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ON A CLASS OF α-CONVEX FUNCTIONS

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Abstract. In this paper we define a general class of α-convex functions with respect to a convex domain D contained in the right half plane by using a generalized Sălăgean operator introduced by F.M. Al-Oboudi in [5] and we give some properties of this class.

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1.Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f$ is univalent in $U\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

Let D^n be the Sălăgean differential operator ([10]) defined as:

$$D^n : A \to A, n \in \mathbf{N}$$

and

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)).$$

Remark 1.1 *If* $f \in S$, $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $j = 2, 3, ...,$ $z \in U$ *then* $D^n f(z) = z + \sum\limits_{j=2}^{\infty} j^n a_j z^j$.

The aim of this paper is to define a general class of α-convex functions with respect to a convex domain D contained in the right half plane by using a generalized Sălăgean operator introduced by F.M. Al-Oboudi in[5] and to obtain some leftoperties of this class.

2. Preliminary results

We recall here the definitions of the well - known classes of starlike functions, convex functions and α-convex functions (see [6])

$$S^* = \left\{ f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0, z \in \mathbf{U} \right\},$$

$$S^c = CV = K = \left\{ f \in H(U);\ f(0) = f'(0) - 1 = 0, Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0,\ z \in U \right\},$$

$$M_\alpha = \{ f \in H(U), f(0) = f'(0) - 1 = 0, \mathrm{Re}J(\alpha, f; z) > 0,\ z \in U\ ,\ \alpha \in \mathbf{R} \}$$

where

$$J\left(\alpha, f; z\right) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

We observe that $M_0 = S^*$ and $M_1 = S^c$ where S^* and S^c are the class of starlike functions, respectively the class of convex functions.

Remark 2.1. *By using the subordination relation, we may define the class M_α thus if $f(z) = z + a_2 z^2 + ...,\ z \in U$, then $f \in M_\alpha$ if and only if $J(\alpha, f; z) \prec \frac{1+z}{1-z},\ z \in U$, where by "$\prec$" we denote the subordination relation.*

Let consider the Libera-Pascu integral operator $L_\alpha : A \to A$ defined as:

$$f(z) = L_\alpha F(z) = \frac{1+a}{z^a}\int_0^z F(t) \cdot t^{a-1} dt, \alpha \in \mathbf{C}\ , \mathrm{Re}\alpha \geq 0. \quad (1)$$

In the case $a = 1, 2, 3, ...$ this operator was introduced by S. D. Bernardi and it was studied by many authors in different general cases.

Definition 2.1.[5] *Let $n \in \mathbf{N}$ and $\lambda \geq 0$. We denote with D_λ^n the operator defined by*

$$D_\lambda^n : A \to A\,,$$

$$D_\lambda^0 f(z) = f(z), D_\lambda^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z),$$

$$D_\lambda^n f(z) = D_\lambda\left(D_\lambda^{n-1} f(z)\right).$$

REMARK 2.2.[5] *We observe that* D_λ^n *is a linear operator and for* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *we have*

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^n a_j z^j.$$

Also, it is easy to observe that if we consider $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator.

DEFINITION 2.2 [3] *Let* $q(z) \in \mathcal{H}_u(U)$, *with* $q(0) = 1$ *and* $q(U) = D$, *where* D *is a convex domain contained in the right half plane,* $n \in \mathbf{N}$ *and* $\lambda \geq 0$. *We say that a function* $f(z) \in A$ *is in the class* $SL_n^*(q)$ *if* $\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \prec q(z), z \in U$.

REMARK 2.3. *Geometric interpretation:* $f(z) \in SL_n^*(q)$ *if and only if* $\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}$ *take all values in the convex domain* D *contained in the right half-plane.*

DEFINITION 2.3 [4] *Let* $q(z) \in \mathcal{H}_u(U)$, *with* $q(0) = 1$ *and* $q(U) = D$, *where* D *is a convex domain contained in the right half plane,* $n \in \mathbf{N}$ *and* $\lambda \geq 0$. *We say that a function* $f(z) \in A$ *is in the class* $SL_n^c(q)$ *if* $\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \prec q(z), z \in U$.

