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SQUARE-STABLE AND WELL-COVERED GRAPHS

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Abstract. The *stability number* of the graph G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set of G. In this paper we characterize the *square-stable graphs*, i.e., the graphs enjoying the property $\alpha(G) = \alpha(G^2)$, where G^2 is the graph with the same vertex set as in G, and an edge of G^2 is joining two distinct vertices, whenever the distance between them in G is at most 2. We show that every square-stable graph is well-covered, and well-covered trees are exactly the square-stable trees.

Keywords: *stable set, square-stable graph, well-covered graph, matching.*

2000 *Mathematics Subject Classification*: 05C75, 05C69, 05C05, 05C70.

1. Introduction

All the graphs considered in this paper are simple, i.e., are finite, undirected, loopless and without multiple edges. For such a graph $G = (V, E)$ we denote its vertex set by $V = V(G)$ and its edge set by $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X.

By $G - W$ we denote the subgraph $G[V - W]$, if $W \subset V(G)$. By $G - F$ we mean the partial subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we use $G - e$, if $W = \{e\}$.

The graph $\overline{G}$ stands for the complement of G, and by $G + e$ we mean the graph $(V(G), E(G) \cup \{e\})$, for any edge $e \in E(\overline{G})$.

By $C_n, P_n, K_n, K_{m,n}$ we denote the chordless cycle on $n \geq 4$ vertices, the chordless path on $n \geq 3$ vertices, the complete graph on $n \geq 1$ vertices, and the complete bipartite graph on $m + n$ vertices, respectively.

A *matching* is a set of non-incident edges of G, and a *perfect matching* is a matching saturating all the vertices of G.

If $|N(v)| = |\{w\}| = 1$, then v is a *pendant vertex* and vw is a *pendant edge* of G, where $N(v) = \{u : u \in V(G), uv \in E(G)\}$ is the *neighborhood* of $v \in V(G)$. If $G[N(v)]$ is a complete subgraph in G, then v is a *simplicial vertex* of G. A clique in G is called a *simplex* if it contains at least a simplicial vertex of G, [2].

A stable set of maximum size will be referred as to a *stability system* of G. The *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a stability system in G.

Let $\Omega(G)$ stand for the family of all stability systems of the graph G, and $core\,(G) = \cap\{S : S \in \Omega(G)\}$ (see [10]).

G is a *well-covered graph* if every maximal stable set of G is also a maximum stable set, i.e., it belongs to $\Omega(G)$ (Plummer, [11]). $G = (V, E)$ is called *very well-covered* provided G is well-covered, without isolated vertices and $|V| = 2\alpha(G)$ (Favaron, [4]). For instance, each $C_{2n}, n \geq 3$, is not well-covered, while C_4, C_5, C_7 are well-covered, but only C_4 is very well-covered.

The following characterization of stability systems in a graph, due to Berge, we shall use in the sequel.

PROPOSITION 1.*([1]) $S \in \Omega(G)$ if and only if every stable set A of G, disjoint from S, can be matched into S.*

By $\theta(G)$ we mean the *clique covering number* of G, i.e., the minimum number of cliques whose union covers $V(G)$. Recall also that:

$$i(G) = \min\{|S| : S \textit{ is a maximal stable set in } G\},$$

$$\gamma(G) = \min\{|D| : D \textit{ is a minimal dominating set in } G\},$$

where $D \subseteq V(G)$ is a *domination set* whenever $\{x, y\} \cap D \neq \emptyset$, for each $xy \in E(G)$.

In general, it can be shown (e.g., see [12]) that these graph invariants are related by the following inequalities:

$$\alpha(G^2) \leq \theta(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \theta(G).$$

For instance,

$$\alpha(C_7^2) = 2 < 3 = \theta(C_7^2) = \gamma(C_7) = i(C_7) = \alpha(C_7) < 4 = \theta(C_7)$$

(see also the graph G from Figure 1).

