GEOMETRIC PREQUANTIZATION OF A GENERALIZED MECHANICAL SYSTEM

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Abstract. In this paper we try to understand some new properties of distributional symplectic geometry and generalized mechanical systems. The paper is divided up as follows. Section 1 presents some general facts on distributional symplectic geometry. In section 2 the central ideas of geometric prequantization are summarized. Section 3 contains the geometric prequantization of a generalized mechanical system.

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1. Distributional symplectic geometry

Let M be a smooth $2n$-dimmensional manifold and ω a symplectic structure on M. We denote by $C^{\infty}(M)$ (resp. $X'(M)$, resp. $\overset{p}{D}'(M)$) the space of smooth (C^{∞}) functions (resp. the spaced of generalized vector fields, resp. the space of p-De Rham currents) on M endowed with the uniform convergence topology. We remind that in local chart a generalized vector field (resp. an p-De Rham current) is a smooth vector field (resp. a smooth p-form) with distributions coefficients instead of smooth ones.

Definition 1 *Let (M, ω) be a symplectic manifold and $H \in \overset{p}{D}'(M)$ a given distribution on M. The generalized vector field X_H determined by*

$$X_H \lrcorner\, \omega + dH = 0$$

is called the generalized Hamiltonian vector field with generalized energy (Hamiltonian) H.

Let $(q^1, ..., q^n, p_1, ..., p_n)$ be canonical coordinates for ω, so $\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$. Then in these coordinates we have:

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

The basic properties of smooth Hamiltonian vector fields as the flow theorem, Lioville theorem and the conservation of energy can be extended in a natural way to generalized Hamiltonian vector fields.

EXAMPLE. Let $(\mathbb{R}^2 = T^*\mathbb{R}, \omega = dp \wedge dq)$ be the canonical symplectic manifold, and consider the generalized vector field

$$X = p\frac{\partial}{\partial q} - \delta(q)\frac{\partial}{\partial p},$$

where $\delta(q)$ is the Dirac function. Then a straightforward calculation yields that X is a generalized vector field associated with the Hamiltonian

$$H = \frac{1}{2}p^2 + V(q),$$

where $V(q)$ is the Heaviside function. In this case the flow of X corresponds to reflection of a wall at the origin.

DEFINITION 2 *Let (M, ω) be a $2n$-dimmensional symplectic manifold, $f \in C^\infty(M)$ a smooth function on M and $T \in \overset{0}{D}{}'(M)$ a distribution on M. Then the Poisson bracket of f and T is the distribution $\{f, T\}$ given by*

$$\{f, T\}\omega^n = ndf \wedge dT \wedge \omega^{n-1}.$$

Let $(q^1, ..., q^n, p_1, ..., p_n)$ be canonical coordinates for ω. Then in these coordinates we have

$$\{f, T\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial T}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial T}{\partial q^i} \right).$$

The basic results needed to do mechanics continue to hold, e.g. $X_{\{f,T\}} = -\{X_f, X_T\}$, $d\{f, T\} = \{df, dT\}$, $\{f, T\} = -L_{X_f}T$, $\int_M \{f, T\} = 0$.

2. The Konstant geometric prequantization process

The first step in geometric quantization is the prequantization. The principal ideas of prequantization can be summarized as follows. Let (M, ω) be a symplectic manifold and (L, π, M) an C^∞ hermitian complex line bundle over M with hermitian structure $< \cdot, \cdot >$. We suppose that L is endowed with a connection ∇ such that $< \cdot, \cdot >$ is preserved under parallel translation with respect to ∇. Such connections ∇ are in $1-1$ correspondence with 1-forms α on the complement L^* of the zero-section in L which are invariant under multiplication of L^* by non-zero complex numbers, whose restriction to any fibre of L^* is $i^{-1}z^{-1}dz$, and which satisfy

$$\nabla_\xi s = i < s^*\alpha, \xi >,$$

for all vector fields $\xi \in X(M)$ and all sections $s \in \Gamma(L)$ of L. We have also:

$$i(\alpha - \bar{\alpha}) = d \log |H|^2,$$

where $|H(L)|^2 =< l, l >$, for all $l \in L_x$, $x \in M$.

The curvature of ∇ is a 2-form on M and satisfies

$$\pi^*\Omega = d\alpha.$$

We shall make the fundamental assumption

$$\Omega = -\frac{1}{\hbar}\omega,$$

where $\hbar$ is a positive constant fixed throughout this chapter. This requires that $(2\pi\hbar)^{-1}\omega$ defines an integral De Rham class on M.

Prequantization associates to each $f \in C^\infty(M)$ a first order differential δ_f on L defined by

$$\delta_f = \nabla_{\xi_f} - \frac{1}{i\hbar} f.$$

We have

$$\delta_{\{f,g\}} = [\delta_f, \delta_g],$$

so $\delta : f \in C^\infty(M) \longrightarrow \delta_f : \Gamma(L) \longrightarrow \Gamma(L)$ is a representation of the Lie algebra $C^\infty(M)$ by first order differential operators. δ is called the prequantization map.

