},$$

which is the desired result (18). This completes the proof of Theorem 3.

3. Conclusions

- If we put $\mu = 0$ in Theorem 2 and Theorem 3; we obtain Theorem 5 and Theorem 7 given by Patil and Naik [8].
- If we put $\mu = 0$ and $\delta = 1$ in Theorem 2 and Theorem 3; we obtain Theorem 2.1 and Theorem 3.1 given by Porwal and Darus [10].
- If we put $\mu = 0$, $\delta = 1$ and $n = 0$ in Theorem 2 and Theorem 3; we obtain Theorem 2.2 and Theorem 3.2 given by Frasin and Aouf [5].
- If we put $\mu = 0$, $\delta = 1$, $n = 0$ and $\lambda = 1$ in Theorem 2 and Theorem 3; we obtain Theorem 1 and Theorem 2 given by Srivastava et al.[14].

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