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COEFFICIENT ESTIMATES FOR A UNIFICATION OF SOME SUBCLASSES OF ANALYTIC AND BI-UNIVALENT FUNCTIONS OF MA-MINDA TYPE

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Abstract. In the present investigation, we consider a new general subclass $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi)$ of the class Σ consisting of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class introduced here, we find estimates on the Taylor-Maclaurin coeffcients $|a_2|$ and $|a_3|$. Several connections to some of the earlier known results are also pointed out.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

It is well-known that if $f(z)$ is an analytic univalent function from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$, then the inverse function $g(z)$ defined by

$$g(f(z)) = z \qquad (z \in \mathbb{D}_1)$$

is an analytic and univalent mapping from $\mathbb{D}_2$ to $\mathbb{D}_1$. Moreover, by the familiar *Koebe One-Quarter Theorem* (see [3]), we know that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Therefore, every univalent function $f \in \mathbb{U}$ has an inverse f^{-1} that satisfies the following conditions:

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \qquad \left(w < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right).$$

The inverse of the function $f(z)$ has a series expansion in some disk about the origin of the form:

$$f^{-1}(w) = w + \rho_2 w^2 + \rho_3 w^3 + \cdots . \tag{2}$$

The inverse of the Koebe function provides the best bound for all $|\rho_k|$ in (2) (see [8, 12]).

An univalent function $f(z)$ in a neighborhood of the origin and its inverse $f^{-1}(w)$ satisfy the following condition:

$$f\left(f^{-1}(w)\right) = w$$

or, equivalently,

$$w = f^{-1}(w) + a_2\left[f^{-1}(w)\right]^2 + a_3\left[f^{-1}(w)\right]^3 + \cdots . \tag{3}$$

Using (1) and (2) in (3), we obtain

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of bi-univalent functions in $\mathbb{U}$ given by (1).

It is worth noting that the familiar Koebe function is not a member of Σ since it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane minus a slit along the line $-\frac{1}{4}$ to $-\infty$. Thus, the image of the domain does not contain the unit disk $\mathbb{U}$.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [9] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. To this end, they considered an analytic function ϕ with positive real part in the unit disk $\mathbb{U}$ such that $\phi(0) = 1$,

$\phi'(0) > 0$, and ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$.

We now introduce the following unification of some subclasses of bi-univalent functions of Ma-Minda type.

Definition 1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{N}_\Sigma^\mu(\lambda, \gamma; \phi)$, $\mu \geqq 0$, $\lambda \geqq 1$ and $0 \leqq \gamma \leqq 1$, if the following subordinations hold:*

$$(1-\lambda)\left(\frac{(1-\gamma)z + \gamma f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{(1-\gamma)z + \gamma f(z)}{z}\right)^{\mu-1} \prec \phi(z) \tag{5}$$

and

$$(1-\lambda)\left(\frac{(1-\gamma)w + \gamma g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{(1-\gamma)w + \gamma g(w)}{w}\right)^{\mu-1} \prec \phi(w), \tag{6}$$

where the function g is given by (4).

A function in the class $\mathcal{N}_\Sigma^\mu(\lambda, \gamma; \phi)$ is called bi-starlike of Ma-Minda type. This class unifies the subclass $\mathcal{N}_\Sigma^{\phi,\phi}(\lambda, \mu)$ introduced recently by Srivastava *et al.* [13] and the subclass $\mathcal{S}_\Sigma^{a,1,a}(1, \gamma, \phi)$ investigated by Peng *et al.* [11]. These subclasses are defined respectively as follows:

Definition 2. *A function $f \in \Sigma$ is said to be in the class $\mathcal{N}_\Sigma^{\phi,\phi}(\lambda, \mu)$, $\mu \geqq 0$ and $\lambda \geqq 1$, if the following subordinations hold:*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \phi(z) \tag{7}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \phi(w), \tag{8}$$

where the function g is given by (4).

Definition 3. *A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_\Sigma^{a,1,a}(1, \gamma, \phi)$, $0 \leqq \gamma \leqq 1$, if the following subordinations hold:*

$$\left(\frac{zf'(z)}{(1-\gamma)z + \gamma f(z)}\right) \prec \phi(z) \tag{9}$$

and

$$\left(\frac{wg'(w)}{(1-\gamma)w+\gamma g(w)}\right) \prec \phi(w), \tag{10}$$

where the function g is given by (4).

It is easy to see that setting $\gamma = 1$ in Definition 1 leads us to Definition 2 and putting $\mu = 0$ and $\lambda = 1$ in Definition 1 leads us to Definition 3.

