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THE STRUCTURE OF $\mathbb{Q}$-GROUPS WITH IRREDUCIBLE ELEMENTS OF ORDER 2

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Abstract. A finite group whose irreducible complex characters are rational is called a $\mathbb{Q}$-group, and element of second order in the group is called irreducible if it cannot write the combination of two elements of second order. In this paper we will classify $\mathbb{Q}$-groups which elements of second order are irreducible.

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1. Introduction

One way of the effective for studying finite group is matrix representation theory specially characters theory. This theory concept was mastermind by Frobenius, and then have been developed by mathematician as Shor, Burnside and Brauer. For example can divide groups with regard to the their value character what they are field.

In this paper, G is a group finite and χ a complex character of G. The field generated by all $\chi(g)$ such that $g \in G$, is denoted by $\mathbb{Q}(\chi)$. By definition a complex character χ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$. A finite group is called a rational or a $\mathbb{Q}$-group if every irreducible complex character of G is rational. Examples of $\mathbb{Q}$-group are the dihedral group D_{2n} only for $n = 1, 2, 3, 4, 6$ and symmetric group S_n and quaternion group Q_8. Also it is shown in [3] that if G is a solvable $\mathbb{Q}$-group, then $\pi(G) \subseteq \{2, 3, 5\}$ where $\pi(G)$ denote the set of prime divisors of $|G|$. But classifying finite $\mathbb{Q}$-group still remains an open research problem. In the book [6] several open problems have been raised concerning $\mathbb{Q}$-group.

Throughout the paper, the semi- direct product of groups H and K is denoted by $H : K$, and a cyclic group of order n by C_n. Also, if p is a prime number, E_p denotes the elementary abelian p-group of order p^n and greatest normal subgroup of odd order is denoted by $O(G)$, and $N_G(< g >)$ the normalizer of $< g >$ in G, $C_G(< g >)$ the centralizer of $< g >$ in G.

2. Preliminaries

Definition 1. *Let G be a group and I is the set of members second order in G. $g \in G$ is called a member of irreducible second order if $I_g = \{g' \in I : gg' \in I\} = \varnothing$.*

Definition 2. *Let G be a group. $g \in G$ is called rational if all generators cyclic subgroup $< g >$ conjugate together in G.*

Before starting our main theorem, we will mention some well-known results about $\mathbb{Q}$-groups. An alternative characterization of $\mathbb{Q}$-group is the following result which can be found in [5].

Result 1. A group G is $\mathbb{Q}$-groups if and only if for every $g \in G$, of order n the members g and g^m are conjugation in G, whenever $(m, n) = 1$. Equivalently is for each $g \in G$ we must have:

$$\frac{N_G(< g >)}{C_G(< g >} \cong Aut(< g >)$$

Also, in [7] it is proved, if p be a prim number then

$$\frac{N_G(< g >)_p}{C_G(< g >)_p} \cong Aut(< g >)_p$$

Result 2. Quotients and direct products of $\mathbb{Q}$-groups are $\mathbb{Q}$-groups.

Theorem 1. *Let G be a finite 2-group such that only has a member of second order than G is a cyclic group or generalized quaternion group.*

Proof. [1] (theorem 6.19).

Lemma 1. *Let G be a $\mathbb{Q}$-group then there is an member of irreducible second order in G if and only if that sylow 2-subgroup be C_2 or Q_8. (i.e members of irreducible second order f a $\mathbb{Q}$-groups is a type of C_2 or Q_8).*

Proof. Let g be member of irreducible second order in G and $P \in Syl_2(G)$, then $g \in P$. Since G is $\mathbb{Q}$-group and $P \in Syl_2(G)$ therefore $Z(P)$ is an elementary abelian 2-group and $Z(P) \leq C_G(< g >)$, hence g is the only member of second order in $C_G(< g >)$, thus $Z(P) =< g >$, therefore $P \leq C_G(< g >)$. So P is a 2-group and only has an member of second order. Therefore by theorem 1, P is cyclic group C_{2^n} or generalized quaternion group Q_{2^n}.

