AN EXTENSIONS OF THE RIEMANN-LEBESGUE LEMMA AND SOME APLICATIONS

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Abstract. We show that a similar relation as (1.2) holds for all continuous and bounded functions $g:[0,\infty)\to R$ of finite Cesaro mean. A result concerning the asymptotic behavior in (2.10) when $a = 0$, $b = T$ and $f\in C^1[0,T]$ is given in Theorem 3.1. Some concrete applications and examples are presented in the last section of the paper.
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1. Introduction

The well-known Riemann-Lebesgue Lemma (see for instance [6, pp.22, Problem 1.4.11(a) and (b)] or [9, Problem9.48]) states that if $f:[a,b]\to R$ is a Riemann integrable function on [a, b] then

$$\lim_{n\to\infty}\int_a^b f(x)\sin nxdx = \lim_{n\to\infty}\int_a^b f(x)\cos nxdx = 0 \qquad (1.1)$$

That implies a classical result in Fourier analysis, i.e. if $f:R\to R$ is a 2π-periodic with Fourier coefficients a_n, b_n, then under suitable conditions $S_n(x)\to f(x)$, where

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx).$$

The following result is well-known and represents a nice generalization of the previous one. It is also called the Riemann-Lebesque Lemma: *Let f:* $[a,b]\to R$

be a continuous function, where 0 ≤ a < b. Suppose the function g:$[0,\infty) \to R$ *to be continuous and T-periodic. Then*

$$\lim_{n\to\infty} \int_a^b f(x)g(nx)dx = \frac{1}{T}\int_0^T g(x)dx \int_a^b f(x)dx \tag{1.2}$$

(see [5] for the special case *a* = 0, *b* = *T*).

There are other extensions and generalization of the important result contained in (1.1). We mention here *Knyazyuk A.V.*[8] which gives necessary and sufficient conditions for the existence of the limit.

$$\lim_{\lambda\to\infty} \int_a^b h(x, g(\lambda\ x)dx \tag{1.3}$$

for a given function $g : [0,\infty) \to R$ and all continuous functions $h : [a,b] \times g([0,\infty)) \to$ R. The following multidimensional generalization of Riemann-Lebesgue Lemma is given in the recent paper *Canada, A., Urena, A.J.* [2, Lemma 3.1] and it is used to study the asymptotic behavior of the solvability set for pendulum-type equations with linear damping and homogeneous Dirichlet conditions: *Let g : R →R be a continuous and T-periodic function with zero mean value and let* u_1 *be given functions satisfying the following property are real numbers sucs that*

$$meas\left\{ x \in [0,\pi] : \sum_{i=1}^{N} \rho_i u'_i\,(x) = 0 \right\} > 0$$

then $\rho_1 = ... = \rho_N = 0$. *Let* $B \subset C^1[0,\pi]$ *be such that the set* $\{b' : b \in B\}$ *is uniformly bounded in* $C[0,\pi]$. *Then for any given function* $r \in L^1[0,\pi]$ *we have u*

$$\lim_{\|\rho\|\to\infty} \int_0^\pi g\left(\sum_{i=1}^{N} \rho_i u_i(x) + b(x) \right) r(x)dx = 0 \tag{1.4}$$

uniformly with respect to b∈ B. Other multidimensional version of Riemann-Lebesgue Lemma is mention in *Canada, A.,Ruiz,D.*[3, Lemma 2.1] and it is applied to the study of periodic perturbations of a class of resonant problems.

Some other extensions and generalizations of Riemann-Lebesgue Lemma were obtained by *Bleistein, N., Handelsman, R. A., Lew, J. S.* [1], *Kantor, P. A.* [7] and *Stadije, W.* [10].

In this paper we extend the relation (1.2). Our main result is contained in Theorem 2.1 and it shows that a similar relation as (1.2) holds for all continuous and bounded function $g : [0,\infty) \to R$ of finite Cesaro mean. Some applications are also given.

2. The main results

Let us begin whit some auxiliary results which will help us to derive our main result.