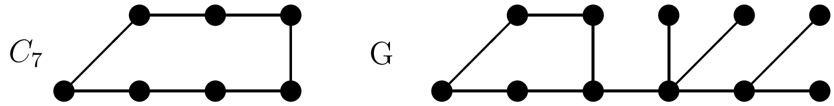

Figure 1: $\alpha(G^2) = \theta(G^2) = 3 < \gamma(G) < i(G) < \alpha(G) < \theta(G) = 7$.

Recall from [5] that a graph G is called:
(i) α^-*-stable* if $\alpha(G - e) = \alpha(G)$, for every $e \in E(G)$, and
(ii) α^+*-stable* if $\alpha(G + e) = \alpha(G)$, for each edge $e \in E(\overline{G})$, .
Recall the following results.

PROPOSITION 2.*([6]) A graph G is:*
(i) α^+*-stable if and only if* $|core\,(G)| \leq 1$*;*
(ii) α^-*-stable if and only if* $|N(v) \cap S| \geq 2$ *is true for every* $S \in \Omega(G)$ *and each* $v \in V(G) - S$*.*

By Proposition 2, an α^+-stable graph G may have either $|core\,(G)| = 0$ or $|core\,(G)| = 1$. This motivates the following definition.

DEFINITION 1.*([8]) A graph G is called:*
(i) α_0^+*-stable whenever* $|core\,(G)| = 0$*;*
(ii) α_1^+*-stable provided* $|core\,(G)| = 1$*.*

Any $C_n, n \geq 4$, is α^+-stable, and all $C_{2n}, n \geq 2$, are α^--stable. For other examples of α_0^+-stable and α_1^+-stable graphs, see Figure 2.

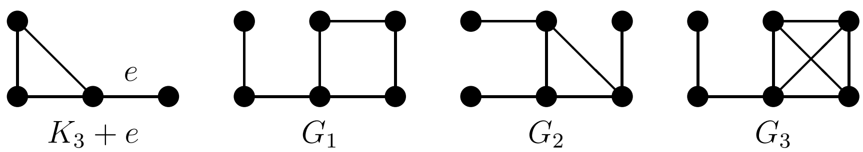

Figure 2: $K_3 + e$ is α_1^+-stable, while the graphs G_1, G_2, G_3 are α_0^+-stable.

In [6] it was shown that an α^+-stable tree $T \neq K_1$ can be only α_0^+-stable, and this is exactly the case of trees possessing a perfect matching. This result was generalized to bipartite graphs in [7].

The distance between two vertices $v, w \in V(G)$ is denoted by $dist_G(v, w)$, or simply $dist(v, w)$, if there is no ambiguity. By G^2 we denote the *second power* of the graph $G = (V, E)$, i.e., the graph having:

$$V(G^2) = V \text{ and } E(G^2) = \{vw : v, w \in V(G^2), 1 \leq dist_G(v, w) \leq 2\}.$$

Clearly, any stable set of G^2 is stable in G, as well, while the converse is not generally true. Therefore, one may assert that

$$1 \leq \alpha(G^2) \leq \alpha(G).$$

Let us notice that the both bounds are sharp.

For instance, it is easy to see that, if:

- G is not a complete graph and $dist(a,b) \leq 2$ holds for any $a, b \in V(G)$, then $\alpha(G) \geq 2 > 1 = \alpha(G^2)$; e.g., for the n-star graph $G = K_{1,n}$, with $n \geq 2$, we have $\alpha(G) = n > \alpha(G^2) = 1$;
- $G = P_4$, then $\alpha(G) = \alpha(G^2) = 2$.

Randerath and Volkmann proved the following theorem.

THEOREM 1.*([12]) For a graph G the following statements are equivalent:*
(i) *every vertex of G belongs to exactly one simplex of G;*
(ii) *G satisfies $\alpha(G) = \alpha(G^2)$;*
(iii) *G satisfies $\theta(G) = \theta(G^2)$;*
(iv) *G satisfies $\alpha(G^2) = \theta(G^2) = \gamma(G) = i(G) = \alpha(G) = \theta(G)$.*

We call a graph G *square-stable* if $\alpha(G) = \alpha(G^2)$. In this paper we continue to investigate square-stable graphs. For instance, we show that any square-stable graph having non-empty edge-set is also α_0^+-stable, and that none of them is α^--stable. We deduce that the square-stable trees coincide with the well-covered trees.

