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ON A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper, we introduce and study a subclass of harmonic univalent functions defined by convolution and integral convolution. Coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combination are determined for functions in this family.

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1. Introduction

A continuous function f is said to be a complex-valued harmonic function in a simply connected domain D in complex plane $\mathbb{C}$ if both real part of f and imaginary part of f are real harmonic in D. Such functions can be expressed as

$$f = h + \bar{g} \tag{1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D, see [3].

Every harmonic function $f = h + \bar{g}$ is uniquely determined by the coefficients of power series expansions in the unit disk $U = \{z : |z| < 1\}$ given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in U, |b_1| < 1, \tag{2}$$

where $a_n \in \mathbb{C}$ for $n = 2, 3, 4, \ldots$ and $b_n \in \mathbb{C}$ for $n = 1, 2, 3, \ldots$. For further information about these mappings, one may refer to [1, 3, 5, 8, 10, 11].

In 1984, Clunie and Sheil-Small [3] studied the family S_H of all univalent sense-preserving harmonic functions f of the form (1) in U, such that h and g are represented by (2). Note that S_H reduces to the well-known family S, the class of all normalized analytic univalent functions h given in (2), whenever the co-analytic part g of f is zero. Let K and K_H denote the respective subclasses of S and S_H where the images of $f(U)$ are convex.

The convolution of two functions of the form

$$\Phi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n \text{ and } \Psi(z) = z + \sum_{n=2}^{\infty} \nu_n z^n, \ \ \mu_n, \nu_n \geq 0 \tag{3}$$

is given by

$$(\Phi * \Psi)(z) = \Phi(z) * \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n \nu_n z^n$$

and the integral convolution is defined by

$$(\Phi \diamond \Psi)(z) = \Phi(z) \diamond \Psi(z) = z + \sum_{n=2}^{\infty} \frac{\mu_n \nu_n}{n} z^n$$

Towards the end of last century, Jahangiri [8], Frasin [7], Silverman [10], and Silverman and Silvia [11] were amongst those who focused on the harmonic starlike functions. Later Ozturk S. et. al [9] defined the class $S_H^*(\lambda, \alpha)$ consisting of functions $f = h + \bar{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{4}$$

which satisfy the condition

$$Re\left\{ \frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} \right\} \geq \alpha,$$

for some $0 \leq \alpha < 1, \ 0 \leq \lambda \leq 1$ and for all $z \in U$.

Let $S_H^*(\Phi, \Psi, \lambda, \alpha)$ denote the subclass of S_H of functions of the form $f = h + \bar{g} \in S_H$ that satisfy the condition

$$Re\left\{ \frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{\lambda(h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}) + (1-\lambda)(h(z) \diamond \Phi(z) + \overline{g(z) \diamond \Psi(z)})} \right\} \geq \alpha, \tag{5}$$

where $0 \leq \alpha < 1, \ 0 \leq \lambda \leq 1$ and Φ, Ψ are as given in (3). We further let $TS_H^*(\Phi, \Psi, \lambda, \alpha)$ denote the subclass of $S_H^*(\Phi, \Psi, \lambda, \alpha)$ consisting of functions $f = h + \bar{g} \in S_H$ such that h and g are of the form (4). We note that the family

$TS^*_H(\Phi,\Psi,\lambda,\alpha)$ is of special interest because it contains various classes of well-known harmonic univalent functions as well as many new ones. For different choice of Φ,Ψ,λ and α we obtain following various classes introduced by other authors:

(i) $TS^*_H(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2},\lambda,\alpha)=TS^*_H(\lambda,\alpha)$ (see Ozturk et al.[9]).

(ii) $TS^*_H(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2},0,\alpha)=TS^*_H(\alpha)$ (see Jahangiri [8]).

(iii) $TS^*_H(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2},0,0)=TS^*_H$ (see Silverman et al. [11]).