REMARK 2.4. *Geometric interpretation:* $f(z) \in SL_n^c(q)$ *if and only if* $\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)}$ *take all values in the convex domain* D *contained in the right half-plane.*

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [7], [8], [9]).

THEOREM 2.1. *Let* h *convex in* U *and* $\mathrm{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. *If* $p \in H(U)$ *with* $p(0) = h(0)$ *and* p *satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \text{then } p(z) \prec h(z).$$

3. Main results

Definition 3.1 *Let $q(z) \in \mathcal{H}_u(U)$, with $q(0) = 1$, $q(U) = D$, where D is a convex domain contained in the right half plane, $n \in \mathbf{N}$, $\lambda \geq 0$ and $\alpha \in [0, 1]$. We say that a function $f(z) \in A$ is in the class $ML_{n,\alpha}(q)$ if*

$$J_{n,\lambda}(\alpha, f; z) = (1-\alpha)\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \prec q(z),\ z \in U.$$

Remark 3.1 *Geometric interpretation: $f(z) \in ML_{n,\alpha}(q)$ if and only if $J_{n,\lambda}(\alpha, f; z)$ take all values in the convex domain D contained in the right half-plane.*

Remark 3.2 *It is easy to observe that if we choose different function $q(z)$ we obtain variously classes of α-convex functions, such as (for example), for $\lambda = 1$ and $n = 0$, the class of α-convex functions, the class of α-uniform convex functions with respect to a convex domain (see [2]), and, for $\lambda = 1$, the class $UD_{n,\alpha}(\beta, \gamma)$, $\beta \geq 0$, $\gamma \in [-1, 1)$, $\beta + \gamma \geq 0$ (see [1]), the class of α-n-uniformly convex functions with respect to a convex domain (see [2]).*

Remark 3.3 *We have $ML_{n,0}(q) = SL_n^*(q)$ and $ML_{n,1}(q) = SL_n^c(q)$.*

Remark 3.4 *For $q_1(z) \prec q_2(z)$ we have $ML_{n,\alpha}(q_1) \subset ML_{n,\alpha}(q_2)$. From the above we obtain $ML_{n,\alpha}(q) \subset ML_{n,\alpha}\left(\frac{1+z}{1-z}\right)$*

Theorem 3.1 *For all $\alpha, \alpha' \in [0, 1]$, with $\alpha < \alpha'$, we have $ML_{n,\alpha'}(q) \subset ML_{n,\alpha}(q)$.*

Proof. From $f(z) \in ML_{n,\alpha'}(q)$ we have

$$J_{n,\lambda}(\alpha, f; z) = (1-\alpha)\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \prec q(z), \tag{2}$$

where $q(z)$ is univalent in U with $q(0) = 1$ and maps the unit disc U into the convex domain D contained in the right half-plane.

With notation

$$p(z) = \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}$$

where

$$p(z) = 1 + p_1 z + \ldots and f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

we have

$$p(z) + \alpha' \lambda \cdot \frac{zp'(z)}{p(z)} =$$

$$= \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha' \lambda \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \cdot z \frac{\left(D_\lambda^{n+1} f(z)\right)' D_\lambda^n f(z) - D_\lambda^{n+1} f(z) \left(D_\lambda^n f(z)\right)'}{\left(D_\lambda^n f(z)\right)^2} =$$

$$= \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha' \lambda \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \left(\frac{z \left(D_\lambda^{n+1} f(z)\right)'}{D_\lambda^n f(z)} - \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \cdot \frac{z \left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} \right) =$$

$$= \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha' \lambda \cdot \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \left(\frac{z \left(z + \sum\limits_{j=2}^{\infty} (1 + (j-1)\lambda)^{n+1} a_j z^j \right)'}{D_\lambda^n f(z)} - \right.$$

$$\left. - \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \cdot \frac{z \left(z + \sum\limits_{j=2}^{\infty} (1 + (j-1)\lambda)^{n} a_j z^j \right)'}{D_\lambda^n f(z)} \right) =$$