Clearly, any complete graph is square-stable. Moreover, since $K_n^2 = K_n$, we get that

$$\Omega(K_n) = \Omega(K_n^2) = \{\{v\} : v \in V(K_n)\}.$$

Some other examples of (non-)square-stable graphs are depicted in Figure 3.

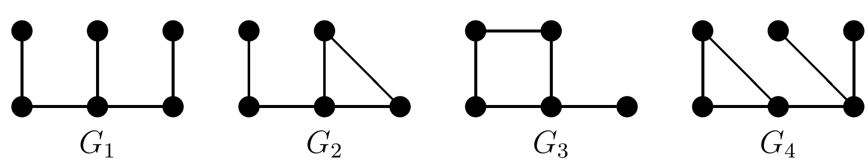

Figure 3: G_1, G_2 are square-stable graphs, while G_3, G_4 are not square-stable.

PROPOSITION 3.*A graph G is square-stable if and only if $\Omega(G^2) \subseteq \Omega(G)$.*

Proof. Clearly, each stable set A of G^2 is stable in G, too. Consequently, if G is square-stable, then every stability system of G^2 is a stability system of G, as well, i.e., $\Omega(G^2) \subseteq \Omega(G)$.

The converse is clear. □

Let us notice that if $H_i, 1 \leq i \leq k$, are the connected components of graph G, then $S \in \Omega(G)$ if and only if $S \cap V(H_i) \in \Omega(H_i), 1 \leq i \leq k$. Since, in addition, G and G^2 are simultaneously connected or disconnected, Proposition 3 assures that a disconnected graph is square-stable if and only if each of its connected components is square-stable. Therefore, in the rest of the paper all the graphs are connected, unless otherwise stated.

2. Main results

PROPOSITION 4.*For any non-complete graph G, he following statements are true:*

(i) *if $S \in \Omega(G^2)$, then $dist_G(a,b) \geq 3$ holds for any distinct $a, b \in S$;*

(ii) *if G is square-stable, then for every $S \in \Omega(G^2)$ and each $a \in S$, there is $b \in S$ with $dist_G(a,b) = 3$;*

(iii) *G is square-stable if and only if there is some $S \in \Omega(G)$ such that $dist_G(a,b) \geq 3$ holds for all distinct $a, b \in S$.*

Proof. *(i)* If $S \in \Omega(G^2)$ and $a, b \in S, a \neq b$, then $dist_G(a,b) \geq 3$, since otherwise $ab \in E(G^2)$, contradicting the stability of S in G^2.

(ii) Suppose, on the contrary, that there are $S \in \Omega(G^2)$ and some $a \in S$, such that $dist_G(a,b) \geq 4$ holds for any $b \in S$. Let $v \in V$ be such that $dist_G(a,v) = 2$. Hence, $dist_G(v,w) \geq 2$ is valid for any $w \in S$, and consequently, $S \cup \{v\}$ is stable in G, thus contradicting the fact that S is a maximum stable set in G, as well.

(iii) If G is square-stable, then Proposition 3 ensures that $\Omega(G^2) \subseteq \Omega(G)$, and, by part *(i)*, $dist(a,b) \geq 3$ holds for every $S \in \Omega(G^2)$ and all distinct $a, b \in S$.

Conversely, let $S \in \Omega(G)$ be such that $dist_G(a,b) \geq 3$ holds for any $a, b \in S$. Hence, S is stable in G^2, as well, and consequently, we obtain

$$|S| \leq \alpha(G^2) \leq \alpha(G) = |S|,$$

which clearly implies $\alpha(G^2) = \alpha(G)$, i.e., G is square-stable. □

PROPOSITION 5.*$\Omega(G^2) = \Omega(G)$ if and only if G is a complete graph.*

Proof. Suppose, on the contrary, that $\Omega(G^2) = \Omega(G)$ holds for some non-complete graph G. Let $S \in \Omega(G)$ and $a \in S$.