(iv) $TS^*_H(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2},0,0)=TS^{*0}_H$ (see Avci et al.[2] and Silverman [10]).

(v) $TS^*_H(\frac{z+z^2}{(1-z)^3},\frac{z+z^2}{(1-z)^3},0,\alpha)=K^*_H(\alpha)$ (see Jahangiri [8]).

(vi) $TS^*_H(\frac{z+z^2}{(1-z)^3},\frac{z+z^2}{(1-z)^3},0,0)=K^{*0}_H$ (see Silverman [10]).

(vii) $TS^*_H(\Phi,\Psi,0,\alpha)=TS^*_H(\Phi,\Psi,\alpha)$ (see Dixit et al.[4]).

(viii) $TS^*_H(\Phi,\Psi,0,\alpha)=\overline{HST}(\phi,\chi,0,\alpha)$ (see El-Ashwah[6] and Dixit et al.[4])

In this paper, we obtain coefficient bounds for the subclasses $S^*_H(\Phi,\Psi,\lambda,\alpha)$ and $TS^*_H(\Phi,\Psi,\lambda,\alpha)$, we also obtain distortion bounds, extreme points, convolution conditions, and convex combination for functions in $TS^*_H(\Phi,\Psi,\lambda,\alpha)$.

2. Main Results

We begin with a sufficient condition for functions in $S^*_H(\Phi,\Psi,\lambda,\alpha)$.

Theorem 1. *Let $f=h+\bar{g}$ be of the form (2). Furthermore, let*

$$\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_n|+\sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_n|\leq 1, \quad (6)$$

where $0\leq\alpha<1$, $0\leq\lambda\leq 1$, $n^2(1-\alpha)\leq\mu_n[n-(1+\alpha)(\lambda n-\lambda+1)]\leq\nu_n[n-(1+\alpha)(\lambda n+\lambda-1)]$. Then f is sense-preserving harmonic univalent in U and for $\lambda\leq\frac{1-\alpha}{1+\alpha}$, $f\in S^*_H(\Phi,\Psi,\lambda,\alpha)$.

Proof. We first note that f is sense-preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1-\sum_{n=2}^{\infty} n|a_n| \\
&\geq 1-\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_n| \\
&\geq \sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_n| > \sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_n||z|^{n-1} \\
&\geq \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|,
\end{aligned}$$

where we have used hypothesis of the theorem.

Now to show that f is univalent in U , suppose $z_1, z_2 \in U$ so that $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1-\left|\frac{\sum\limits_{n=1}^{\infty} b_n(z_1^n-z_2^n)}{(z_1-z_2)+\sum\limits_{n=2}^{\infty} a_n(z_1^n-z_2^n)}\right| \\
&> 1-\frac{\sum\limits_{n=1}^{\infty} n|b_n|}{1-\sum\limits_{n=2}^{\infty} n|a_n|} \\
&\geq 1-\frac{\sum\limits_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_n|}{1-\sum\limits_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_n|} \\
&\geq 0.
\end{aligned}$$

Now, we show that $f \in S_H^*(\Phi, \Psi, \lambda, \alpha)$. By using the fact that $Re(w) > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, it suffices to show that,

$$|(1-\alpha)B(z)+A(z)| - |(1+\alpha)B(z)-A(z)| > 0, \tag{7}$$

where $A(z) = h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}$ and $B(z) = \lambda A(z) + (1-\lambda)(h(z) \diamond \Phi(z) + \overline{g(z) \diamond \Psi(z)})$.