If $P = C_{2^n}$ then $n = 1$, because $Z(P)$ is a elementary abelian 2-group, therefore $P = C_2$. But if $P = Q_{2^n}$ then there is generators a, b such that $b^{-1}ab = a^{-1}$, $a^{2^{n-1}} = 1$, $a^{2^{n-2}} = b$. By Result 1, we have,

$$[N_G(< a >) : C_G(< a >)] =| Aut(< a >) |= \phi(|a|) = 2^{n-2}.$$

Also by part two of result 1, and above relation we have

$$|N_G(< a >)_2| = 2^{n-2}|C_G(< a >)_2| \geq 2^{n-2}|a| = 2^{n-2} \times 2^{n-1} = 2^{2n-3}.$$

Since $P = Q_{2^n}$ and $N_G(< g >)_2 \leq Q_{2^n}$, therefore $2^{2n-3} \leq 2^2$, and hence $n \leq 3$. Furthermore we defined generalized quaternion for $n \geq 3$, so it is $P = Q_8$.

Conversely: Let C_2 or Q_8 be in $Syl_2(G)$ and x be member of second order in P. If $g \neq 1$ and $g \neq x$ be an other element of second order in $C_G(< x >)$ then $< x > \times < g >$ is 2-subgroup in G. Therefore by theorem sylow, $< x > \times < g >$ in P or conjugation of P. If $P = C_2$, then $< x > \times < g >$ is in C_2 which is a contradiction because $< x > \times < g >$ is a member of four order that is not in C_2. But if $P = Q_8$ then a^2 is only member of second order in Q_8, which a is a generator Q_8, thus $a^2 \notin < x >$, therefore $< x > \times < g > \notin Q_8$, i.e $< x >$ is not product to members of second order. Therefore x is irreducible.

Lemma 2. *Let G be a $\mathbb{Q}$-group with member of irreducible second order of type Q_8. If $O(G)$ be abelian then $O(G)$ is elementary abelian p-group, such that $p = 3$ or $p = 5$.*

Proof. Since $|O(G)| = 3^n \times 5^m$. But 3 and 5 dose not appear together in $|O(G)|$. Because otherwise, there are members x, y in $O(G)$ such that $|x| = 3$, $|y| = 5$, Since $O(G)$ is abelian so $(|x|, |y|) = 1$, therefore $x.y$ is a member of 15 order in $O(G)$. Since G is a $\mathbb{Q}$-group, by result 1 we have

$$[N_G(< xy >)_2 : C_G(< xy >)_2] = |Aut(< xy >)_2| = \phi(15) = 8 = 2^3.$$

Since Q_8 is sylow 2-group in G, therefore $\frac{N_G(<xy>)_2}{C_G(<xy>)_2} \cong Q_8$ which this is a contradiction. because Q_8 is not abelian but $Aut(< xy >)$ is abelian. Therefore $O(G)$ is a 3-group or 5-group. Since $O(G)$ is abelian thus $O(G)$ is the elementary ableian p-group, such that $p = 3$, $p = 5$.

Lemma 3. *Let G be a $\mathbb{Q}$-group with member of irreducible second order of type Q_8. Then G has a elementary abelian normal p-group as E_p, such that $G \cong E_p : Q_8$ ($p = 3$, $p = 5$).*

Proof. Since G is a $\mathbb{Q}$-group with members of irreducible second order of type Q_8. Then N has a normal subgroup similar N such that $G \cong N : Q_8$ and $|N| = 3^n \times 5^m$. Therefore with supposed that $N = O(G)$, furthermore $O(G)$ is abelian group, by lemma 2, then N is an elementary abelian p-group. We know $N = O(G)$ is a normal in G. Thus N is a E_p. Therefore $G \cong E_p : Q_8$ $(p = 3, p = 5)$

Lemma 4. *G is a $\mathbb{Q}$-group with members of irreducible second order of type C_2 if and only if $G \cong E_3 :< g >$ such that E_3 may by trivial and g be the member of second order which inverse every member of E_3.*

Proof. Let $P =< g >$ then P is abelian group. Since G is a $\mathbb{Q}$-group therefore $G = G'P$ such that G' is 3-group. Now we will show G' is elementary abelian 3-group and g is a member of second order which inverse every the member G'. Let $a \in G'$ and $a \neq 1$. Since G' is 3-group so $|a| = 3^n$. Since G is a $\mathbb{Q}$-goup, by result 1, we have

$$[N_G(< a >)_r 2 : C_G(< a >)_2] = |Aut(< a >)_2| = \phi(|a|) = 2.$$

Since $|P| = 2$, so $C_G(< a >)_2 =< 1 >$. Therefore $a^g = gag^{-1} = gag \neq a$. Now we define of the following function:

$$f : G' \to G'$$

$$f(a) \to a^g = gag$$

It is clear, f is a automorphism such that any member does not fixed. Since $|g| = 2$ then $|f| = 2$. Since f is a endomorphism without fixed point of second order. Therefore by theorem 4.1.10 of [4], G' is abelian group and f project every member of G' it is inverse. Since G' is abelian group then $Z(G') = G'$. Furthermore G' is a 3-sylow subgroup in G therefore $Z(G')$ is a elementary abelian 3-group. (namely G' is a elementary abelian 3-group).