Since $\Omega(G) = \Omega(G^2)$, Proposition 4 *(ii)* implies that $dist_G(a, v) \geq 3$ holds for every $v \in S - \{a\}$, and, according to Proposition 4 *(iii)*, there is some $b \in S$ with $dist_G(a, b) = 3$. Now, if $c \in N_G(a)$ and $dist_G(c, b) = 2$, Proposition 4 *(iii)* implies that $S \cup \{c\} - \{a\} \in \Omega(G) - \Omega(G^2)$, contradicting the equality $\Omega(G^2) = \Omega(G)$.

The converse is clear. □

Let $A \triangle B$ denotes the symmetric difference of the sets A, B, i.e., the set

$$A \triangle B = (A - B) \cup (B - A).$$

THEOREM 2.*For a graph G the following assertions are equivalent:*

(i) *G is square-stable;*

(ii) *there exists $S \in \Omega(G)$ that satisfies the property*

$P1$: any stable set A of G disjoint from S can be uniquely matched into S;

(iii) *every $S \in \Omega(G^2)$ has property $P1$;*

(iv) *for each $S_1 \in \Omega(G)$ and every $S_2 \in \Omega(G^2), G[S_1 \triangle S_2]$ has a unique perfect matching.*

Proof. *(i) ⇒ (ii), (iii)* By Proposition 3 we get that $\Omega(G^2) \subseteq \Omega(G)$. Now, every $S \in \Omega(G^2)$ belongs also to $\Omega(G)$, and consequently, if A is a stable set in G disjoint from S, Proposition 1 implies that A can be matched into S. If there exists another matching of A into S, then at least one vertex $a \in A$ has two neighbors in S, say b, c. Hence, $bc \in E(G^2)$ and this contradicts the stability of S. Therefore, each $S \in \Omega(G^2) \subseteq \Omega(G)$ has property $P1$.

(ii) ⇒ (i) Let $S_0 \in \Omega(G)$ be a stability system of G that satisfies the property $P1$. Suppose, on the contrary, that G is not square-stable. It follows that $S_0 \notin \Omega(G^2)$, i.e., there are $v, w \in S_0$ with $vw \in E(G^2)$. Hence, there must be some $u \in V - \{v, w\}$, such that $uv, uw \in E(G)$. Consequently, there are two matchings of $A = \{u\}$ into S_0, contradicting the fact that S_0 has property $P1$.

(iii) ⇒ (iv) Let $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$. Then $|S_2| \leq |S_1|$, and since $S_1 - S_2$ is stable in G and disjoint from S_2, we infer that $S_1 - S_2$ can be uniquely matched into S_2, precisely into $S_2 - S_1$, and because $|S_2 - S_1| \leq |S_1 - S_2|$, this matching is perfect. In conclusion, $G[S_1 \triangle S_2]$ has a unique perfect matching.

(iv) $\Rightarrow$ *(i)* If $G[S_1 \bigtriangleup S_2]$ has a perfect matching, for every $S_1 \in \Omega(G)$ and each $S_2 \in \Omega(G^2)$, it follows that $|S_1 - S_2| = |S_2 - S_1|$, and this implies $|S_1| = |S_2|$, i.e., $\alpha(G) = \alpha(G^2)$ is valid. □

COROLLARY 1. *There exists no α^--stable graph having non-empty edge set, that is square-stable.*

Proof. According to Proposition 2, G is α^--stable provided $|N(v) \cap S| \geq 2$ holds for every $S \in \Omega(G)$ and each $v \in V(G) - S$. If, in addition, G is also square-stable, then Theorem 2 assures that there exists some $S_0 \in \Omega(G)$ satisfying property $P1$, which implies that $|N(v) \cap S_0| = 1$ holds for every $v \in V(G) - S_0$. This incompatibility concerning S_0 proves that G can not be simultaneously square-stable and α^--stable. □

In Figure 4 are presented some non-square-stable graphs: $K_4 - e$, which is also α^--stable, C_6, which is both α^--stable and α^+-stable, and H, which is neither α^--stable, nor α^+-stable.