$$= \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha' \lambda \cdot \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \left(\frac{z \left(1 + \sum\limits_{j=2}^{\infty} j (1 + (j-1)\lambda)^{n+1} a_j z^{j-1} \right)}{D_\lambda^n f(z)} - \right.$$

$$\left. - \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \cdot \frac{z \left(1 + \sum\limits_{j=2}^{\infty} j (1 + (j-1)\lambda)^{n} a_j z^{j-1} \right)}{D_\lambda^n f(z)} \right)$$

or

$$p(z)+\alpha'\cdot\lambda\cdot\frac{zp^{(z)}}{p(z)}=\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}+\alpha'\lambda\cdot\frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)}\left(\frac{z+\sum\limits_{j=2}^{\infty}j\left(1+(j-1)\lambda\right)^{n+1}a_jz^j}{D_\lambda^n f(z)}-\right.$$

$$\left.-\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}\frac{z+\sum\limits_{j=2}^{\infty}j\left(1+(j-1)\lambda\right)^{n}a_jz^j}{D_\lambda^n f(z)}\right) \tag{3}$$

We have

$$z+\sum_{j=2}^{\infty}j\left(1+(j-1)\lambda\right)^{n+1}a_jz^j=z+\sum_{j=2}^{\infty}\left((j-1)+1\right)\left(1+(j-1)\lambda\right)^{n+1}a_jz^j=$$

$$=z+\sum_{j=2}^{\infty}\left(1+(j-1)\lambda\right)^{n+1}a_jz^j+\sum_{j=2}^{\infty}(j-1)\left(1+(j-1)\lambda\right)^{n+1}a_jz^j=$$

$$=z+D_\lambda^{n+1}f(z)-z+\sum_{j=2}^{\infty}(j-1)\left(1+(j-1)\lambda\right)^{n+1}a_jz^j=$$

$$=D_\lambda^{n+1}f(z)+\frac{1}{\lambda}\sum_{j=2}^{\infty}\left((j-1)\lambda\right)\left(1+(j-1)\lambda\right)^{n+1}a_jz^j=$$

$$=D_\lambda^{n+1}f(z)+\frac{1}{\lambda}\sum_{j=2}^{\infty}\left(1+(j-1)\lambda-1\right)\left(1+(j-1)\lambda\right)^{n+1}a_jz^j=$$

$$=D_\lambda^{n+1}f(z)-\frac{1}{\lambda}\sum_{j=2}^{\infty}\left(1+(j-1)\lambda\right)^{n+1}a_jz^j+\frac{1}{\lambda}\sum_{j=2}^{\infty}\left(1+(j-1)\lambda\right)^{n+2}a_jz^j=$$

$$=D_\lambda^{n+1}f(z)-\frac{1}{\lambda}\left(D_\lambda^{n+1}f(z)-z\right)+\frac{1}{\lambda}\left(D_\lambda^{n+2}f(z)-z\right)=$$

$$=D_\lambda^{n+1}f(z)-\frac{1}{\lambda}D_\lambda^{n+1}f(z)+\frac{z}{\lambda}+\frac{1}{\lambda}D_\lambda^{n+2}f(z)-\frac{z}{\lambda}=$$

$$=\frac{\lambda-1}{\lambda}D_\lambda^{n+1}f(z)+\frac{1}{\lambda}D_\lambda^{n+2}f(z)=$$

$$= \frac{1}{\lambda}\left(\lambda - 1)D_\lambda^{n+1} f(z) + D_\lambda^{n+2} f(z)\right).$$

Similarly we have

$$z + \sum_{j=2}^{\infty} j\left(1 + (j-1)\lambda\right)^n a_j z^j = \frac{1}{\lambda}\left(\lambda - 1)D_\lambda^{n} f(z) + D_\lambda^{n+1} f(z)\right).$$

From (3) we obtain

$$p(z) + \alpha' \cdot \lambda \cdot \frac{zp'(z)}{p(z)} =$$

$$= \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha'\lambda \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \frac{1}{\lambda} \cdot \left((\lambda - 1)\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \right.$$

$$\left. + \frac{D_\lambda^{n+2} f(z)}{D_\lambda^n f(z)} - \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}(\lambda - 1) - \left(\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}\right)^2 \right) =$$

$$= \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + \alpha' \frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} - \alpha' \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} =$$

$$= \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}(1 - \alpha') + \alpha' \frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} = J_{n,\lambda}(\alpha', f; z)$$