Conversely: Let $G = E_3 :< g >$ and E_3 be a elementary abelian 3-group therefore for all $x \in G$, we have $x = ag$ such that $a \in E_3$, than $x^2 = (ag)^2 = (ag)(ag) = a(gag) = aa^{-1} = 1$ (because, we define that endomorphism f in above is without fixed point of second order and project every member it is inverse).

Also $g \neq 1$. According to above we have if the member be form product the member of E_3 in G then is a member of second order and only the member which their order are unequal 2, they are in E_3. Since E_3 is a elementary abelian 3-group, and also, G is a $\mathbb{Q}$-group because generators cyclic group are conjugate. *(Because with hypothesis $a \in E_3$, thus a and a^{-1} is a generators $< a >$, such that a and a^{-1} conjugate. By definition 2, G is a $\mathbb{Q}$-group).

Now prove g is member of irreducible second order. Since G is a $\mathbb{Q}$-group and the order each $\mathbb{Q}$-group is even, so g is only member of second order in $C_G(< g >)$. Thus that is enough which show $C_G(< g >) =< g >$.

If $x = ag$ such that $a \in E_3$ and $x \in C_G(< g >)$, then with regrded to *, we have

$$xg = gx \Rightarrow (ag)g = g(ga) \Rightarrow a = gag \Rightarrow a = a^{-1} \Rightarrow a = 1 \Rightarrow x = g.$$

Since x was a member of arbitrary of $C_G(< g >)$ thus $C_G(< g >) = (< g >)$. Therefore g is a member of second order such that $C_2 \cong< g >$. Then by 2.6, g is a member of irreducible second order. Also $g \neq 1$ because, if $g = 1$ then $ag = ga$, and therefore $a^2 = 1$, which is a contradiction ($a \in E_3$).

Remark 1. *Copy of 2-dimension irreducible representation Q_8 on field C_3 namely irreducible FG-submodule which analogous with irreducible representation $\rho : Q_8 \to GL(2, C_3)$.*

3. Main Theorem

Theorem 2. *Let G be a $\mathbb{Q}$-group with members of irreducible second order then for G exactly one of the following occurs:*

(1) If members of second order of type C_2 then $G \cong E_3 : C_2$, such that E_3 is a elementary abelian 3-group and C_2 inverse every members E_3.

(2) If members of second order are type of Q_8 then one of the following possibilities holds:

- *(i) $G \cong E_3 : Q_8$, such that E_3 is trivial or direct addition copies of 2-dimension irreducible representation Q_8 on field C_3*
- *(ii) $G \cong (C_5 \times C_5) : Q_8$, such that action Q_8 over $C_5 \times C_5$ is to shape 2-dimension irreducible representation Q_8 on field C_5.*

Proof. **(1):** It is similar to Lemma 4.

(2-i): G is a $\mathbb{Q}$-group with members of irreducible second order of type Q_8. Therefore by 2.8, G has an elementary abelian normal p-group is similar to E_p, such that $G = E_p : Q_8$ and $p = 3$ or $p = 5$.

Suppose Q_8 acts on E_p by conjugately therefore there is a homomorphism $\rho : Q_8 \to Aut(E_p)$, $i \mapsto \rho_i$ such that $\rho_i : E_p \to E_p$, $x \mapsto x^i$.

Also, $Aut(E_p) \cong GL(V) \cong GL(n.F)$, therefore $\rho : Q_8 \to GL(n, F)$ is a representations on field F with p members. Of course V^+ space is equivalent G). Furthermore by theorem Maschke, V is the direct addition of irreducible FG-submodules.