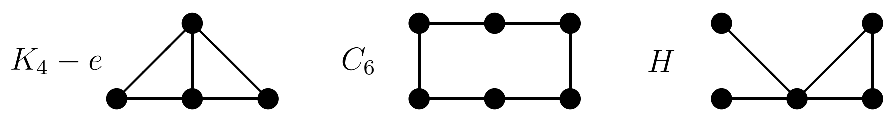

Figure 4: Non-square-stable graphs: $K_4 - e$ and C_6 are also α^--stable graphs, while H is not α^--stable.

Recall the following characterizations of well-covered trees.

THEOREM 3. *([13])* (i) *A tree having at least two vertices is well-covered if and only if it has a perfect matching consisting of pendant edges.*

(ii) *([9]) A tree $T \neq K_1$ is well-covered if and only if either T is a well-covered spider, or T is obtained from a well-covered tree T_1 and a well-covered spider T_2, by adding an edge joining two non-pendant vertices of T_1, T_2.*

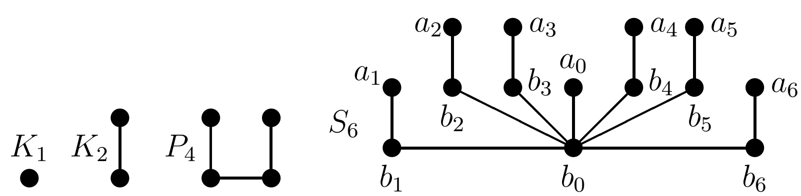

Figure 5: Well-covered spiders.

It turns out that a tree $T \neq K_1$ is well-covered if and only if it is very well-covered. Clearly, K_1 is both well-covered and square-stable, but is not very well-covered.

THEOREM 4. (i) *Any square-stable graph is well-covered.*

(ii) *Any square-stable graph with non-empty edge set is* α_0^+*-stable.*

(iii) *A tree of order at least two is square-stable if and only if it is very well-covered.*

Proof. *(i)* Assume, on the contrary, that there exists a square-stable graph G which is not well-covered. Hence, there is in G some maximal stable set A having $|A| < \alpha(G)$. According to Theorem 2 *(iii)*, for every $S \in \Omega(G^2)$, there is a unique matching from $B = A - S \cap A$ into S, in fact, into $S - A$. Consequently, $S \cup B - N(B) \cap S$ is a stability system of G that includes A, contradicting the fact that A is a maximal stable set.

(ii) Suppose, on the contrary, that G is a square-stable graph, but is not α_0^+-stable, i.e., there exists an $a \in core\,(G)$. Hence, every maximal stable set containing some $b \in N(a)$ can not be maximum, in contradiction with the fact, by part *(i)*, G is also well-covered.

(iii) According to part *(i)*, every square-stable tree T is well-covered, and, by Theorem 3, T is very well-covered, since it has at least two vertices.

Conversely, if T is a very well-covered tree, then, by Theorem 3, it has a perfect matching

$$\{a_i b_i : 1 \leq i \leq |V(T)|/2, \deg(a_i) = 1\},$$

consisting of pendant edges only. Hence, $S = \{a_i : 1 \leq i \leq |V(T)|/2\}$ is a stable set in T of size $|V(T)|/2$, i.e., $S \in \Omega(T)$, because $\alpha(T) = |V(T)|/2$. Moreover, $S \in \Omega(T^2)$, since $dist_T(a_i, a_j) \geq 3$, for $i \neq j$. □

Actually, Theorem 4 *(i)* is stated implicitly in the proof of Theorem 1 from [12]. The converse of Theorem 4 *(i)* is not generally true; e.g., C_5 is well-covered, but is not square-stable. The square-stable graphs do not coincide with the very well-covered graphs. For instance, P_4 is both square-stable and very well-covered, C_4 is very well-covered and non-square-stable, but there are square-stable graphs that are not very well-covered; for example, the graph G in Figure 6. Let us also remark that there are α_0^+-stable graphs that are not square-stable, e.g., C_6.

