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STABILITY OF A VARIATIONAL INEQUALITY WITH RESPECT TO DOMAIN PERTURBATIONS

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Abstract.We study a class of parametric nonlinear variational inequalities, in the special case when the parameter is the underlying domain. The stability of the solution with respect to the perturbations of the domain is investigated.

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1. Introduction

Parametric variational inequalities were studied in many papers, among which we mention [7], [8], where there were established some results concerning the stability under small perturbations of the parameter, for the solutions of some abstract variational inequalities.

A special case is when the parameter to be perturbed is the geometric domain on which the variational problem is defined. Continuity with respect to the domain is an important issue in shape optimization and was studied a lot, see [3], [4], [5], [6], [1] and the references therein.

More precisely, consider a family of variational inequalities:

$$\text{Find} \quad u_\Omega \in K(\Omega), \quad \text{such that} \quad \langle \mathcal{A}(\Omega, u_\Omega), v - u_\Omega \rangle \geq 0, \ \forall\, v \in K(\Omega),$$

where $\Omega \subset \mathbf{R}^N$ is bounded and open , $K(\Omega) \subset H_0^1(\Omega)$ and $\mathcal{A}(\Omega, u) \in (H_0^1(\Omega))^*$.

For Ω_0 fixed in the class of admissible domains and u_0 a solution of the corresponding inequality, the following problem appears: Is there a neighborhood W_0 of Ω_0 and a mapping u defined on it, continuous at Ω_0, with $u(\Omega_0) = u_0$, and such that for each $\Omega \in W_0$, $u(\Omega)$ is a solution of the corresponding variational inequality ?

We studied such a problem in a previous paper [2], with the topology on the class of admissible domains given by the mapping method (see [5]). In this paper we are concerned with the stability with respect to the domain when the topology is the one defined by the Hausdorff complementary metric.

2. Abstract parametric variational inequalities

We expose here a particular case of the results proved in [7], which will be used in what follows.

Let H be a real, reflexive Banach space and H^* its dual. Let W be a topological space, $T : W \times H \to H^*$ a mapping and $K : W \to 2^H$ a set-valued mapping. For each parameter $w \in W$, a variational inequality is formulated:

$(VI)_w$ Find $x(w) \in K(w)$, such that $\langle T(w, x(w)), y - x(w)\rangle \geq 0$, $\forall\, y \in K(w)$.

Let $w_0 \in W$ be fixed and let $x_0 \in K(w_0)$ be the solution of the corresponding problem $(VI)_{w_0}$. The problem $(VI)_{w_0}$ is called *stable under perturbations* if there exists a neighborhood W_0 of w_0 and a mapping $x : W_0 \to H$, continuous at w_0, with $x(w_0) = x_0$, such that, for each $w \in W_0$, $x(w)$ is a solution of $(VI)_w$.

Definition 1. *The mapping $T : W \times H \to H^*$ is called* consistent in w at (w_0, x_0) *if, for each $0 < r \leq 1$, there exists a neighborhood W_r of w_0 and a function $\beta : W_r \to \mathbf{R}$, continuous at w_0, with $\beta(w_0) = 0$ such that, for each $w \in W_r$, there exists $y_w \in K(w)$ such that*

$$\|y_w - x_0\| \leq \beta(w)$$

and

$$\langle T(w, y_w), z - y_w\rangle + \beta(w)\|z - y_w\| \geq 0,$$

for each $z \in K(w)$ such that $r < \|z - y_w\| \leq 2$.

Definition 2. *The mappings $T(w, \cdot) : H \to H^*$ are called* uniformly strongly monotone *on $W_0 \subset W$ if there exists a positive constant α, such that for all $w \in W_0$ and $x, y \in H$, $x \neq y$ we have :*

$$\langle T(w, x) - T(w, y), x - y\rangle \geq \alpha\|x - y\|^2.$$

Theorem 1. *With the above notations, let the set $K(w)$ be closed and convex for each parameter $w \in W$. Consider $w_0 \in W$ and $x_0 \in K(w_0)$ fixed. Suppose that:*
(i) x_0 is a solution of $(VI)_{w_0}$;
(ii) T is consistent in w at (w_0, x_0);

(iii) there exists a neighborhood V of w_0 such that the maps $T(w,\cdot)$ are uniformly strongly monotone, continuous from the line segments of H to H^* with the weak topology, for all $w \in V$ and $x \in K(w)$.
Then the problem $(VI)_{w_0}$ is stable under perturbations.