Substituting $A(z)$ and $B(z)$ in (7) as well as making use of (6) and $\lambda \leq \frac{1-\alpha}{1+\alpha}$, we obtain

$$
\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\
&= \left|[1+\lambda(1-\alpha)](h(z)*\Phi(z)-\overline{g(z)*\Psi(z)})+(1-\alpha)(1-\lambda)(h(z)\diamond\Phi(z)+\overline{g(z)\diamond\Psi(z)})\right| \\
&\quad -\left|[1-\lambda(1+\alpha)](h(z)*\Phi(z)-\overline{g(z)*\Psi(z)})-(1+\alpha)(1-\lambda)(h(z)\diamond\Phi(z)+\overline{g(z)\diamond\Psi(z)})\right| \\
&= \left|(2-\alpha)z+\sum_{n=2}^{\infty}\left[1+(1-\alpha)\left(\lambda+\frac{1-\lambda}{n}\right)\right]\mu_n a_n z^n\right. \\
&\quad \left.-\sum_{n=1}^{\infty}\left[1+(1-\alpha)\left(\lambda-\frac{1-\lambda}{n}\right)\right]\nu_n\overline{b_n z^n}\right| \qquad (\text{where } \nu_1=1) \\
&\quad -\left|-\alpha z+\sum_{n=2}^{\infty}\left[1-(1+\alpha)\left(\lambda+\frac{1-\lambda}{n}\right)\right]\mu_n a_n z^n\right. \\
&\quad \left.-\sum_{n=1}^{\infty}\left[1-(1+\alpha)\left(\lambda-\frac{1-\lambda}{n}\right)\right]\nu_n\overline{b_n z^n}\right| \qquad (\text{where } \nu_1=1) \\
&\geq 2(1-\alpha)|z|\left\{1-\sum_{n=2}^{\infty}\frac{n-\alpha-\alpha\lambda(n-1)}{n(1-\alpha)}\mu_n|a_n||z|^{n-1}\right. \\
&\quad \left.-\sum_{n=1}^{\infty}\frac{n+\alpha-\alpha\lambda(n+1)}{n(1-\alpha)}\nu_n|b_n||z|^{n-1}\right\} \\
&> 2(1-\alpha)|z|\left\{1-\sum_{n=2}^{\infty}\frac{n-\alpha-\alpha\lambda(n-1)}{n(1-\alpha)}\mu_n|a_n|\right. \\
&\quad \left.-\sum_{n=1}^{\infty}\frac{n+\alpha-\alpha\lambda(n+1)}{n(1-\alpha)}\nu_n|b_n|\right\} \\
&\geq 0 \quad from\ (6).
\end{aligned}
$$

The coefficient bound (6) is sharp for the functions

$$
f(z)=z+\sum_{n=2}^{\infty}\frac{n}{\mu_n}\left(\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}\right)x_n z^n+\sum_{n=1}^{\infty}\frac{n}{\nu_n}\left(\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}\right)\overline{y_n z^n},
$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1.$

Next, we show that the above sufficient condition is also necessary for functions in $TS_H^*(\Phi,\Psi,\lambda,\alpha)$.

Theorem 2. *Let $f = h + \bar{g}$ be of the form (4). Then $f \in TS_H^*(\Phi,\Psi,\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_n| + \sum_{n=1}^{\infty} \frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_n| \leq 1, \qquad (8)$$

where, $0 \leq \alpha < 1,\ 0 \leq \lambda \leq 1,\ n^2(1-\alpha) \leq \mu_n[n-(1+\alpha)(\lambda n - \lambda + 1)] \leq \nu_n[n-(1+\alpha)(\lambda n + \lambda - 1)]$.

Proof. The if part, follows from Theorem 1. To prove the only if part, let $f \in TS_H^*(\Phi,\Psi,\lambda,\alpha)$ then from (5) we have

$$Re\left\{\frac{h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}}{\lambda(h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}) + (1-\lambda)(h(z)\diamond\Phi(z) + \overline{g(z)\diamond\Psi(z)})} - \alpha\right\}$$

$$= Re\left\{\frac{(1-\alpha)z - \sum_{n=2}^{\infty}\mu_n \frac{[n-\alpha-\alpha\lambda(n-1)]}{n}|a_n|z^n - \sum_{n=1}^{\infty}\nu_n\frac{[n+\alpha-\alpha\lambda(n+1)]}{n}|b_n|\bar{z}^n}{z - \sum_{n=2}^{\infty}\mu_n\left[\lambda + \left(\frac{1-\lambda}{n}\right)\right]|a_n|z^n + \sum_{n=1}^{\infty}\nu_n\left[\left(\frac{1-\lambda}{n}\right) - \lambda\right]|b_n|\bar{z}^n}\right\}$$
$$> 0.$$