THEOREM 5.*For a graph G the following statements are equivalent:*

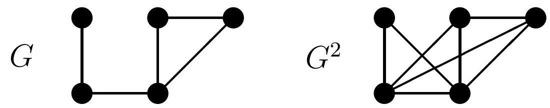

Figure 6: A square-stable graph which is not very well-covered.

(i) *G is square-stable;*
(ii) *there is $S_0 \in \Omega(G)$ that has the property*
$P2$: for any stable set A of G disjoint from S_0, $A \cup S^* \in \Omega(G)$ holds for some $S^* \subset S_0$.

Proof. *(i) ⇒ (ii)* By Theorem 2, for every $S \in \Omega(G^2)$ and each stable set A in G, disjoint from S, there is a unique matching of A into S. Consequently, $S^* = S - N(A) \cap S$ has $|S^*| = |S| - |A|$ and $S^* \cup A \in \Omega(G)$.

(ii) ⇒ (i) It suffices to show that $S_0 \in \Omega(G^2)$. If $S_0 \notin \Omega(G)$, there must exist $a, b \in S_0$ such that $ab \in E(G^2)$, and this is possible provided $a, b \in N(c) \cap S_0$ for some $c \in V - S_0$. Hence, $|S_0 \cup \{c\} - \{a, b\}| < |S_0|$ and this implies that $\{c\} \cup S^* \notin \Omega(G)$ holds for any $S^* \subset S$, contradicting the fact that S_0 has the property $P2$. Therefore, we deduce that $S_0 \in \Omega(G^2)$, and this implies that $\alpha(G) = \alpha(G^2)$. □

As a consequence of Theorem 5, we obtain that $\Omega(G)$ is the set of bases of a matroid on $V(G)$ provided G is a complete graph.

Corollary 2.$\Omega(G)$ *is the set of bases of a matroid on $V(G)$ if and only if $\Omega(G^2) = \Omega(G)$.*

Proof. If $\Omega(G)$ is the set of bases of a matroid on V, then any $S \in \Omega(G)$ must have the property $P2$. By Theorem 5, G is square-stable and therefore $\Omega(G^2) \subseteq \Omega(G)$. Suppose that there exists $S_0 \in \Omega(G) - \Omega(G^2)$. It follows that there are $a, b \in S_0$ and $c \in N(a) \cap N(b)$. Hence, $\{c\}$ is stable in G and disjoint from S_0, but $S^* \cup \{c\} \notin \Omega(G)$ for any $S^* \subset S_0$, and this is a contradiction, since S_0 has property $P2$. Consequently, the equality $\Omega(G^2) = \Omega(G)$ is true.

Conversely, according to Theorem 5, any $S \in \Omega(G^2) = \Omega(G)$ has the property $P2$. Therefore, $\Omega(G)$ is the set of bases of a matroid on V. □

Combining Proposition 5 and Corollary 2, we get the following result.

Corollary 3.*([3]) Let G be a disconnected graph. Then $\Omega(G)$ is the set of bases of a matroid on $V(G)$ if and only if G is a disjoint union of cliques.*

3. Conclusions

In this paper we continue the research, started by Randerath and Volkmann [12] in 1997, on the class of square-stable graphs, by emphasizing a number of new properties. It turns out that any of the two equalities: $\alpha(G^2) = \alpha(G)$ and $\theta(G^2) = \theta(G)$, is equivalent to $\alpha(G^2) = \theta(G^2) = \gamma(G) = i(G) = \alpha(G) = \theta(G)$, and it could be interesting to study graphs satisfying other equalities between the invariants appearing in the relation:

$$\alpha(G^2) \leq \theta(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \theta(G).$$

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