3.Stability with respect to perturbations of the underlying domain

Let D be a bounded and open subset of $\mathbf{R}^N$.

Denote $\mathcal{G}(D) = \{\Omega \subset D \mid \Omega \text{ open }\}$.

On each $\Omega \in \mathcal{G}(D)$ we formulate a variational inequality, obtaining the following family of problems:

$$(VI_\Omega) \qquad \text{Find } u_\Omega \in K(\Omega), \quad \text{such that}$$
$$\int_\Omega A(x, u_\Omega(x), \nabla u_\Omega(x))(\nabla v(x) - \nabla u_\Omega(x))dx$$
$$+ \int_\Omega a(x, u_\Omega(x), \nabla u_\Omega(x))(v(x) - u_\Omega(x))dx \geq 0, \ \forall\, v \in K(\Omega),$$

where

(H1) $K(\Omega)$ is a closed, convex, nonempty subset of the Sobolev space $H_0^1(\Omega)$,

(H2) $A = (a_1, \ldots, a_N)$, $a_j, a : D \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ are functions with the properties:

(P1) $a_j(x, \eta, \xi), a(x, \eta, \xi)$ are measurable with respect to the first variable and continuous with respect to the pair (η, ξ),

(P2) $|a_j(x, \eta, \xi)| + |a(x, \eta, \xi)| \leq c_1(k(x) + |\eta| + \|\xi\|_N)$, with $c_1 > 0$ a constant and $k \in L^2(D)$ a positive function,

(P3) $\sum_{j=1}^N a_j(x, \eta, \xi)\xi_j \geq c_2\|\xi\|_N^2 - c_3$ and $a(x, \eta, \xi)\eta \geq c_4|\eta|^2 - c_5$, with c_j positive constants,

(P4) $\sum_{j=1}^N [a_j(x, \eta, \xi) - a_j(x, \tilde{\eta}, \tilde{\xi})](\xi_j - \tilde{\xi}_j) + [a(x, \eta, \xi) - a(x, \tilde{\eta}, \tilde{\xi})](\eta - \tilde{\eta}) \geq \alpha(|\eta - \tilde{\eta}|^2 + \|\xi - \tilde{\xi}\|_N^2)$, for a.e. $x \in D$, for all $\eta, \tilde{\eta} \in \mathbf{R}$ and $\xi, \tilde{\xi} \in \mathbf{R}^N$.

In the conditions stated above, there exists a solution to the variational inequality (VI_Ω) (see for example [9], pg. 74).

For each inequality from the family (VI_Ω), the space where the solution lies ($H_0^1(\Omega)$) changes together with Ω. In order to apply Theorem 1, we will write equivalent forms of the variational problems by extending the functions on the hold-all domain D:

$$(VI_{D\Omega}) \qquad \text{Find } u^\Omega \in e_0(K(\Omega)), \quad \text{such that}$$
$$\int_D A(x, u^\Omega(x), \nabla u^\Omega(x))(\nabla v(x) - \nabla u^\Omega(x))dx$$
$$+ \int_D a(x, u^\Omega(x), \nabla u^\Omega(x))(v(x) - u^\Omega(x))dx \geq 0, \ \forall\, v \in e_0(K(\Omega)),$$

where $u^\Omega = e_0(u_\Omega)$ is the extension with 0 to D of the function $u_\Omega \in H_0^1(\Omega)$ and $e_0(K(\Omega)) = \{e_0(u) \mid u \in K(\Omega)\} \subset H_0^1(D)$.

According to [1], the extension operator $e_0 : H_0^1(\Omega) \to H_0^1(D)$ is linear and isometric. Also, it takes place:

THEOREM 2. [1] *Let Ω and D be open subsets of $\mathbf{R}^N$, with $\Omega \subset D$. The linear subspace $e_0(K(\Omega))$ of $H_0^1(D)$ is closed and isometric isomorphic with $H_0^1(\Omega)$. For each $\psi \in e_0(K(\Omega))$, we have $\psi|_\Omega \in H_0^1(\Omega)$ and $\partial^\alpha \psi = 0$, a.e. on $D \setminus \Omega$, for every α, $|\alpha| \leq 1$.*

The topology that we use on the class of geometric domains $\mathcal{G}(D)$ is the Hausdorff complementary topology.