We shall mention that by suitably choosing $\phi(z)$, the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi)$ reduces to interesting and important special cases. Let us give some examples.

Example 1. *If we set $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, then the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi) \equiv \mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; A, B)$ which is defined as $f \in \Sigma$,*

$$(1-\lambda)\left(\frac{(1-\gamma)z+\gamma f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{(1-\gamma)z+\gamma f(z)}{z}\right)^{\mu-1} \prec \frac{1+Az}{1+Bz} \tag{11}$$

and

$$(1-\lambda)\left(\frac{(1-\gamma)w+\gamma g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{(1-\gamma)w+\gamma g(w)}{w}\right)^{\mu-1} \prec \frac{1+Aw}{1+Bw}, \tag{12}$$

where the function g is given by (4).

Example 2. *Letting $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, then the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi) \equiv \mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \beta)$ which is defined as $f \in \Sigma$,*

$$\Re\left((1-\lambda)\left(\frac{(1-\gamma)z+\gamma f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{(1-\gamma)z+\gamma f(z)}{z}\right)^{\mu-1}\right) > \beta \tag{13}$$

and

$$\Re\left((1-\lambda)\left(\frac{(1-\gamma)w+\gamma g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{(1-\gamma)w+\gamma g(w)}{w}\right)^{\mu-1}\right) > \beta, \tag{14}$$

where the function g is given by (4).

Example 3. *If we put $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, $0 < \alpha \leq 1$, then the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi) \equiv \mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \alpha)$ which is defined as $f \in \Sigma$,*

$$\left|\arg\left((1-\lambda)\left(\frac{(1-\gamma)z+\gamma f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{(1-\gamma)z+\gamma f(z)}{z}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \tag{15}$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{(1-\gamma)w+\gamma g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{(1-\gamma)w+\gamma g(w)}{w}\right)^{\mu-1}\right)\right|<\frac{\alpha\pi}{2}, \tag{16}$$

where the function g is given by (4).

In 1967, Lewin [6] investigated the class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [1] conjectured that $|a_2| \leqq \sqrt{2}$. On the other hand, Netanyahu [10] showed that

$$\max_{f\in\Sigma}|a_2| = \frac{4}{3}.$$

Afterwards in 1981, Styer and Wright [18] showed that there exist functions $f(z) \in \Sigma$ for which $|a_2| > \frac{4}{3}$. The best known estimate for functions in Σ has been obtained in 1984 by Tan [19], that is, $|a_2| \leqq 1.485$. The coefficient estimate problem involving the bound of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\})$ for each $f \in \Sigma$ given by (1) is still an open problem.

Recently, many researchers [4, 5, 7, 14, 15, 16, 17, 20, 21], following the work of Brannan and Taha [2], introduced and investigated a lot of interesting subclasses of the bi-univalent function class Σ and they obtained non-sharp estimates of the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

In this paper, we derive estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions belonging to the unifying subclass $\mathcal{N}_{\Sigma}^{\mu}(\lambda,\gamma;\phi)$ of Σ. Several connections to earlier known results are made.

The following lemma [3] will be required in order to derive our main results.

Lemma 1. *If $h \in \mathcal{P}$, then $|c_k| \leqq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\left(h(z)\right) > 0, \qquad (z \in \mathbb{U}),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

2. Coefficient Bounds for the functions class $\mathcal{N}_{\Sigma}^{\mu}(\lambda,\gamma;\phi)$

We begin by finding the estimates on the coefficient $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda,\gamma;\phi)$.

Let ϕ be an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0) = 1$, $\phi'(0) > 0$, and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the following form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \qquad (B_1 > 0,\ z \in \mathbb{U}). \tag{17}$$

Define the functions p_1 and p_2 in $\mathcal{P}$ given by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{18}$$

and

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots. \tag{19}$$

It follows that

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots \tag{20}$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{d_1}{2}z + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)z^2 + \cdots. \tag{21}$$

Using (20) and (21) with (17) lead us to

$$\phi(u(z)) = 1 + \frac{B_1 c_1}{2}z + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)B_1 + \frac{1}{4}c_1^2 B_2\right\}z^2 + \cdots \tag{22}$$

and

$$\phi(v(z)) = 1 + \frac{B_1 d_1}{2}z + \left\{\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)B_1 + \frac{1}{4}d_1^2 B_2\right\}z^2 + \cdots. \tag{23}$$

The following coefficient estimates hold for functions in the class $\mathcal{N}_\Sigma^\mu(\lambda,\gamma;\phi)$.