Lemma 2.1. Let $\omega : [0,\infty)\to R$ be a continuous function such that $\lim\limits_{x\to\infty}\frac{\omega(x)}{x} = 0$. If $(c_n)_{n\geq 1}$ is a sequence of non-negative real numbers such that $\left(\frac{c_n}{n}\right)_{n\geq 1}$ is bounded, then $\lim\limits_{n\to\infty}\frac{\omega(c_n)}{n} = 0$.

Proof. We consider the following cases.

Case 1. There exists $\lim\limits_{n\to\infty} c_n$ and it is $+\infty$. In that case we have

$$\left|\frac{\omega(c_n)}{n}\right| = \left|\frac{\omega(c_n)}{c_n}\cdot\frac{c_n}{n}\right| \leq M\left|\frac{\omega(c_n)}{c_n}\right|, \tag{2.1}$$

where $M = \sup\left\{\frac{c_n}{n} : n\geq 1\right\}$. Since $\lim\limits_{x\to\infty}\frac{\omega(x)}{x} = 0$, from (2.1) it follows $\lim\limits_{n\to\infty}\left|\frac{\omega(c_n)}{n}\right| = 0$ hence the desired conclusion.

Case 2. The sequence $(c_n)_{n\geq 1}$ is bounded. Consider $A > 0$ such that $c_n \leq A$ for all positive integers $n\geq 1$, and define $K = \sup\limits_{x\in[0,A]}|\omega(x)|$. It is clear that $\left|\frac{\omega(c_n)}{n}\right| \leq \frac{K}{n}$ for all $n \geq 1$, i.e. $\lim\limits_{n\to\infty}\frac{\omega(c_n)}{n} = 0$.

Case 3. The sequence $(c_n)_{n\geq 1}$ is unbounded and $\lim_{n\to\infty} c_n$ does not exist. For $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|\frac{\omega(x)}{x}\right| < \frac{\varepsilon}{M}$ for any $x > \delta$, where

$M = \sup\left\{\frac{c_n}{n} : n \geq 1\right\}$. Consider the sets

$$A_\delta = \{n \in N^* : c_n \leq \delta\} \quad \text{and} \quad B_\delta = \{n \in N^* : c_n > \delta\}$$

If one of these sets is finite, then we immediately derive the desired result. Assume that A_δ and B_δ are both infinite. Since $\lim_{n\in A_\delta}\frac{\omega(c_n)}{n} = 0$, it follows that there exists $N_1(\varepsilon)$ with the property $\left|\frac{\omega(c_n)}{n}\right| < \varepsilon$ for all $n \in A_\delta$ with $n \geq N_1(\varepsilon)$. For $n \in B_\delta$, we have

$$\left|\frac{\omega(c_n)}{n}\right| = \left|\frac{\omega(c_n)}{c_n}\cdot\frac{c_n}{n}\right| < \frac{\varepsilon}{M}M = \varepsilon, \tag{2.2}$$

i.e $\left|\frac{\omega(c_n)}{n}\right| < \varepsilon$ for any $n \geq N_1(\varepsilon)$. Finally $\lim_{n\to\infty}\frac{\omega(c_n)}{n} = 0$.

Lemma 2.2 Let $f : R \to R$ be a continuous non-constant periodic function. If F is an antiderivative of f, then

$$F(x) = \left(\frac{1}{T}\int_o^T f(t)dt\right)x + g(x), \tag{2.2}$$

where $T > 0$ is the period of f and $g : R \to R$ is a T-periodic function.

Proof. Using the relation $f(t + T) = f(t)$ for any $t \in R$ it follows $F(x + t) - F(x) = \int_o^T f(t)dt$ for all $x \in R$. Considering the function $h(x) = \left(\frac{1}{T}\int_o^T f(t)dt\right)x$, we have $h(x + T) – h(x) = \int_o^T f(t)dt$, i.e. $F(x + T) - h(x + T)=F(x) - h(x)$. That is the function defined by $g(x) = F(x) - h(x)$, $x\in R$, is periodic of period T. Moreover the formula (2.2) holds.