For a set $A \subset \mathbf{R}^N$, the distance function is defined by

$$d_A(x) = \begin{cases} \inf_{y \in A} \|y - x\|_N, & A \neq \emptyset \\ \infty, & A = \emptyset \end{cases}$$

The Hausdorff complementary metric on $\mathcal{G}(D)$ is:

$$\rho_H^C(\Omega_1, \Omega_2) = \|d_{C\Omega_1} - d_{C\Omega_2}\|_{C(D)},$$

for each $\Omega_1, \Omega_2 \in \mathcal{G}(D)$, $C\Omega_i = D \setminus \Omega_i$.

THEOREM 3.[1] *$(\mathcal{G}(D), \rho_H^C)$ is a compact metric space.*

Defining the operator $\mathcal{A} : H_0^1(D) \to (H_0^1(D))^*$ by

$$\langle \mathcal{A}(u), w \rangle = \int_D A(x, u(x), \nabla u(x)) \nabla w(x) dx + \int_D a(x, u(x), \nabla u(x)) w(x) dx,$$

we can write the family of problems $(VI_{D\Omega})$ like:

Find $\ u^\Omega \in e_0(K(\Omega)) \ $ s. t. $\ \langle \mathcal{A}(u^\Omega), v - u^\Omega \rangle \geq 0, \ \forall\, v \in e_0(K(\Omega)) \subset H_0^1(D)$.

With these notations, the theory for the abstract variational inequalities can be applied: $H = H^1_0(D)$ (reflexive Banach space); $W = \mathcal{G}(D)$ endowed with the Hausdorff complementary topology; $T = \mathcal{A}$; $\tilde{K} : \mathcal{G}(D) \to 2^{H^1_0(D)}$, $\tilde{K}(\Omega) = e_0(K(\Omega))$.

Let $\Omega_0 \in \mathcal{G}(D)$ be fixed and let $u^{\Omega_0} = e_0(u_{\Omega_0})$ be the solution of the problem $(VI_{D\Omega_0})$. We make the additional hypotheses:

(H3) There exists a neighborhood W_0 of Ω_0 and a positive constant δ such that, for every $\Omega_1, \Omega_2 \in W_0$ and every $u_1 \in e_0(K(\Omega_1))$, there exists $u_2 \in e_0(K(\Omega_2))$ such that

$$\|u_1 - u_2\|_{H^1_0(D)} \leq \delta \rho^C_H(\Omega_1, \Omega_2).$$

(H4) $|a_j(x, \eta, \xi) - a_j(x, \tilde{\eta}, \tilde{\xi})| \leq L_1|\eta - \tilde{\eta}| + L_2\|\xi - \tilde{\xi}\|_N$ and $|a(x, \eta, \xi) - a(x, \tilde{\eta}, \tilde{\xi})| \leq L_3|\eta - \tilde{\eta}| + L_4\|\xi - \tilde{\xi}\|_N$, with L_i positive constants. In these conditions, the main result of this section can be proved:

THEOREM 4. *Let $\Omega_0 \in \mathcal{G}(D)$ be fixed and let $u^{\Omega_0} \in e_0(K(\Omega_0))$ be a solution of the variational inequality $(VI_{D\Omega_0})$. If **(H1)-(H4)** are satisfied, then $(VI_{D\Omega_0})$ is stable under perturbations, that is: there exists a neighborhood W_0 of Ω_0 and a mapping $u : W_0 \to H^1_0(D)$ such that for each $\Omega \in W_0$, $u(\Omega)$ is a solution of $(VI_{D\Omega})$, $u(\Omega_0) = u^{\Omega_0}$ and u is continuous at Ω_0.*

Proof:

(a) We prove that $\mathcal{A}$ is consistent in Ω at (Ω_0, u^{Ω_0}). Let $r \in]0, 1]$, W_0 the neighborhood from **(H3)** and $\Omega \in W_0$. For $\Omega_1 := \Omega_0$, $\Omega_2 := \Omega$, $u_1 = u^{\Omega_0}$ there exists, according to **(H3)**, $v_\Omega \in e_0(K(\Omega))$ such that

$$\|v_\Omega - u^{\Omega_0}\|_{H^1_0(D)} \leq \delta \rho^C_H(\Omega, \Omega_0).$$

