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LINEAR QUASI-MCCOY RINGS

M.J. Nikmehr, M. Deldar, H. Daneshmand

Abstract. In this paper, we introduce linear quasi-McCoy rings which are a generalization of weak quasi-Armendariz rings. It is shown, for a semiprime ring R, $\frac{R[x]}{(x^n)}$ and $R[x]$ are linear quasi-McCoy. Also, it is shown $M_n(R)$ is linear quasi-McCoy if R is linear quasi McCoy ring. Various properties of linear quasi-McCoy rings are also observed.

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1. Introduction

Throughout this paper R denote an associative ring with identity. Given a ring R, the polynomial ring with an indeterminate x over R is denoted by $R[x]$. Rege and Chhawchharia [18] introduced the notion of an Armendariz ring. A ring R is an Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0+b_1x+\cdots+b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i, j. The name Armendariz ring was chosen because Armendariz (1974) had shown that a reduced ring (i. e., a ring without nonzero nilpotent elements) satisfies this condition. Some properties of Armendariz rings have been studied in [1, 2, 9, 18, 16, 17]. According to Hirano [4], a ring is called quasi-Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_iRb_j = 0$ for all i, j.

Recall that a ring R is called *reversible* if $ab = 0$ implies $ba = 0$, for all a, $b \in R$. R is called *semicommutative* if for all a, $b \in R$, $ab = 0$ implies $aRb = 0$. In [15] has shown reduced rings are reversible and reversible rings are semicommutative, but the converse is not true in general. According to Nielsen [15], a ring R is called *right McCoy* (resp., *left McCoy*) if for any polynomials $f(x)$, $g(x) \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies $f(x)r = 0$ (resp., $sg(x) = 0$) for some $0 \neq r \in R$ (resp., for some $0 \neq s \in R$).

A ring is called *McCoy* if it is both left and right McCoy. By McCoy [14], commutative rings are McCoy rings. Reduced rings are Armendariz and Armendariz rings are McCoy. In [3] Baser and Kaynarca studied a generalization of quasi Armendariz rings, which is called weak quasi Armendariz.

A ring R is called weak quasi Armendariz if for $f(x) = a_0 + a_1x$, $g(x) = b_0 + b_1x \in R[x]$, $f(x)R[x]g(x) = 0$ implies $a_iRb_j = 0$ for all $0 \leq i, j \leq 1$. They showed $M_n(R)$, $T_n(R)$ and $R[x]$ over a weak quasi-Armendariz ring are too. Motivated by the above results, we investigate a generalization of weak quasi-Armendariz rings which we call a linear quasi-McCoy ring and study several results.

2. Linear Quasi-McCoy Rings

We begin this section by the following definition and also we study properties of linear quasi-McCoy rings.

Definition 1. *A ring R is called a* right linear quasi-McCoy ring *if for $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$ in $R[x]$, $f(x)R[x]g(x) = 0$ implies $f(x)Rs = 0$ for some nonzero $s \in R$. (i. e. $a_iRs = 0$ for $0 \leq i \leq 1$). Left linear quasi McCoy rings are defined analogously.*

The following lemma will be used very frequently in this paper.

Lemma 1. *[4, Lemma 2.1] Let $f(x)$ and $g(x)$ be two elements of $R[x]$. Then $f(x)R[x]g(x) = 0$ if and only if $f(x)Rg(x) = 0$.*

Clearly, any weak quasi-Armendariz ring is linear quasi-McCoy. In the following, we will see that the converse is not true.

Recall that for a ring R and an (R, R)-bimodule M, the trivial extention of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$, $m \in M$ and the usual matrix operations are used.

Example 1. *The commutative rings are linear quasi-McCoy but need not be weak quasi Armendariz. Consider the polynomial $f(x) = (\overline{4}, \overline{0}) + (\overline{4}, \overline{1})x$ over the ring $\frac{\mathbb{Z}}{8\mathbb{Z}} \oplus \frac{\mathbb{Z}}{8\mathbb{Z}}$. The square of this polynomial is zero but the product $(\overline{4}, \overline{0})(\overline{4}, \overline{1}) = (\overline{0}, \overline{4})$ is not zero.*

By [3, Theorem 2.7], if R is a semiprime ring, then R, $R[x]$, $S_n(R)$, $R[x]$, $\frac{R[x]}{(x^n)}$ and $V_n(R)$ (for $n \geq 2$) are weak quasi-Armendariz rings, and so linear quasi-McCoy rings.