Lemma 2.3 Consider $0\le a < b$ to be real numbers and let $f:[a,b]\to R$ be a function of class C^1. Let $g : [0,\infty) \to R$ be a continuous function such that $\lim_{x\to\infty}\frac{1}{x}\int_0^x g(t)dt = L$ (finite). Then the following relation holds:

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)dx = L\int_a^b f(x)dx \tag{2.3}$$

Proof. If $G(x)= \int_0^x g(t)dt$, then define the function $\omega(x) = G(x) - Lx$, $x \ge 0$. It is clear that ω is differentiable and it satisfies $\lim_{x\to\infty}\frac{\omega(x)}{x} = 0$.

We have

$$\int_b^a f(x)g(nx)dx = \frac{1}{n}\ (G(nb)f(b) - f(a)G(na)) - \frac{1}{n}\int_a^b f'(x)G(nx)dx =$$

$$= \frac{1}{n}(G(nb(f(b) - f(a)G(na)) - \frac{1}{n}\int_a^b f'(x)(Lnx + \omega(nx))dx \quad =$$

$$= \frac{1}{n}(G(nb)f(b) - f(a)G(na)) - L\int_a^b f'(x)xdx - \frac{1}{n}\int_b^a \omega(nx)f'(x)dx =$$

$$\frac{1}{n}(G(nb)f(b) - f(a)G(na)) - L(bf(b) - af(a)) +$$

$$+\ L\int_a^b f(x)dx - \frac{1}{n}\int_a^b \omega(nx)f'(x)dx$$

Let us show that

$$\lim_{n\to\infty}\frac{1}{n}\int_a^b \omega(nx)f'(x)dx = 0 \qquad (2.4)$$

Indeed, from the relations

$$\left|\frac{1}{n}\int_a^b \omega(nx)f'(x)dx\right| \le \frac{1}{n}\int_a^b |\,\omega(nx)\,||\,f'(x)\,|\,dx =$$
$$= \frac{|\,\omega(c_n)\,|}{n}\int_a^b |\,f'(x)\,|\,dx,$$

where $na < c_n < nb$, by applying Lemma 2.1 it follows $\lim_{n\to\infty}\frac{|\omega(c_n)|}{n} = 0$, i.e. the equality (2.4) holds.

Moreover we have

$$\lim_{n\to\infty}\frac{1}{n}(G(nb)f(b) - G(na)f(a)) = \lim_{n\to\infty}\left(b\frac{G(nb)}{nb}f(b) - a\frac{G(na)}{na}f(a)\right) =$$
$$= L(bf(b) - af(a))$$

Using (2.4) and (2.5) the relation (2.3) follows.

Theorem 2.1 Let $0 \le a < b$, be real numbers and let $f : [a,b] \to R$ be a continuous function. Consider the function $g : [0,\infty) \to R$ to be continuous and bounded such that $\lim_{x\to\infty}\frac{1}{x}\int_0^x g(t)dt = L$ (finite). Then

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)dx = L\int_a^b f(x)dx \qquad (2.6)$$

Proof. Without loss of generality we can assume that $0 \le g(x) \le 1$for any $x \in [0,\infty)$. Applying the well-known Weierstrass approximation Theorem, it follows that there exists a sequence $(f_m)_{m\ge 1}$ of polynomials, $f_m:[a,b]\to R$, which converges uniformly to f. Consider $\varepsilon > 0$ and define $\varepsilon' =$

