CALCULATION OF THE CRITICAL STOKES NUMBER FOR WIDE-STREAM IMPACTION OF POTENTIAL FLOW OVER SYMMETRIC ARC-NOSED COLLECTORS

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Abstract.The wide-stream impaction on a class of semi-infinite two-dimensional symmetric bodies having flat or circular-arc noses placed in a uniform potential flow of aerosols is investigated. The governing Stokes equations of motion are nonlinear differential equations involving a parameter called the Stokes number. The study calculates analytically the critical value of the Stokes number on the centre-line, k_{cr}, below which no particles reach the stagnation point in a finite time. This in turn can help the experimentalist in designing appropriate collector shapes for obtaining a better collection of small aerosol particles.

Keywords. Critical Stokes number, Wide-stream impaction, potential flow, Arc-nosed collectors.

Running Title. Critical Stokes number.

1.Introduction

The problem of particle deposition from an aersol stream onto obstacle (collectors) of various shapes is of great importance in the mechanics of aerosols, see Fuchs (1964). Travelling with the fluid flow at large distances, a particle will arrive in the region of disturbance caused by the obstacle's presence and due to its inertia will cross the streamlines, so permitting a possible collision with the obstacle. This phenomenon deals with the collection efficiency concept, η, defined as the ratio of the number of particles actually deposited on the obstacle to the number of particles which would have been deposited if they had not been deviated by the fluid flow. Theoretically, the calculation of η involves determining the fluid flow field and then finding the particle trajectories by integrating the Stokes equations for the curvilinear motion of a spherical particle, which in non-dimensional form are, see e.g. Dunnett and Ingham (1988),

$$\frac{d\underline{r}}{dt} = \underline{u}, \quad k\frac{d\underline{u}}{dt} = \underline{u}_f - \underline{u} \tag{1}$$

with the conditions at infinity

$$\underline{r}(t) \to (-\infty, h), \quad \underline{u}(\underline{r}(t)) \to (1, 0), \quad \text{as } t \to -\infty \tag{2}$$

where $\underline{r} = (x, y)$ is the position vector of the particle, $\underline{u}_f = (u_f, v_f)$ is the fluid velocity, $\underline{u} = (u, v)$ is the particle velocity, h is a given positive number, $k = d^2U_0\rho_p/(18\mu l)$ is the Stokes number, d is the particle diameter, ρ_p is the particle density, μ is the fluid dynamic viscosity, l is a characteristic dimension of the obstacle taken to be half of the width of the wide-stream collection surface. All distances and velocities have been non-dimensionalised with respect to l and U_0, respectively. For $h = 0$ we have $y(t) \equiv 0$ and then one obtains the rectilinear motion of aerosol particles on the centre-line $y = 0$, namely,

$$\frac{dx}{dt} = u(x(t)), \quad k\frac{du}{dt} = \phi(x(t)) - u(x(t)) \tag{3}$$

with the conditions at infinity

$$x(t) \to -\infty, \quad x'(t) = u(x(t)) \to 1, \quad \text{as } t \to -\infty \tag{4}$$

where $\phi(x) = u_f(x, 0)$ denotes the centre-line fluid velocity and, for simplicity, we have written $u(x)$ instead of $u(x, 0)$.

The essential problem in the phenomenon under consideration, described by eqns (1) and (2), and (3) and (4), is to determine the critical Stokes numbers, $k = K_{cr}$ and $k = k_{cr}$, below which no particles may be deposited on the obstacle or arrive at the stagnation point in a finite time, respectively.