We know that irreducible representation of Q_8 are 1-dimension or 2-dimension. But thus irreducible representation are not 1-dimension, because with suppose that N is 1-dimension ($N =< x >$). Since Q_8 have four liner character, hence they are corresponds with four irreducible representation of 1-dimension. Therefore we can suppose representation of $\rho_{(2)} : Q_8 \to GL(1, \mathbb{C})$ is corresponds with liner character χ_2. By $\rho_{(2)}$ and $i \in Q_8$ we have $x\rho_{(2)i} = x^i$ then $x^i = x\rho_{(2)i} = x[1]_i = x.\chi_2(1) = x.1 = x$, namely for every liner character there is similar member $i \in Q_8$ such that $x^i = x$. Thus $x \in C_{Q_8}(< i >)$. which is a contraction because in lemma 4 we proved that in such condition we must have $C_{Q_8}(< i >) =< i >$. So $V = E_p$ can not 1-dimension.

With regarded to remark 1, E_p is direct addition of copies of the 2-dimension irreducible representation Q_8 on field C_p. By lemma 3, $p = 3$ or $p = 5$.

If $p = 3$ then $G = E_3 : Q_8$. Since Q_8 acts on E_3 by conjugately therefore $\rho : Q_8 \to Aut(E_3)$ and $g \mapsto \rho_g$ such that $\rho_g : E_3 \to E_3$ and $x \mapsto x^g$ is an homomorphism that is not fixed point, in other words, $x^g \neq x$ for all $x \in E_3$ and $1 \neq g \in Q_8$.

Now we assume (x, g) be generator $E_3 : Q_8$. Since the generator any cyclic group are conjugate then by definition 2, $E_3 : Q_8$ is a $\mathbb{Q}$-group such that E_3 is direct addition of copies of 2-dimension irreducible representation of Q_8 on field C_3 and case (2-i) of the theorem is proved now.

(2-ii): If $p = 5$ then $G = E3 : Q_8$. It is similar to proof $(2 - i)$. We have G is a $\mathbb{Q}$-group and E_5 is direct addition copies of 2-dimension irreducible representations of Q_8 on feild five member C_5. Now we show $E_5 = V = C_5 \times C_5$. Since $G = E_5 : Q_8$ and E_5 is direct addition copies of the 2-dimension irreducible representation then $E_5 = V \oplus ... \oplus V$, such that any V is copy of the 2-dimension irreducible representation. Now if we define representation of $\rho : Q_8 \to GL(2, \mathbb{C})$ such that:

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \to \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, j \to \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, k \to \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $E_5 = V = C_5 \times C_5$ because, in otherwise, if we have two copy the vector space V such that $< V_1, V_2 > \oplus < V_3, V_4 >\subseteq E_5$ and $V_1 + V_3 + V_4$ be a generator of 5 order then $3V_1 + 3V_3 + 3V_4$ is a another generator for this cyclic group. Since the generators any cyclic group are conjugately and $E_5 : Q_8$ is a $\mathbb{Q}$-group and Q_8 acts on E_5 by conjugately thus there is similar member $g \in Q_8$ such that $(V_1 + V_3 + V_4)^g = 3V_1 + 3V_3 + 3V_4$, so $V_1^g = 3V_1$ and $(V_3 + V_4)^g = 3V_3 + 3V_4$, furthermore by definition homomorphism $\rho : Q_8 \to GL(2, \mathbb{C})$, on member Q_8 and $V_1^g = 3V_1$ if and only if $g = k$. But if $g = k$, we have $(V_3 + V_4)^k = 3V_3 + 2V_4 \neq 3V_3 + 3V_4$. So there is not any g in Q_8. Thus $E_5 = C_5 \times C_5$, therefore $G \cong (C_5 \times C_5) : Q_8$.

References

[1] L. Dornhoff, *Group resnatation theory*, Part A, Marcel Dekker. (1971).

[2] W. FEit, J. G. Thompson, *Solvabillhty of groups of odd order*, Pacific. J. Math. Soc. 13 (1988), 775-1029.

[3] R. Gow, *Group whose characters are rational-valued*, J.Algebra. 40 (1976), 280-299.

[4] D. Gorenstein, *Finet groups*, Harper and row, (1988).

[5] B. Hupper, *Endliche Gruppen*, New York. (1987).

[6] D. Keetzing, *Structare and representation of $\mathbb{Q}$-group*, Berlin-Heidelbreg-New York-Tokyo, (1984).

[7] D.S. Passman, *Permutation groups*, New York, (1968).

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