$\frac{\varepsilon}{(b-a)(L+2)}$. Then we can find a positive integer $M(\varepsilon)$ such that $|f_m(x)-f(x)|<\varepsilon'$ for any $m \geq M(\varepsilon)$ and for all $x\in[a,b]$. If $m\geq \mathrm{I}(\varepsilon)$, then we have

$$\left|\int_a^b f_m(x)g(nx)dx-\int_a^b f(xg(nx)dx\right| \leq \int_a^b |f_m(x)-f(x)|\, g(nx)dx \leq \varepsilon^{'}(b-a),$$

for all positive n. Therefore, for $m\geq M(\varepsilon)$ and $n\in N^*$

$$\int_a^b f_m(x)g(nx)dx-\varepsilon^{'}(b-a)\leq \int_a^b f(x)g(nx)dx\leq$$
$$\leq \varepsilon^{'}(b-a)+\int_a^b f_m(x)g(nx)dx \qquad (2.7)$$

from Lemma 2.3. it follows

$$\lim_{n\to\infty}\int_a^b f_m(x)g(nx)dx = L\int_a^b f_m(x)dx.$$

Hence, for $\varepsilon>0$ one can find a positive integer $N(\varepsilon,m)$ such that for all $n\geq N(\varepsilon,m)$ we have

$$L\int_a^b f_m(x)-\varepsilon^{'}(b-a)<\int_a^b f_m(x)g(nx)dx<L\int_a^b f_m(x)dx+\varepsilon^{'}(b-a).$$

Using the last inequalities and (2.7) we get

$$L\int_a^b f_m(x)dx-2\varepsilon^{'}(b-a)<\int_a^b f(x)g(nx)dx<$$
$$<L\int_a^b f_m(x)dx+2\varepsilon^{'}(b-a). \qquad (2.8)$$

But for all $m\geq M(\varepsilon)$ we have $f(x)-\varepsilon^{'} < f_m(x) < f(x)+\varepsilon^{'}$ for any $x\in[a,b]$. Thus

$$L\int_a^b f(x)dx-L\varepsilon^{'}(b-a)<\int_a^b f_m(x)dx<L\int_a^b f(x)dx+L\varepsilon^{'}(b-a) \qquad (2.9)$$

From (2.8) and (2.9) it follows

$$L\int_a^b f(x)dx-(b-a)(L+2)\varepsilon' < \int_a^b f(x)g(nx)dx < L\int_a^b f(x)dx+(b-a)(L+2)\varepsilon'$$

for all positive integers $n \geq N(\varepsilon, m)$, i.e.

$$\left|\int_a^b f(x)g(nx)dx - L\int_a^b f(x)dx\right| < (b-a)(L+2)\varepsilon' = \varepsilon$$

and the desired conclusion is obtained.

Corollary 2.1 Let $0 \leq a < b$ be real numbers and let $f : [a,b] \longrightarrow R$ be a continuous function. Consider the function $g : [0,\infty) \to R$ to be continuous such that $\lim_{x\to\infty} g(x) = L$ (finite). Then

$$\lim_{x\to\infty} \int_a^b f(x)g(nx)dx = L\int_a^b f(x)dx.$$

Proof. Because g is continuous and $\lim_{x\to\infty} g(x) = L$ we obtain that g is bounded. Moreover, applying L'Hospital rule it follows

$$\lim_{x\to\infty} \int_0^x g(t)dt = \lim_{x\longrightarrow\infty} g(x) = L$$

and the conclusion follows from Theorem 2.1.

Corollary 2.2 (Riemann-Lebesgue Lemma) Let $f : [0,b] \to R$ be a continuous function, where $0 \leq a \leq b$. Consider the function $g : [0,\infty) \to R$ to be a continuous and periodic of period T. Then

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)dx = \frac{1}{T}\int_0^T g(x)dx\int_a^b f(x)dx \quad . \tag{2.10}$$

Proof. Using Lemma 2.2 it follows that the function $G(x) = \left(\frac{1}{T}\int_0^x g(t)dt\right)$ has the representation

$$G(x) = \left(\frac{1}{T}\int_0^T g(t)dt\right)x + h(x), \quad x\in[0,\infty)$$

where h is continuous and periodic. We have

$$\frac{G(x)}{x} = \frac{1}{T}\int_0^T g(t)dt + \frac{h(x)}{x},$$

hence $\lim_{x\to\infty}\frac{1}{x}\int_0^x g(t)dt = \frac{1}{T}\int_0^T g(t)dt$, and the relation (2.10) follows from Theorem 2.1.