Proposition 1. *Let R be a ring and Δ be a multiplicative closed subset of R consisting of central regular elements. Then R is linear quasi-McCoy if and only if $\Delta^{-1}R$ is linear quasi-McCoy.*

Proof. Let $S = \Delta^{-1}R$. Assume that S is linear quasi-McCoy. Let $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x \in R[x]$ such that $f(x)Rg(x) = 0$. For any $r \in R$ with $w \in \Delta$, $0 = w^{-1}f(x)rg(x) = f(x)(w^{-1}r)g(x)$. So we have $f(x)Sg(x) = 0$. Since S is linear quasi-McCoy, $a_iSu^{-1}c = 0$ for some nonzero $u^{-1}c \in S$ $(0 \leq i \leq 1)$, and so $a_iRc = 0$. Therefore, R is linear quasi-McCoy.

Conversely, suppose that R is linear quasi-McCoy. Let $F(x) = \alpha_0 + \alpha_1x$ and $G(x) = \beta_0 + \beta_1x \in S[x]$ such that $F(x)SG(x) = 0$, where $\alpha_i = u^{-1}a_i$ and $\beta_j = v^{-1}b_j$ with a_i, $b_j \in R$ and regular elements $u, v \in R$. Since Δ is contained in the center of R and $F(x)SG(x) = 0$, for any $w^{-1}r \in S$, we have

$$0 = u^{-1}(a_0 + a_1x)(w^{-1}r)v^{-1}(b_0 + b_1x) = (uvw)^{-1}(a_0 + a_1x)r(b_0 + b_1x).$$

Let $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$. Then $f(x),\ g(x) \in R[x]$ with $f(x)Rg(x) = 0$. Since R is linear quasi-McCoy, $a_iRc = 0$ for some nonzero $c \in R$ $(0 \leq i \leq 1)$.

This shows that $\alpha_iSv^{-1}c = 0$ $(0 \leq i \leq 1)$. Therefore, S is linear quasi-McCoy.

Corollary 2. *Let R be a ring. Then $R[x]$ is linear quasi-McCoy if and only if $R[x; x^{-1}]$ is linear quasi-McCoy.*

Proof. It follows directly from since $\Delta = \{1, x, x^2, \dots\}$ is clearly a multiplicatively closed subset of $R[x]$ and $R[x, x^{-1}] = \Delta^{-1}R[x]$.

Proposition 2. *Let e be a central idempotent of a ring R. If eR and $(1-e)R$ are linear quasi-McCoy, then R is linear quasi-McCoy.*

Proof. Suppose that both eR and $(1-e)R$ are linear quasi-McCoy. Let $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x \in R[x]$ with $f(x)R[x]g(x) = 0$. Then for any $r \in R$, $0 = e(f(x)rg(x)) = ef(x)(er)eg(x)$, $(ef(x) = f(x),\ g(x)e = g(x))$ and $(1-e)f(x)((1-e)r)(1-e)g(x) = 0$, and so $ef(x)(eR)[x]eg(x) = 0$ and $(1-e)f(x)((1-e)R)[x](1-e)g(x) = 0$.

Since eR and $(1-e)R$ are linear quasi-McCoy, for all i we have $ea_i(eR)ec = 0$ and $(1-e)a_i((1-e)R)(1-e)t = 0$ for some s, $t \in R$. Thus, $e(a_iRc) = 0$ and $(1-e)(a_iRt) = 0$ for all i, and hence $a_iRct = (1-e)a_iRct + e(a_iRct) = 0$. Therefore, R is linear quasi -McCoy.

For a nonempty subset S of a ring R, we write $r_R(S) = \{c \in R | Sc = 0\}$ and $\ell_R(S) = \{c \in R | cS = 0\}$, which are called the right and left annihilators of S in R, respectively.

Proposition 3. *If R is a linear quasi-McCoy and the one-sided annihilator A of a nonempty subset in R is a two-sided ideal of R, then R/A is linear quasi-McCoy.*

Proof. Let $A = r_R(S)$ be a two -sided ideal of a linear quasi-McCoy ring R for $\emptyset \neq S \subseteq R$. Let $\overline{a} = a + A$ for $a \in R$. Suppose $f(x) = \overline{a_o} + \overline{a_1}x$ and $g(x) = \overline{b_0} + \overline{b_1}x \in (R/A)[x]$ with $f(x)(R/A)[x]g(x) = \overline{0}$. From $f(x)(R/A)[x]g(x) = \overline{0}$, we get $f(x)\overline{r}g(x) = \overline{0}$ for any $\overline{r} \in R/A$. Hence, a_0rb_0, $a_0rb_1 + a_1rb_0$, $a_1rb_1 \in A$, and so $sa_0rb_0 = 0$, $s(a_0rb_1 + a_1rb_0) = 0$ and $sa_1rb_1 = 0$ for any $r \in R$ and $s \in S$. Thus, $(sa_0 + sa_1x)R[x](b_0 + b_1x) = 0$. Since R is linear quasi-McCoy, we have $s(a_iRt) = 0$ for some $t \in R$, for any i and $s \in S$, and hence $a_iRt \subseteq A$. Thus $\overline{a_i}(R/A)\overline{t} = \overline{0}$ for any i, and therefore R/A is linear quasi-McCoy.