Theorem 2. *Let $f(z) \in \mathcal{N}_\Sigma^\mu(\lambda,\gamma;\phi)$ be of the form* (1). *Then*

$$|a_2| \leqq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|B_1^2\Omega(\mu,\lambda,\gamma) + 2(B_1 - B_2)(2\lambda - \gamma\lambda + \mu\gamma)^2\right|}} \tag{24}$$

and

$$|a_3| \leqq \frac{B_1^2}{(2\lambda - \gamma\lambda + \mu\gamma)^2} + \frac{B_1}{|3\lambda + \mu\gamma - \gamma\lambda|} \tag{25}$$

where

$$\Omega(\mu,\lambda,\gamma) = 4\mu\gamma\lambda - 6\gamma\lambda - 2\mu\gamma^2\lambda + 2\lambda\gamma^2 - \mu\gamma^2 + \mu^2\gamma^2 + 6\lambda + 2\mu\gamma \tag{26}$$

and the coefficients B_1 *and* B_2 *are given as in* (17).

Proof. Let $f \in \mathcal{N}_\Sigma^\mu(\lambda,\gamma;\phi)$. Then there are analytic functions $u,\ v\ : \mathbb{U} \to \mathbb{U}$, with $u(0) = v(0) = 0$, satisfying

$$(1-\lambda)\left(\frac{(1-\gamma)z + \gamma f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{(1-\gamma)z + \gamma f(z)}{z}\right)^{\mu-1} \prec \phi(z) \tag{27}$$

and

$$(1-\lambda)\left(\frac{(1-\gamma)w + \gamma g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{(1-\gamma)w + \gamma g(w)}{w}\right)^{\mu-1} \prec \phi(w), \tag{28}$$

where $g(w) := f^{-1}(w)$.
Now, equating the coefficients in (22), (23), (27) and (28), we obtain

$$(2\lambda - \gamma\lambda + \mu\gamma)\ a_2 = \frac{B_1 c_1}{2}, \tag{29}$$

$$\frac{1}{2}\gamma(\mu-1)(4\lambda + \mu\gamma - 2\gamma\lambda)\ a_2^2 + (3\lambda + \mu\gamma - \gamma\lambda)\ a_3$$

$$= \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4}c_1^2 B_2, \tag{30}$$

$$-(2\lambda - \gamma\lambda + \mu\gamma)\ a_2 = \frac{B_1 d_1}{2}, \tag{31}$$

and

$$\frac{1}{2}\gamma(\mu-1)(4\lambda + \mu\gamma - 2\gamma\lambda)\ a_2^2 + (3\lambda + \mu\gamma - \gamma\lambda)\ (2a_2^2 - a_3)$$

$$= \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4}c_1^2 B_2, \tag{32}$$

From (29) and (31), we find that

$$c_1 = -d_1. \tag{33}$$

Adding (30) and (32) and then using (33), we get

$$(4\mu\gamma\lambda - 6\gamma\lambda - 2\mu\gamma^2\lambda + 2\lambda\gamma^2 - \mu\gamma^2 + \mu^2\gamma^2 + 6\lambda + 2\mu\gamma)a_2^2$$

$$= \frac{c_1^2}{2}(B_2 - B_1) + \frac{B_1}{2}(c_2 + d_2). \tag{34}$$

For the sake of brevity, we will use the notation given in (26).
Now, using the notation defined above and combining (29) and (34), we obtain

$$a_2^2 = \frac{B_1^3(c_2 + d_2)}{2\left[B_1^2\Omega(\mu,\lambda,\gamma) + 2(B_1 - B_2)(2\lambda - \gamma\lambda + \mu\gamma)^2\right]}. \tag{35}$$

Applying Lemma 1 for the coefficients c_2 and d_2, we find

$$|a_2|^2 \leqq \frac{2B_1^3}{\left[B_1^2\Omega(\mu,\lambda,\gamma) + 2(B_1 - B_2)(2\lambda - \gamma\lambda + \mu\gamma)^2\right]} \tag{36}$$

and thus

$$|a_2| \leqq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|B_1^2\Omega(\mu,\lambda,\gamma) + 2(B_1 - B_2)(2\lambda - \gamma\lambda + \mu\gamma)^2\right|}}, \tag{37}$$

where $\Omega(\mu,\lambda,\gamma)$ is given by (26).