Remark. If $g : [0,\infty) \to R$ is continuous and almost-periodic, then its Cesaro mean $\lim_{x\to\infty}\frac{1}{x}\int_0^x g(t)dt$ exists and it is finite (see for instance our paper [1] or any book on almost-periodic functions). Therefore the class of continuous functions g in Theorem 2.1 is more general than the class of functions g in Corollary 2.2.

3. Some applications

First of all we derive a result concerning the asymptotic behavior in (2.10) when $a = 0$, $b = T$ and $f \in C^1[0,T]$.

Theorem 3.1. Let $g : [0,\infty) \to R$ be continuous and T-periodic and let C^1 $[0,T]$. Then

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)dx = \frac{1}{T}\int_0^T g(x)dx\int_a^b f(x)dx$$

$$=\frac{1}{T}(f(T)-f(0))\left(G(t)-\int_0^T G(x)dx\right) \tag{3.1}$$

where $G(x)=\int_0^x g(t)dt.$

Proof. Denote by $M_g=\frac{1}{T}\int_0^T g(t)dt,$ the Cesaro's mean of g on the interval $[0,T]$. According to Lemma 2.2 we have $G(x)=M_g\cdot x+h(h)$, $x\in[0,\infty)$, where h is continuous and T-periodic.

We can write

$$n\int_0^T f(x)g(nx)dx=\int_0^T f(x)G'(nx)dx= f(x)G(nx)\Big|_0^T-\int_0^T f'(x)G(nx)dx=$$

$$= f(T)G(nT)-\int_0^T f'(x)G(nx)dx= f(T)(M_g nT+h(nT))-$$

$$-nM_g\int_0^T xf'(x)dx-\int_0^T f'(x)h(nx)dx = nM_g Tf(T)-$$

$$-nM_g Tf(T)+nM_g\int_0^T f(x)dx-\int_0^T f'(x)h(nx)dx=$$

$$=nM_g\int_0^T f(x)dx-\int_0^T f'(x)h(nx)dx.$$

That is equivalent to

$$n\left(\int_0^T f(x)g(nx)dx-\frac{1}{T}\int_0^T g(x)dx\int_0^T f(x)dx\right)=-\int_0^T f'(x)h(nx)dx. \tag{3.2}$$

From Corollary 2.2 it follows that

$$\lim_{n\to\infty}\int_0^T f'(x)h(nx)dx = \frac{1}{T}\int_0^T h(x)dx\int_0^T f'(x)dx =$$
$$= \frac{1}{T}(f(T)-f(0))\int_0^T h(x)dx\int_0^T f'(x)dx =$$
$$= \frac{1}{T}(f(T)-f(0))\left(\int_0^T G(x)dx - G(T)\right)$$

and the desired formula is obtained from (3.2).

Remark. In the similar way we get the following result when $a = kT$, $b = (k+1)T$, where k is a fixed positive integer:

$$\lim_{n\to\infty}\left(\int_{kT}^{(k+1)T} f(x)g(nx)dx\right) - \frac{1}{T}\int_0^T g(x)dx\int_{kT}^{(k+1)T} f(x)dx =$$
$$= \frac{1}{T}(f((k+1)T) - f(xT))\left(G(T) - \int_{kT}^{(k+1)T} G(x)dx\right). \qquad (3.3)$$

The asymptotic behavior of the Fourier coefficients is given in the following result.

Corolary 3.2. Let $f : [0,\infty) \to R$ be a function of class C^1. Then

$$\lim_{n\to\infty} n\int_{2k\pi}^{2(k+1)\pi} f(x)\sin nxdx = f(2k\pi) - f(2(k+1)\pi) \qquad (3.4)$$

$$\lim_{n\to\infty} n\int_{2k\pi}^{2(k+1)\pi} f(x)\cos nxdx = 0. \qquad (3.5)$$

Proof. In formula (3.3) take $g(x) = \sin x$, respectively $g(x) = \cos x$ and obtain the desired results.

In what follows we shall present some concrete applications.