If we choose z to be real and $z \to 1^-$, we get

$$\frac{(1-\alpha) - \sum_{n=2}^{\infty}\mu_n\frac{[n-\alpha-\alpha\lambda(n-1)]}{n}|a_n| - \sum_{n=1}^{\infty}\nu_n\frac{[n+\alpha-\alpha\lambda(n+1)]}{n}|b_n|}{1 - \sum_{n=2}^{\infty}\mu_n\left[\lambda + \left(\frac{1-\lambda}{n}\right)\right]|a_n| + \sum_{n=1}^{\infty}\nu_n\left[\left(\frac{1-\lambda}{n}\right)-\lambda\right]|b_n|} \geq 0,$$

or, equivalently,

$$\sum_{n=2}^{\infty}\mu_n\frac{[n-\alpha-\alpha\lambda(n-1)]}{n}|a_n| + \sum_{n=1}^{\infty}\nu_n\frac{[n+\alpha-\alpha\lambda(n+1)]}{n}|b_n| \leq 1-\alpha,$$

which is the required condition (8).

In addition to the above main result, the following results are further properties concerning the class $TS_H^*(\Phi,\Psi,\lambda,\alpha)$. These results agree with previously obtained ones by other authors.

Theorem 3. *If $f \in TS_H^*(\Phi, \Psi, \lambda, \alpha)$ and $\mu_2(2-\alpha-\alpha\lambda) \leq \mu_n(n-\alpha-\alpha\lambda(n-1)) \leq \nu_n(n+\alpha-\alpha\lambda(n+1))$ for $n \geq 2$. Then we have,*

$$|f(z)| \leq (1+|b_1|)r + 2\left(\frac{(1-\alpha)}{\mu_2(2-\alpha-\alpha\lambda)} - \frac{1+\alpha-2\alpha\lambda}{\mu_2(2-\alpha-\alpha\lambda)}\nu_1|b_1|\right)r^2, \quad |z| = r < 1,$$

and

$$|f(z)| \geq (1-|b_1|)r - 2\left(\frac{(1-\alpha)}{\mu_2(2-\alpha-\alpha\lambda)} - \frac{1+\alpha-2\alpha\lambda}{\mu_2(2-\alpha-\alpha\lambda)}\nu_1|b_1|\right)r^2, \quad |z| = r < 1,$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted.

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&\leq (1+|b_1|)r + \frac{2(1-\alpha)}{\mu_2(2-\alpha-\alpha\lambda)}\sum_{n=2}^{\infty}\frac{\mu_2(2-\alpha-\alpha\lambda)}{2(1-\alpha)}(|a_n|+|b_n|)r^2 \\
&\leq (1+|b_1|)r + \frac{2(1-\alpha)}{\mu_2(2-\alpha-\alpha\lambda)}\times \\
&\quad \sum_{n=2}^{\infty}\left(\frac{\mu_n}{n}\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}|a_n| + \frac{\nu_n}{n}\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}|b_n|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{2(1-\alpha)}{\mu_2(2-\alpha-\alpha\lambda)}\left(1 - \frac{1+\alpha-2\alpha\lambda}{1-\alpha}\nu_1|b_1|\right)r^2 \\
&\leq (1+|b_1|)r + 2\left(\frac{(1-\alpha)}{\mu_2(2-\alpha-\alpha\lambda)} - \frac{1+\alpha-2\alpha\lambda}{\mu_2(2-\alpha-\alpha\lambda)}\nu_1|b_1|\right)r^2.
\end{aligned}$$