We define, following the idea from [7],

$$\beta(\Omega) = \max\{\sqrt{\delta \rho^C_H(\Omega, \Omega_0)}, 2\|\mathcal{A}(v_\Omega) - \mathcal{A}(u^{\Omega_0})\|_{(H^1_0(D))^*}\}$$

Obviously, $\beta(\Omega_0) = 0$ (taking $v_{\Omega_0} = u^{\Omega_0}$).
To prove the continuity of β at Ω_0, let $\Omega_n \to \Omega_0$ in the Hausdorff complementary topology, that is $\rho^C_H(\Omega_n, \Omega_0) \to 0$. This also implies (by **(H3)**) that there exists v_{Ω_n} such that $v_{\Omega_n} \to u^{\Omega_0}$ in $H^1_0(D)$.

$$\|\mathcal{A}(v_{\Omega_n}) - \mathcal{A}(u^{\Omega_0})\|_{(H^1_0(D))^*}$$
$$= \sup\{|\langle \mathcal{A}(v_{\Omega_n}) - \mathcal{A}(u^{\Omega_0}), w\rangle| \mid w \in H^1_0(D), \|w\|_{H^1_0(D)} \leq 1\}$$

We evaluate, using **(H4)** and Hölder inequality:

$$\begin{aligned}
&|\langle \mathcal{A}(v_{\Omega_n}) - \mathcal{A}(u^{\Omega_0}), w\rangle| \\
&\leq \int_D |[A(x, v_{\Omega_n}(x), \nabla v_{\Omega_n}(x)) - A(x, u^{\Omega_0}(x), \nabla u^{\Omega_0}(x))]\nabla w(x)|dx \\
&+ \int_D |[a(x, v_{\Omega_n}(x), \nabla v_{\Omega_n}(x)) - a(x, u^{\Omega_0}(x), \nabla u^{\Omega_0}(x))]w(x)|dx \\
&\leq \int_D \Big(L_1 |v_{\Omega_n}(x) - u^{\Omega_0}(x)| \\
&\quad + L_2 \|\nabla v_{\Omega_n}(x) - \nabla u^{\Omega_0}(x)\|_N\Big) \left(|w(x)| + \|\nabla w(x)\|_N\right) dx \\
&\leq L\|v_{\Omega_n} - u^{\Omega_0}\|_{H_0^1(D)} \|w\|_{H_0^1(D)} \to 0,
\end{aligned}$$

when $\Omega_n \to \Omega_0$. This implies the continuity of β at Ω_0.
Let now W_r be a neighborhood of Ω_0 such that for each $\Omega \in W_r$, $\beta(\Omega) \leq 1$ (this is possible because $\beta(\Omega_0) = 0$ and β is continuous at Ω_0) and $r - 4\beta(\Omega)\|\mathcal{A}(u^{\Omega_0})\|_{(H_0^1(D))^*} \geq 0$ (β can be sufficiently small in a neighborhood of Ω_0 and r and $\|\mathcal{A}(u^{\Omega_0})\|$ are fixed).
From **(H3)** and the definition of β we have that

$$\|v_\Omega - u^{\Omega_0}\|_{H_0^1(D)} \leq \beta^2(\Omega) \leq \beta(\Omega).$$

For $\Omega \in W_r$, let $z \in e_0(K(\Omega))$ such that $r \leq \|z - v_\Omega\| \leq 2$. We have:

$$\begin{aligned}
&\langle \mathcal{A}(v_\Omega), z - v_\Omega\rangle + \beta(\Omega)\|z - v_\Omega\|_{H_0^1(D)} \\
&= \langle \mathcal{A}(v_\Omega) - \mathcal{A}(u^{\Omega_0}), z - v_\Omega\rangle + \langle \mathcal{A}(u^{\Omega_0}), z - v_\Omega\rangle + \beta(\Omega)\|z - v_\Omega\|_{H_0^1(D)} \\
&\geq -\|\mathcal{A}(v_\Omega) - \mathcal{A}(u^{\Omega_0})\|_{(H_0^1(D))^*}\|z - v_\Omega\|_{H_0^1(D)} + \langle \mathcal{A}(u^{\Omega_0}), z - z_0\rangle \\
&\quad + \langle \mathcal{A}(u^{\Omega_0}), z_0 - u^{\Omega_0}\rangle + \langle \mathcal{A}(u^{\Omega_0}), u^{\Omega_0} - v_\Omega\rangle + \beta(\Omega)r \\
&\geq -\frac{1}{2}\beta(\Omega)r - \|\mathcal{A}(u^{\Omega_0})\|\|z - z_0\| - \|\mathcal{A}(u^{\Omega_0})\|\|u^{\Omega_0} - v_\Omega\| + \beta(\Omega)r \\
&\geq \frac{1}{2}\beta(\Omega)r - 2\beta^2(\Omega)\|\mathcal{A}(u^{\Omega_0})\| = \frac{1}{2}\beta(\Omega)[r - 4\beta(\Omega)\|\mathcal{A}(u^{\Omega_0})\|] \geq 0.
\end{aligned}$$