Assuming that the inertial impaction is the principal mechanism of deposition, hence neglecting interception effects, brownian motion and diffusion of aersol particles, for a given configuration of the obstacle, the value of K_{cr} determines the minimum size D_{min} of the particles settling on the obstacle, whilst the value of k_{cr} determines the minimum size d_{min} of particles arriving at the stagnation point. According to the definition of the Stokes number, the minimum diameters of the collected particles, D_{min} and d_{min}, are directly proportional to the square root of the characteristic dimension of the obstacle, l, and inversely proportional to the square root of the freestream velocity, U_0, and particle density, ρ_p, namely,

$$D_{min} = \left(\frac{18\mu l K_{cr}}{\rho_p U_0}\right)^{1/2}, \quad d_{min} = \left(\frac{18\mu l k_{cr}}{\rho_p U_0}\right)^{1/2} \tag{5}$$

Consequently, the critical Stokes numbers K_{cr} and k_{cr} may serve as a criterion for the collection of small particles by the obstacle and at the stagnation point, respectively. From expression (5) it can be observed that, for the same conditions, the larger K_{cr} and k_{cr} the more poorly will the given obstacle collect small particles. Also from eqn.(5) it can be seen that for an obstacle (collector) of a given shape, particles are collected more efficiently by the obstacle the higher the freestream velocity U_0, or the density of particles ρ_p, and the smaller the characteristic dimension of the obstacle l, or the dynamic viscosity, μ, of the fluid.

Even the experimental investigations, which present numerous difficulties because of the particle interception effect, cannot determine exactly the cut-off value of the Stokes number for which the collection efficiency, η, becomes zero.

In the previous years, there has been much controversy regarding the inconsistency of the values of k_{cr} and K_{cr} in the empirical, numerical and theoretical works, but all these have been elucidated (theoretically) by Lesnic *et al.* (1994a) and to summarise, the main result is as follows.

Theorem 1.
(i) For the potential flow past symmetrically convex collectors

$$k_{cr} = \frac{1}{4a} \tag{6}$$

where $a = -\phi'(x_0)$ *and* x_0 *is the stagnation point of the centre-line where* $\phi(x_0) = 0$. *If further the fluid velocity is finite everywhere on the obstacle then*

$$K_{cr} = \frac{1}{4a}. \tag{7}$$

(ii) For the potential flow past symmetrically concave collectors

$$\frac{1}{4m} \leq k_{cr} \leq \frac{1}{4a} \tag{8}$$

where $m = -\phi'(x_0')$ *and* x_0' *is the inflexion point of the centre-line fluid velocity where* $\phi''(x_0') = 0$.

(iii) For the slow viscous flow past symmetrically collectors

$$\frac{1}{4m} \leq k_{cr}. \tag{9}$$

The purpose of this study is to apply this theorem for calculating mainly the critical Stokes number k_{cr} for a wide-class of semi-infinite two-dimensional symmetric collectors having flat and circular-arc noses as placed in a potential free stream of aerosols. These bodies are streamlined semi-infinite collectors of Rankine-type for which analytical solutions for the fluid flow field were developed by Hess (1973). Various types of shaped-collectors and other classical potential theories on wide-stream impaction onto obstacle are discussed and compared.

2.Streamlined collectors

A large class of analytic solutions for the fluid flow field can be generated indirectly via Rankine's idea based on the use of flow singularities, such as sources, vortices and dipoles. Each singularity gives rise to a velocity field that satisfies the basic potential-flow equations except at the singularity itself, similarly as a Green's function. Such flows are superimposed upon a uniform stream. Any streamline of the resulting flow may be considered as the boundary of an obstacle, the flow about which is given by adding the individual flows of the singularities of the uniform stream. Proper distribution of singularities and proper selection of a streamline yield flows about interesting families of bodies (collectors), which in turn can be used by experimentalists for the practical design of collectors with respect to maximizing their collection efficiencies. More clearly, one can think inversely (indirectly) of the fluid flow field generating the collector rather viceversa as is the traditional direct approach in which the collector gives the flow field. In this way, one can control the properties of the field such that they generate optimal shapes of collectors. This class of analytical solutions were derived by Hess (1973) using methods that have features of both direct and inverse solutions. While the general method of classical potential flow without separation distributes a source density on a *complete closed body*,e.g. a Rankine body, the new method distributes sources on a *partial open* body, e.g. a Rankine half body. Furthermore, the importance of including separation in the potential flow models past objects has been stressed in Lesnic *et al.* (1994b). For arbitrary shaped collectors the fluid velocity can be evaluated numerically using the boundary element method, as described by Hess and Smith (1966), although analytical solutions can be obtained for a circular arc and for a straight line, as given in section 4. In this study, we aim to calculate the critical Stokes number on the centre-line, k_{cr}, associated with this class of potential flow fields and to compare the results obtained with those from the other theories based on the potential flow over bodies with or without separation.