Similarly, upon subtracting (32) from (30), we get

$$2(3\lambda + \mu\gamma - \gamma\lambda)(a_3 - a_2^2) = \frac{1}{2}B_1(d_2 - c_2). \tag{38}$$

It follows from (29) and (38) that

$$a_3 = \frac{B_1^2\, c_1^2}{4\,(2\lambda - \gamma\lambda + \mu\gamma)^2} + \frac{B_1(d_2 - c_2)}{4(3\lambda + \mu\gamma - \gamma\lambda)}. \tag{39}$$

Finally, applying Lemma 1 for the coefficients c_1, c_2 and d_2, we readily obtain

$$|a_3| \leqq \frac{B_1^2}{(2\lambda - \gamma\lambda + \mu\gamma)^2} + \frac{B_1}{|(3\lambda + \mu\gamma - \gamma\lambda)|}. \tag{40}$$

3. Corollaries and Consequences

This section is devoted to the presentation of some interesting special cases of Theorem 1.

Let $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, $0 < \alpha \leqq 1$ ($B_1 = 2\alpha,\ B_2 = 2\alpha^2$), in Theorem 1. Then, the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi)$ reduces to $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \alpha)$ given in Example 3 and thus, we get the following corollary:

Corollary 3. *Let $f(z) \in \mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \alpha)$ be of the form (1). Then*

$$|a_2| \leqq \frac{2\alpha}{\sqrt{\left|\alpha\Omega(\mu, \lambda, \gamma) - (\alpha - 1)(2\lambda - \gamma\lambda + \mu\gamma)^2\right|}} \tag{41}$$

and

$$|a_3| \leqq \frac{4\alpha^2}{(2\lambda - \gamma\lambda + \mu\gamma)^2} + \frac{2\alpha}{|3\lambda + \mu\gamma - \gamma\lambda|} \tag{42}$$

where $\Omega(\mu, \lambda, \gamma)$ is given by (26).

Now, if we set $\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z}$, $0 \leqq \beta < 1$ ($B_1 = B_2 = 2 - 2\beta$), in Theorem 1, then the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi)$ reduces to $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \beta)$ given in Example 2 and then we obtain the following corollary:

Corollary 4.

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{|\Omega(\mu, \lambda, \gamma)|}} \tag{43}$$

and

$$|a_3| \leqq \frac{4(1-\beta)^2}{(2\lambda - \gamma\lambda + \mu\gamma)^2} + \frac{2(1-\beta)}{|3\lambda + \mu\gamma - \gamma\lambda|} \tag{44}$$

where $\Omega(\mu, \lambda, \gamma)$ is given by (26).

Numerous other (presumably new) corollaries and consequences of our main result can also be deduced by specializing the different parameters involved in the class $\mathcal{N}_{\Sigma}^{\mu}(\lambda, \gamma; \phi)$. For example, letting $\lambda = 1$ in Theorem 1 leads us to the following corollary:

Corollary 5. *Let* $f(z) \in \mathcal{N}_{\Sigma}^{\mu}(1, \gamma; \phi)$ *be of the form* (1)*. Then*

$$|a_2| \leqq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|B_1^2\left[6+\gamma(\mu-1)\Big(\gamma(\mu-2)+6\Big)\right]+2(B_1-B_2)(2-\gamma+\mu\gamma)^2\right|}} \tag{45}$$

and

$$|a_3| \leqq \frac{B_1^2}{(2-\gamma+\mu\gamma)^2}+\frac{B_1}{(3+\mu\gamma-\gamma)} \tag{46}$$

where the coefficients B_1 *and* B_2 *are given as in* (17)*.*

The class $\mathcal{N}_{\Sigma}^{\mu}(1, \gamma; \phi)$ is explicitly defined as follows:

Definition 4. *A function* $f \in \Sigma$ *is said to be in the class* $\mathcal{N}_{\Sigma}^{\mu}(1, \gamma; \phi)$*,* $\mu \geqq 0$ *and* $0 \leqq \gamma \leqq 1$*, if the following subordinations hold:*

$$f'(z)\left(\frac{(1-\gamma)z+\gamma f(z)}{z}\right)^{\mu-1} \prec \phi(z) \tag{47}$$

and

$$g'(w)\left(\frac{(1-\gamma)w+\gamma g(w)}{w}\right)^{\mu-1} \prec \phi(w), \tag{48}$$

where the function g *is given by* (4)*.*

Obviously, by setting $\gamma = 1$ in Theorem 1, we recover the result obtained by Srivastava *et al.* [13]. Also, letting $\mu = 0$ and $\lambda = 1$ in Theorem 1, we find the result given recently by Peng *et al.* [11].

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