Here we used the fact that z_0 is chosen in $e_0(K(\Omega_0))$ such that $\|z - z_0\| \leq \delta\rho_H^C(\Omega_0, \Omega) \leq \beta^2(\Omega)$ and $\|u^{\Omega_0} - v_\Omega\| \leq \delta\rho_H^C(\Omega_0, \Omega) \leq \beta^2(\Omega)$. So the consistency of $\mathcal{A}$ is proved.

(b) $\mathcal{A}$ is uniformly strongly monotone. Indeed, for every $u, v \in H_0^1(D)$,

$u \neq v$ we have, using the property *(P4)*:

$$\begin{aligned}
&\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle \\
&= \int_D \sum_{j=1}^{N} [a_j(x, u(x), \nabla u(x)) - a_j(x, v(x), \nabla v(x))](\nabla u_j(x) - \nabla v_j(x)) \\
&\quad +[a(x, u(x), \nabla u(x)) - a(x, v(x), \nabla v(x))](u(x) - v(x))dx \\
&\geq \int_D \alpha(|u(x) - v(x)|^2 + \|\nabla u(x) - \nabla v(x)\|_N^2)dx = \alpha \|u - v\|^2_{H_0^1(D)}.
\end{aligned}$$

(c) The operator $\mathcal{A}$ is continuous from $H_0^1(D)$ with the strong topology to $(H_0^1(D))^*$ with the weak* topology.

Indeed, for $u_n \to u$ in $H_0^1(D)$ and $v \in H_0^1(D)$, we have

$$|\langle \mathcal{A}(u_n), v\rangle - \langle \mathcal{A}(u), v\rangle| \leq L\|u_n - u\|_{H_0^1(D)} \|v\|_{H_0^1(D)} \to 0.$$

All the hypotheses of Theorem 1 are satisfied, so applying this result we get the desired conclusion.

Remark. *The classical linear case*

$$\langle \mathcal{B}(u), w\rangle = \int_D B(x)\nabla u(x)\nabla w(x)dx + \int_D b(x)u(x)w(x)dx$$

with $B \in C(\mathbf{R}^N)^{N^2} \cap L^\infty(\mathbf{R}^N)^{N^2}$ *and* $b \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, *satisfies the hypotheses* ***H1, H2, H4***.

References

[1] M.C. Delfour and J.-P. Zolésio, *Shapes and Geometries. Analysis, Differential Calculus and Optimization*, Advances in Design and Control, SIAM, 2001.

[2] D. Inoan, *Stability with respect to the domain of a nonlinear variational inequality*, Mathematical Notes, Miskolc, vol. 4, no. 1, (2003), pg. 25-34.

[3] J. Sokolowski and J.P. Zolesio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer-Verlag, 1992.

[4] O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, New-York, 1984.

[5] F.Murat and J. Simon, *Sur le Controle par un Domaine Geometrique*, preprint No. 76015, University of Paris, 1976.

[6] W.B. Liu and J.E. Rubio, *Optimal Shape Design for Systems Governed by Variational Inequalities, Part 1:Existence theory for Elliptic Case, Part 2: Existence Theory for Evolution Case*, J. of Optimization Theory and Applications, 69 (1991) pg. 351-371, pg. 373-396.

[7] G. Kassay and J. Kolumbán, *Multivalued Parametric Variational Inequalities with α-Pseudomonotone Maps*, J. of Optimization Theory and Applications 107, no.1 (2000), pg. 35-50.

[8] W. Alt and I. Kolumbán, *An Implicit Function Theorem for a Class of Monotone Generalized Equations*, Kybernetica, vol. 29 (1993), pg. 210-221.

[9] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, Mathematical Surveys and Monographs, vol. 49, (1997).

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