3. Rankine-type flows

The idea of obtaining an inverse solution by superposition of point sources was put forward by Rankine in 1871. The simplest Rankine-type flow is obtained from a uniform stream velocity $\underline{U}_0 = (U_0, 0)$ parallel to the $x-$axis and a point source of strength Q located at the origin $(0, 0)$. The fluid velocity field $\underline{u}_f = (u_f, v_f)$, in non-dimensional quantities, is then given by, see Hess (1973),

$$\underline{u}_f = (u_f, v_f) = \left(1 + \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right). \tag{10}$$

The velocity becomes zero at the stagnation point $(x_0, y_0) = (-1, 0)$. The streamline which bifurcates at this stagnation point is taken as a body contour and is given by, see Milne-Thomson (1950, p.196),

$$x^{-1} = y^{-1} tan(y). \tag{11}$$

This body is semi-infinite and symmetric about the $x-$axis. Being convex to the uncoming flow Levin's theorem, see Levin (1961), applies and thus

$$k_{cr} = K_{cr} = -\frac{1}{4\phi'(x_0)} = \frac{1}{4}. \tag{12}$$

For comparison, it is interesting to compare the values (12) with the critical Stokes values obtained for the potential flow without separation past a circular cylinder, namely, $k_{cr} = K_{cr} = 1/8$, see Langmuir and Blodgett (1946).

4. Other semi-infinite two-dimensional symmetric collectors having flat and circular arc-noses

4.1 Flat-Nosed Collector

Consider a flat-nosed collector (partial body, obstacle) consisting of a straight line lying along $y-$axis from $-l$ to l. Then the potential fluid velocity field of a uniform stream of velocity U_0 along the $x-$axis, in non-dimensional quantities, is given by, see Hess (1973),

$$u_f(x,y) = 1 + \frac{1}{\pi}\left[tan^{-1}\left(\frac{1-y}{x}\right) + tan^{-1}\left(\frac{1+y}{x}\right)\right]$$
$$v_f(x,y) = \frac{1}{2\pi} ln\left(\frac{(1+y)^2+x^2}{(1-y)^2+x^2}\right). \qquad (13)$$

The velocity becomes zero at the stagnation point $(x_0, y_0) = (0, 0)$. The upper streamline which bifurcates at this stagnation point is taken as the body contour and it is given implicitly by, see Hess (1973), for $y > 0$

$$3\pi P - 2P tan^{-1}(P) + ln(1+P^2) = \pi Q - 2Q tan^{-1}(Q) + ln(1+Q^2) \qquad (14)$$

where $P = \frac{y-1}{x}$, $Q = \frac{y+1}{x}$. This body is semi-infinite and symmetric about the $x-$axis. Being convex to the oncoming flow Levin's theorem, see Levin (1961), applies and thus