The upper bound given for $f \in TS_H^*(\Phi, \Psi, \lambda, \alpha)$ is sharp and equality occurs for the function

$$f(z) = z + |b_1|\bar{z} + 2\left(\frac{(1-\alpha)}{\mu_2(2-\alpha-\alpha\lambda)} - \frac{1+\alpha-2\alpha\lambda}{\mu_2(2-\alpha-\alpha\lambda)}\nu_1|b_1|\right)\bar{z}^2, \quad |b_1| \leq \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\nu_1}.$$

The following covering result follows from the left hand inequality in Theorem 3.

Corollary 4. *Let* $f \in TS_H^*(\Phi, \Psi, \lambda, \alpha)$*, then*

$$\left\{ w : |w| < \frac{1}{A}[A - (1-\alpha) + ((1+\alpha-2\alpha\lambda)\nu_1 - A)|b_1|] \right\} \subset f(U),$$

where $A = \frac{\mu_2}{2}(2-\alpha-\alpha\lambda)$.

Now we determine the extreme points of $TS_H^*(\Phi, \Psi, \lambda, \alpha)$

Theorem 5. *Let*

$$h_1(z) = z, \;\; h_n(z) = z - \frac{n}{\mu_n}\left(\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}\right) z^n \quad (n = 2, 3, ...)$$

and

$$g_n(z) = z + \frac{n}{\nu_n}\left(\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}\right) \bar{z}^n \quad (n = 1, 2...).$$

Then $f \in TS_H^*(\Phi, \Psi, \lambda, \alpha)$ *if and only if it can be expressed as*

$$f(z) = \sum_{n=1}^{\infty} x_n h_n + y_n g_n,$$

where $x_n \geq 0, y_n \geq 0, x_1 = 1 - \sum\limits_{n=2}^{\infty} x_n + y_n \geq 0,$ *and* $y_1 = 0$*. In particular, the extreme points of* $TS_H^*(\Phi, \Psi, \lambda, \alpha)$ *are* h_n *and* g_n*.*

Proof. Suppose

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} x_n h_n + y_n g_n \\ &= \sum_{n=1}^{\infty}(x_n + y_n)z - \sum_{n=2}^{\infty}\frac{n}{\mu_n}\left(\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}\right) x_n z^n \\ &\quad + \sum_{n=1}^{\infty}\frac{n}{\nu_n}\left(\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}\right) y_n \bar{z}^n. \end{aligned}$$

Then

$$\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)\frac{n}{\mu_n}\left(\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}\right) x_n$$

$$+\sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)\frac{n}{\nu_n}\left(\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}\right)y_n$$

$$=\sum_{n=2}^{\infty}x_n+\sum_{n=1}^{\infty}y_n=1-x_1\leq 1$$

and so $f\in TS_H^*(\Phi,\Psi,\lambda,\alpha)$. Conversely, if $f\in TS_H^*(\Phi,\Psi,\lambda,\alpha)$, then

$$|a_n|\leq\frac{n}{\mu_n}\left(\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}\right)\text{ and }|b_n|\leq\frac{n}{\nu_n}\left(\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}\right).$$

Setting

$$x_n=\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_n|\quad(n=2,3...)$$

and

$$y_n=\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_n|\quad(n=1,2...).$$

Then note that by Theorem 2, $0\leq x_n\leq 1$ $(n=2,3...)$ and $0\leq y_n\leq 1$ $(n=1,2...)$. We define $x_1=1-\sum\limits_{n=2}^{\infty}x_n-\sum\limits_{n=1}^{\infty}y_n$, by Theorem 2 we obtain $f(z)=\sum\limits_{n=1}^{\infty}x_nh_n+y_ng_n$. This completes the proof of Theorem 5.