From (2) we have

$$p(z) + \frac{zp'(z)}{\frac{1}{\alpha'\lambda} \cdot p(z)} \prec q(z)$$

with $p(0) = q(0)$, $\mathrm{Re}q(z) > 0, z \in U$, $\alpha' > 0$ and $\lambda \geq 0$. In this conditions from Theorem 2.1 we obtain $p(z) \prec q(z)$ or $p(z)$ take all values in D.

If we consider the function $g : [0, \alpha'] \to \mathbf{C}$,

$$g(u) = p(z) + u \cdot \frac{\lambda z p'(z)}{p(z)},$$

with $g(0) = p(z) \in D$ and $g(\alpha') = J_{n,\lambda}(\alpha', f; z) \in D$, it easy to see that

$$g(\alpha) = p(z) + \alpha \cdot \frac{\lambda z p'(z)}{p(z)} \in D, 0 \leq \alpha < \alpha'.$$

Thus we have

$$J_{n,\lambda}(\alpha, f; z) \prec q(z)$$

or

$$f(z) \in ML_{n,\alpha}(q).$$

From the above theorem we have

COROLLARY 3.1 *For every $n \in \mathbf{N}$ and $\alpha \in [0,1]$, we have*

$$ML_{n,\alpha}(q) \subset ML_{n,0}(q) = SL_n^*(q)$$

.

REMARK 3.5 *If we consider $\lambda = 1$ and $n = 0$ we obtain the Theorem 3.1 from [2]. Also, for $\lambda = 1$ and $n \in \mathbf{N}$, we obtain the Theorem 3.3 from [2].*

REMARK 3.6 *If we consider $\lambda = 1$ and $D = D_{\beta,\gamma}$ (see [1] or [2]) in the above theorem we obtain the Theorem 3.1 from [1].*

THEOREM 3.2 *Let $n \in \mathbf{N}$, $\alpha \in [0,1]$ and $\lambda \geq 1$. If $F(z) \in ML_{n,\alpha}(q)$ then $f(z) = L_a F(z) \in SL_n^*(q)$, where L_a is the Libera-Pascu integral operator defined by (1).*

Proof.

From (1) we have

$$(1+a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator D_λ^{n+1} and if we consider $f(z) = \sum_{j=2}^{\infty} a_j z^j$, we obtain

$$(1+a)D_\lambda^{n+1}F(z) = aD_\lambda^{n+1}f(z) + D_\lambda^{n+1}\left(z + \sum_{j=2}^{\infty} j a_j z^j\right) =$$

$$= aD_\lambda^{n+1}f(z) + z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{n+1} j a_j z^j$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j\left(1+(j-1)\lambda\right)^{n+1} a_j z^j = \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)\right) \qquad (4)$$

Thus

$$(1+a)D_\lambda^{n+1}F(z) = aD_\lambda^{n+1}f(z) + \frac{1}{\lambda}\Big((\lambda-1)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)\Big) =$$

$$= \left(a + \frac{\lambda-1}{\lambda}\right) D_\lambda^{n+1}f(z) + \frac{1}{\lambda}D_\lambda^{n+2}f(z)$$

or

$$\lambda(1+a)D_\lambda^{n+1}F(z) = \Big((a+1)\lambda - 1\Big)\, D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z).$$

Similarly, we obtain

$$\lambda(1+a)D_\lambda^{n}F(z) = \Big((a+1)\lambda - 1\Big)\, D_\lambda^{n}f(z) + D_\lambda^{n+1}f(z).$$

Then

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^{n}F(z)} = \frac{\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)} \cdot \frac{D_\lambda^{n+1}f(z)}{D_\lambda^{n}f(z)} + ((a+1)\,\lambda - 1) \cdot \frac{D_\lambda^{n+1}f(z)}{D_\lambda^{n}f(z)}}{((a+1)\,\lambda - 1) + \frac{D_\lambda^{n+1}f(z)}{D_\lambda^{n}f(z)}}$$