$$k_{cr} = -\frac{1}{4\phi'(x_0)} = \frac{\pi}{8}. \qquad (15)$$

For comparison, it is interesting to compare the value (15) with the critical Stokes values obtained for the potential flow with or without separation past a flat plate. For the potential flow without separation past a flat plate the fluid velocity is given by $-u_f + iv_f = (x+iy)/(x^2-y^2+1+2ixy)^{1/2}$, see Milne-Thomson (1968, p.172), and therefore $\phi(x) = -x/(x^2+1)^{1/2}$ and thus $k_{cr} = 1/4$. However, the speed of the fluid flow becomes infinite at the edges of the plate for both the potential flow without separation and the flat-nosed collector models, so that these solutions cannot represent the complete motion past an actual plate. In fact, Golovin and Putnam (1962) observed that for the potential flow past a flat plate without separation $K_{cr} < k_{cr}$ and numerical calculations of Lesnic *et al.* (1993) showed that $K_{cr} \approx 0.212$. From this discussion, as the speed of the flow past a flat- nosed collector at the edge (0,1) is still infinite but it has a weaker (logarithmic) singularity $v_f(0,y) = \frac{1}{\pi} ln\left(\frac{1+y}{1-y}\right)$ than the (algebraic) singularity of the potential flow without separation past a flat plate $v_f(0,y) = \frac{y}{(1-y^2)^{1/2}}$, one expects that $0.212 \leq K_{cr} \leq \pi/8 \approx 0.392$. For the potential flow with separation past a flat plate, the speed is finite at the edges of the plate giving a more realistic mathematical model for which $k_{cr} = K_{cr} = \frac{4}{4+\pi}$, see Fucs (1964, p.164). For completeness the critical Stokes numbers on the centre-line for the potential flow past a recessed trap collector consisting of a straight line lying along the $y-$axis from $-l$ to l and extending to infinity has been calculated, for which the fluid velocity is given by $u_f(x,y) = -\frac{sinh(\pi x)}{cosh(\pi x)+cos(\pi y)}$, $v_f(x,y) = \frac{sin(\pi y)}{cosh(\pi x)+cos(\pi y)}$, see Brun *et al.* (1948). Using formula (6) $k_{cr} = \frac{1}{2\pi}$ is obtained for a recessed trap. For a flat narrow jet striking a plane at right angles an impingement instrument, see Fucs (1964, p.153, 164), $k_{cr} = \frac{2}{\pi}$.

4.2 Concave Circular Arc

This section considers the case where a circular arc is concave to the oncoming uniform flow and thus part (ii) of Theorem 1 will apply. The arc is assumed to have a unit radius centered at the origin and to be symmetric about the $x-$axis with its angle extending from $-\beta$ to β. The fluid velocity for the concave circular arc-nosed collector in potential flow can be calculated from Hess (1973), and is given by

$$u_f(x,y) = 1 + \frac{1}{\pi}\int_{-\beta}^{\beta} \frac{(x-cos(\phi))(cos(\phi)+\frac{sin(\beta)}{\pi-\beta})}{(x-cos(\phi))^2+(y-sin(\phi))^2} d\phi,$$

$$v_f(x,y) = \frac{1}{\pi}\int_{-\beta}^{\beta} \frac{(y-sin(\phi))(cos(\phi)+\frac{sin(\beta)}{\pi-\beta})}{(x-cos(\phi))^2+(y-sin(\phi))^2} d\phi \qquad (16)$$

After some calculus, the values $a = 0.0211$, $m = 0.28$, $x_0 = -0.55$ for $\beta = \pi/2$ and $a = 0.0001$, $m = 0.274$, $x_0 = -1.48$ for $\beta = 5\pi/6$ are obtained. These values of β have been previously chosen for the design of directional dust gauges, see Bush *et al.* (1976) and Ralph and Hall (1989). Using the inequalities (8) one obtains the estimates shown in Table 1. Of considerable interest is the observation that as β increases towards π, i.e. the collector becomes closer to a blunt body with a nozzle, whilst the lower limit $1/(4m)$ for k_{cr} remains almost constant, the upper limit $1/(4a)$ increases rapidly, highlighting the fact that the critical Stokes number k_{cr} is likely to increase.

4.3 Convex Circular Arc

Again the arc is symmetric about the $x-$ axis, has unit radius, and has an angle extending from $-\beta$ to β, but is oriented convexly to the oncoming uniform flow. Then the fluid velocity for the convex circular arc-nosed collector in potential flow can be calculated from Hess (1973), and is given by

$$u_f(x,y) = 1 + \frac{1}{\pi}\int_{-\beta}^{\beta} \frac{(x-cos(\phi))(cos(\phi)-\frac{sin(\beta)}{\pi+\beta})}{(x-cos(\phi))^2+(y-sin(\phi))^2}d\phi,$$
$$v_f(x,y) = \frac{1}{\pi}\int_{-\beta}^{\beta} \frac{(y-sin(\phi))(cos(\phi)-\frac{sin(\beta)}{\pi+\beta})}{(x-cos(\phi))^2+(y-sin(\phi))^2}d\phi \qquad (17)$$

After some calculus the values $a = 0.0980$ for $\beta = \pi/2$ and $a = 0.0976$ for $\beta = 77.45\pi/180$ are obtained. This latter angle β is a 'natural' point of separation of the incompressible flow from a circular cylinder, see Hess (1973). Using (6) and (7) the values of k_{cr} shown in Table 1 are obtained.