Application 1. 1) Let $f : [0,\infty) \to R$ be a continuous and periodic function of period 1. Then

$$\lim_{n\to\infty}\int_0^1 f(x)f(nx)dx = \left(\int_0^1 f(x)dx\right)^2. \tag{3.6}$$

2) If f is of class C^1 on $[0,\infty)$ then

$$\lim_{n\to\infty} n\left(\int_0^1 f(x)f(nx)dx - \left(\int_0^1 f(x)dx\right)^2\right) = 0. \tag{3.7}$$

Indeed, from Corollary 2.2 we have

$$\lim_{n\to\infty}\int_0^1 f(x)f(nx)dx = \frac{1}{1}\int_0^1 f(x)dx\int_0^1 f(x)dx = \left(\int_0^1 f(x)dx\right)^2,$$

that is (3.6). In order to obtain (3.7) we apply Theorem 3.1 and the relation $f(0) = f(1)$.

Application 2. The following relation holds:

$$\lim_{n\to\infty}\int_\pi^{2\pi}\frac{|\sin nx|}{x}dx = \frac{2}{\pi}\ln 2. \tag{3.8}$$

Let us apply Corollary 2.2 to functions $f:[\pi,2\pi]\to \boldsymbol{R}$, $f(x) = \frac{1}{x}$ and $g:[0,\infty]\to \boldsymbol{R}$, $g(x) = |\sin x|$. Taking into account that g is periodic of period π, it follows

$$\lim_{n\to\infty}\int_\pi^{2\pi}\frac{|\sin x|}{x}dx = \frac{1}{x}\int_0^\pi |\sin x|dx\int_\pi^{2\pi}\frac{1}{x}dx = \frac{2}{\pi}\ln 2.$$

Application 3. The following relation holds

$$\lim_{n\to\infty} n\int_{2\pi}^{4\pi}\frac{\sin nx}{x^2}dx = \frac{3}{16\pi^2}. \tag{3.9}$$

In formula (3.3) take $g(x)=\sin x$, $f(x)=\frac{1}{x^2}$, $T = 2\pi$ and k = 1. We have $G(x)=-\cos x+1$ and $\int_{2\pi}^{4\pi}G(x)dx=2\pi$. Then

$$\lim_{n\to\infty} n\int_{2\pi}^{4\pi}\frac{\sin nx}{x^2}dx=\frac{1}{2\pi}\left(\frac{1}{16\pi^2}-\frac{1}{4\pi^2}\right)(-2\pi)=\frac{3}{16\pi^2}.$$

For a fixed positive integer k we have

$$\lim_{n\to\infty} n\int_{2k\pi}^{2(k+1)\pi}\frac{\sin nx}{x^2}=\frac{2k+1}{4(k(k+1)^2\pi^2} \tag{3.10}$$

(see also formula (3.4) in Corollary 3.2).

Application 4. ([9, Problem 9.2]) The following relation holds:

$$\lim_{n\to\infty}\int_0^{\pi}\frac{\sin x}{1+\cos^2 nx}dx=\sqrt{2}\,. \tag{3.11}$$

We apply Corollary 2.2 to functions $f:[\pi,2\pi]\to \boldsymbol{R}$, $f(x)=\sin x$ and $g:[0,\infty]\to \boldsymbol{R}$, $g(x)=\frac{1}{1+\cos^2 x}$. The function g g is periodic of period π, hence

$$\lim_{n\to\infty}\int_0^{\pi}\frac{\sin x}{1+\cos^2 nx}dx=\frac{1}{\pi}\int_0^{\pi}\frac{1}{1+\cos^2 x}dx\int_0^{\pi}\sin xdx=$$
$$=\frac{2}{\pi}\int_0^{\pi}\frac{dx}{1+\cos^2 x}=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\frac{dx}{1+\cos^2 x}=\frac{4}{\pi}\int_0^{\infty}\frac{du}{1+u^2}$$
$$=\frac{4}{\pi\sqrt{2}}arctg\frac{u}{\sqrt{2}}\Big|_0^{\infty}=\sqrt{2}\,.$$

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