In the following we will show that $M_n(R)$ and $T_n(R)$ over a linear quasi-McCoy ring R are linear quasi-McCoy.

Proposition 4. *For a ring R, we consider the following conditions:*

1. *R is linear quasi-McCoy.*
2. *$M_n(R)$ is linear quasi-McCoy for any $n \geq 1$.*
3. *$M_n(R)$ is linear quasi-McCoy for some $n \geq 1$.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. $(1) \Rightarrow (2)$ Let R be a linear quasi-McCoy ring. Note that $M_n(R)[x] \cong M_n(R[x])$. We let $f(x) = \sum_{i=0}^{1} A_ix^i,\ g(x) = \sum_{i=0}^{1} B_jx^j \in M_n(R[x])$ with $A_i = (a^i_{st})$ and $B_j = (b^j_{vw})$. We write $f(x) = (f_{st}),\ g(x) = (g_{vw}) \in M_n(R[x])$ with $f_{st} = \sum_{i=0}^{1} a^i_{st}x^i$ and $g_{vw} = \sum_{i=0}^{1} b^j_{vw}x^j$. Put $f(x)M_n(R)[x]g(x) = 0$, then equivalently, $f(x)M_n(R[x])g(x) = 0$. Let E_{ij} denote the matrix unit with (i, j)-entry 1 and zero elsewhere. From $f(x)(RE_{hk})g(x) = 0$, we get $f_{\alpha h}Rg_{k\beta} = 0$ for all $1 \leq \alpha, \beta \leq n$. Since R is linear quasi-McCoy, we have $a^i_{st}Rc_{st} = 0$ for some $c_{st} \in R$ and for all i and $1 \leq s,\ t \leq n$. Let

$$S = \begin{pmatrix} \prod_{i=1}^{n} c_{1i} & 0 & \cdots & 0 \\ 0 & \prod_{i=1}^{n} c_{2i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \prod_{i=1}^{n} c_{ni} \end{pmatrix}.$$

It then follows that $A_iM_n(R)S = 0$ for all i, concluding that $M_n(R)$ is linear quasi-McCoy.

$(2) \Rightarrow (3)$ is obvious.

Corollary 3. *Let R be a ring. If R is linear quasi-McCoy then $T_n(R)$ is linear quasi McCoy.*

Proposition 5. *Finite direct product of linear quasi-McCoy rings is linear quasi-McCoy.*

Proof. Let $R_1, R_2, \ldots, R_n$ be linear quasi McCoy rings and $R = \prod_{k=1}^{n} R_k$. Suppose that $f(x) = \sum_{i=0}^{1} a_i x^i$, $g(x) = \sum_{j=0}^{1} b_j x^j \in R[x] \setminus \{0\}$, such that $f(x)R[x]g(x) = 0$, where $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, $b_j = (b_{j1}, b_{j2}, \ldots, b_{jn}) \in R$. Set

$$f_k(x) = \sum_{i=0}^{1} a_{ik} x^i, \; g_k(x) = \sum_{j=0}^{1} b_{jk} x^j,$$

for each $1 \leq k \leq n$. Since $f_k(x)R[x]g_k(x) = 0$ and R_k is linear quasi-McCoy, there exists $s_k \in R_k$ such that $a_{ik}Rs_k = 0$. Let $s = (s_1, s_2, \ldots, s_n)$ then $a_iRs = 0$. Therefore R is linear quasi-McCoy.

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Mohammad Javad Nikmehr
Department of Mathematics,
K. N. Toosi University of Technology,
P.O. Box 16315 – 1618, Tehran, Iran
email: *nikmehr@kntu.ac.ir*

Mansoureh Deldar
Department of Mathematics, Islamic Azad University,
Central Tehran Branch, Iran
email: *Man.Deldar@iauctb.ac.ir*

Hosein Daneshmand
Shamsipour Technical College,
Technical and Vocational University, Tehran, Iran