5. Conclusions

In this paper the calculation of the critical Stokes number for the wide-stream impaction of potential flow over a new class of streamlined, symmetric collectors has been performed. The results for k_{cr} are summarised in Table 1. From Table 1 the performances of each streamlined collector for collecting small size particles at the centre-line can be assessed. In particular, it can be seen that, on the centre-line, the Rankine-type collector will collect small particles, whilst the convex and concave collectors will collect only larger particles. Of course, the proposed new class of arc-nosed collectors remains to be validated experimentally in a future work.

Table 1: The values of the critical Stokes number on the centre-line for various arc-nosed collectors.

Collector	k_{cr}
Rankine	$1/4 = 0.25$
Flat nosed	$\pi/8 \approx 0.392$
Convex circular arc ($\beta = \pi/2$)	2.551
Convex circular arc ($\beta = 77.45\pi/180$)	2.561
Concave circular arc ($\beta = \pi/2$)	$0.89 < k_{cr} < 11.85$
Concave circular arc ($\beta = 5\pi/6$)	$0.91 < k_{cr} < 2500$

References

[1] Brun, Ed., Demon, L. and Vasseur, M. (1948) *Captation mecanique de corpuscules en suspension dans l'air*, La Recherche Aeronautique **1**, 15-19.

[2] Bush, A.W., Cross, M., Gibson, R.D. and Owst, A.P. (1976) *The collection efficiency of the directional dust gauges*, Atmos. Envir. **10**, 997-1000.

[3] Dunnett, S.J. and Ingham, D.B. (1988) *The Mathematics of Blunt Body Sampling*, Lecture Notes in Engineering **38**, Springer-Verlag, Berlin.

[4] Fuchs, N.A. (1964) *The Mechanics of Aerosols*, Pergamon Press, New York.

[5] Golovin, M.N. and Putnam, A.A. (1962) *Inertial impaction on single elements*, Ind. Eng. Chem.- Fundam. **1**, 264-273.

[6] Hess, J.L. and Smith, A.M.O. (1966) *Calculation of potential flow about arbitrary bodies*, Prog. Aeron. Sci. **8**, 1-138.

[7] Hess, J.L. (1973) *Analytical solutions for potential flow over a class of semi-infinite two-dimensional bodies having circular-arc noses*, J.Fluid Mech. **60**, 225-239.

[8] Langmuir, I. and Blodgett, K. (1946) *Mathematical investigation of water droplet trajectories*, Collected Works of I.Langmuir **10**, 348-393.

[9] Lesnic, D., Elliott, L. and Ingham, D.B. (1993) *On Levin's theorem for the critical Stokes number*, J. Aerosol Sci. **24**, Suppl.1, S151-S152.

[10] Lesnic, D., Elliott, L. and Ingham, D.B. (1994a) *Characterizations of the critical Stokes number for potential and viscous flows*, Mathematatika **41**, 277-292.

[11] Lesnic, D., Elliott, L. and Ingham, D.B. (1994b) *The influence of separation on the collection efficiencies of obstacles*, J. Aerosol Sci. **25**, 527-533.

[12] Levin, L.M. (1961) *Investigations in the Physics of Coarse-Dispersed Aerosols*, (English trans. Foreign Div.Document), FTD-HT-23-1593-67.

[13] Michael, D.H. and Norey, P.W. (1970) *Slow motion of a sphere in a two-phase medium*, Canadian J. Phys. **48**, 1607-1616.

[14] Milne-Thomson, L.M. (1968) *Theoretical Hydrodynamics*, 5th edn., MacMillian, London.

[15] Ralph, M.D. and Hall, D.J. (1989) *Performance of the BS directional dust gauge*, Proc. 3rd Ann. UK Aerosol Soc. Conf., Birmingham, pp.115-120.

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