With notation

$$\frac{D_\lambda^{n+1}f(z)}{D_\lambda^{n}f(z)} = p(z), p(0) = 1$$

we obtain

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^{n}F(z)} = \frac{\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)} \cdot p(z) + (a+1)\lambda - 1 \cdot p(z)}{p(z) + (a+1)\lambda - 1} \tag{5}$$

Also, we obtain

$$\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)} = \frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n}f(z)} \cdot \frac{D_\lambda^{n}f(z)}{D_\lambda^{n+1}f(z)} = \frac{1}{p(z)} \cdot \frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n}f(z)} \tag{6}$$

We have

$$\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n}f(z)} = \frac{z + \sum\limits_{j=2}^{\infty} (1+(j-1)\lambda)^{n+2} a_j z^j}{z + \sum\limits_{j=2}^{\infty} (1+(j-1)\lambda)^{n} a_j z^j}$$

and

$$zp'(z) = \frac{z\left(D_\lambda^{n+1}f(z)\right)'}{D_\lambda^n f(z)} - \frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)} \cdot \frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} =$$

$$= \frac{z\left(1 + \sum\limits_{j=2}^{\infty}(1+(j-1)\lambda)^{n+1} ja_j z^{j-1}\right)}{D_\lambda^n f(z)} -$$

$$-p(z) \cdot \frac{z\left(1 + \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{n} ja_j z^{j-1}\right)}{D_\lambda^n f(z)}$$

or

$$zp'(z) = \frac{z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{n+1} a_j z^j}{D_\lambda^n f(z)} - p(z) \cdot \frac{z + \sum\limits_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{n} a_j z^j}{D_\lambda^n f(z)}. \tag{7}$$

By using (4) and (7) we obtain

$$zp'(z) = \frac{1}{\lambda}\left(\frac{(\lambda-1)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)\frac{(\lambda-1)D_\lambda^n f(z) + D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}\right) =$$

$$= \frac{1}{\lambda}\left((\lambda-1)p(z) + \frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)((\lambda-1)+p(z))\right) =$$

$$= \frac{1}{\lambda}\left(\frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)^2\right)$$

Thus

$$\lambda zp'(z) = \frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)^2$$

or

$$\frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} = p(z)^2 + \lambda zp'(z).$$

From (6) we obtain

$$\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)} = \frac{1}{p(z)}(p(z)^2 + \lambda zp'(z)).$$

Then, from (5), we obtain

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} = \frac{p(z)^2+\lambda z p'(z)+((a+1)\lambda-1)\,p(z)}{p(z)+((a+1)\lambda-1)} = p(z)+\lambda\frac{zp'(z)}{p(z)+((a+1)\lambda-1)}$$

where $\alpha \in \mathbf{C}$, Re$a \geq 0$ and $\lambda \geq 1$.

If we denote $\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} = h(z)$, with $h(0)=1$, we have from $F(z) \in ML_{n,\alpha}(q)$ (see the proof of the above Theorem):

$$J_{n,\lambda}(\alpha, F; z) = h(z) + \alpha \cdot \lambda \cdot \frac{zh'(z)}{h(z)} \prec q(z)$$

Using the hypothesis, from Theorem 2.1, we obtain

$$h(z) \prec q(z)$$

or

$$p(z) + \lambda\frac{zp'(z)}{p(z)+((a+1)\lambda-1)} \prec q(z).$$

By using the Theorem 2.1 and the hypothesis we have

$$p(z) \prec q(z)$$

or

$$\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)} \prec q(z).$$

This means $f(z) = L_\alpha F(z) \in SL_n^*(q)$.

Remark 3.7 *If we consider $\lambda = 1$ and $n = 0$ we obtain the Theorem 3.2 from [2]. Also, for $\lambda = 1$ and $n \in \mathbf{N}$, we obtain the Theorem 3.4 from [2].*

Remark 3.8 *If we consider $\lambda = 1$ and $D = D_{\beta,\gamma}$ (see [1] or [2]) in the above theorem we obtain the Theorem 3.2 from [1].*

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