Next, we show that $TS_H^*(\Phi,\Psi,\lambda,\alpha)$ is closed under convex combinations of its members.

Theorem 6. *The class $TS_H^*(\Phi,\Psi,\lambda,\alpha)$ is closed under convex combination.*

Proof. For $i=1,2,3...$ let $f_i\in TS_H^*(\Phi,\Psi,\lambda,\alpha)$, where f_i is given by

$$f_i(z)=z-\sum_{n=2}^{\infty}|a_{i_n}|z^n+\sum_{n=1}^{\infty}|b_{i_n}|\bar{z}^n.$$

Then by Theorem 2,

$$\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_{i_n}|+\sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_{i_n}|\leq 1\qquad(9)$$

For $\sum\limits_{i=1}^{\infty}t_i=1,\ \ 0\leq t_i\leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{i_n}|\right)z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{i_n}|\right)\bar{z}^n.$$

Then by 6,

$$\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)\left|\sum_{i=1}^{\infty}t_i|a_{i_n}|\right|+\sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)\left|\sum_{i=1}^{\infty}t_i|b_{i_n}|\right|$$

$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_{i_n}|+\sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_{i_n}|\right\}$$
$$\leq 1\sum_{i=1}^{\infty}t_i=1.\ \ from\ (9)$$

and so by Theorem 2, we have $\sum\limits_{i=1}^{\infty}t_if_i(z)\in TS_H^*(\Phi,\Psi,\lambda,\alpha)$.

Finally we show that the class $TS_H^*(\Phi,\Psi,\lambda,\alpha)$ is invariant under convolution. For harmonic functions $f(z)=z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}b_n\bar{z}^n$ and $F(z)=z-\sum_{n=2}^{\infty}A_nz^n+\sum_{n=1}^{\infty}B_n\bar{z}^n$, we define the convolution of two harmonic functions f and F as

$$(f*F)(z)=f(z)*F(z)=z-\sum_{n=2}^{\infty}a_nA_nz^n+\sum_{n=1}^{\infty}b_nB_n\bar{z}^n.$$

Theorem 7. *If* $f\in TS_H^*(\Phi,\Psi,\lambda,\alpha)$ *and* $F\in TS_H^*(\Phi,\Psi,\lambda,\alpha)$ *then* $f*F\in TS_H^*(\Phi,\Psi,\lambda,\alpha)$.

Proof. Let $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^n$

and $F(z)=z-\sum_{n=2}^{\infty}|A_n|z^n+\sum_{n=1}^{\infty}|B_n|\bar{z}^n$

be in $TS_H^*(\Phi,\Psi,\lambda,\alpha)$, Then by Theorem 2, we have

$$\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|a_n|+\sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|b_n|\leq 1,$$

and

$$\sum_{n=2}^{\infty}\frac{\mu_n}{n}\left(\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}\right)|A_n|+\sum_{n=1}^{\infty}\frac{\nu_n}{n}\left(\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}\right)|B_n|\leq 1.$$

So for the coefficients of $f * F$ we can write

$$\sum_{n=2}^{\infty} \frac{\mu_n}{n} \left(\frac{n - \alpha - \alpha\lambda(n-1)}{1-\alpha} \right) |a_n A_n| + \sum_{n=1}^{\infty} \frac{\nu_n}{n} \left(\frac{n + \alpha - \alpha\lambda(n+1)}{1-\alpha} \right) |b_n B_n|$$

$$\leq \sum_{n=2}^{\infty} \frac{\mu_n}{n} \left(\frac{n - \alpha - \alpha\lambda(n-1)}{1-\alpha} \right) |a_n| + \sum_{n=1}^{\infty} \frac{\nu_n}{n} \left(\frac{n + \alpha - \alpha\lambda(n+1)}{1-\alpha} \right) |b_n| \leq 1.$$

Thus $f * F \in TS_H^*(\Phi, \Psi, \lambda, \alpha)$.

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