It is not difficult to see that the characteristic curves of δ_f agree with the integral curves of ξ_f and then because the behavior of a classical mechanical system at quantic level is given by δ_f and at classical level by ξ_f, we are lead to consider the above property as a form of Correspondence Principle for δ_f. Therefore the Konstant prequantization process gives a representation of the Lie algebra $C^\infty(M)$ by first order differential operators which satisfy the Correspondence Principle.

Example. Let Q be a configuration space of a classical mechanical system and let M be the cotangent bundle T^*Q with its canonical symplectic structure. Since $\omega = d\theta$, it follows that M is a quantizable manifold and the line bundle (L, π, M) is simply the trivial bundle, $L = M \times \mathbb{C}$. The hermitian structure on L is given by

$$< (x, c_1), (x, c_2) >= c_1\bar{c}_2$$

and if we identify the smooth sections of L with the smooth complex valued functions on M , the connection ∇ is defined globally by

$$\nabla_\xi f = \xi(f) - \frac{i}{\hbar}(\xi \lrcorner \theta)f,$$

and then the prequantization map δ is given by

$$\delta_f = \xi_f - \frac{i}{\hbar}(\xi \lrcorner \theta) - \frac{1}{i\hbar}f,$$

for each $f \in C^\infty(M)$.

Let (q^a) be a local coordinate system on Q and let (q^a, p_a) be the corresponding canonical coordinate system on M . Then it is easy to see that the coordinate functions become the differential operators

$$\begin{cases} \delta_{p_a} = \dfrac{\partial}{\partial q^a} \\ \delta_{q^a} = \dfrac{\partial}{\partial p_a} - \dfrac{1}{i\hbar}q^a \end{cases}.$$

3. Prequantization of a generalized mechanical system

Let Q be a smooth n-dimensional manifold and $M = T^*Q$ its cotangent bundle with the canonical symplectic structure, $\omega = d\theta$.

Definition 3 [1] *A generalized mechanical system is an ensamble* (M, ω, H), *where* $H \in \overset{0}{D'}(M)$ *is a distribution on* M. H *is called the generalized Hamiltonian of the system.*

Examples. 1. Let $(\mathbb{R}^4, \omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2)$ be the classical symplectic manifold and let H be a Dirac function spread along the q^2-axis. Roughly $H(q^1, q^2) = \delta(q^1)$. Then $(\mathbb{R}^4, \omega, H)$ is a generalized mechanical system, the generalized Hamiltonian vector field corresponding to H is

$$X_H = -\frac{\partial \delta(q^1)}{\partial q^1} \frac{\partial}{\partial p_1}.$$

and the corresponding motion in configuration space is just the free motion of particles reflecting (elastically) from a wall along the q^2 axis.

2. For the same symplectic manifold let H be the Heaviside function in q^1-variable

$$H(q^1, q^2) = V(q^1, q^2) = \begin{cases} 0 & \text{iff } q^1 < 0 \\ 1 & \text{iff } q^1 \geq 0 \end{cases}.$$

Then $(\mathbb{R}^4, \omega, H)$ is a generalized mechanical system, the generalized Hamiltonian vector field corresponding to H is

$$X_H = -\delta(q^1) \frac{\partial}{\partial p_1}.$$

and the corresponding motion is configuration space is the refraction of particles according to Snell's law as they cross the interface along the q^2-axis.

Now, the problem is to define the prequantization of a generalized mechanical system, or equivalent to obtain the prequantizing operator δ_H when H is a distribution on M.

For beginning to observe that the connection map ∇ can be extended to generalized vector fields as follows:

PROPOSITION 1 *The map*

$$\nabla : X(M) \times C^{\infty}_{\mathbb{C}}(M) \longrightarrow C^{\infty}_{\mathbb{C}}(M)$$

has a unique continuous extension

$$\bar{\nabla} : X'(M) \times C^{\infty}_{\mathbb{C}}(M) \longrightarrow \overset{0}{D}'(M).$$

In fact

$$\bar{\nabla}_X(f) \overset{def}{=} X(f) - \frac{i}{\hbar}(\xi \lrcorner \theta)f.$$

Moreover we have:

i) $\bar{\nabla}_{X+Y}(f) = \bar{\nabla}_X(f) + \bar{\nabla}_Y(f)$,

ii) $\bar{\nabla}_{fX}(g) = f \cdot \bar{\nabla}_X(g)$,

iii) $\bar{\nabla}_X(f, g) = f \cdot \bar{\nabla}_X(g) + g \cdot \bar{\nabla}_X(f)$,

for each $X, Y \in X'(M)$, $f, g \in C^{\infty}_{\mathbb{C}}(M)$.

REMARK.The proof can be obtained directly using the definition of $\bar{\nabla}$.

In similar way we get:

PROPOSITION 2 *The Konstant geometric prequantization map* δ *has a unique continous extension*

$$\bar{\delta} : T \in \overset{0}{D}'(M) \longrightarrow \bar{\delta}_T : C^{\infty}_{C}(M) \longrightarrow \overset{0}{D}'(M).$$

In fact

$$\bar{\delta}_T \overset{def}{=} \bar{\nabla}_{X_T} - \frac{1}{i\hbar}T.$$

Moreover $\bar{\delta}$ *is* $\mathbb{R}$*-linear.*

Definition 4 *The map $\bar{\delta}$ is called the generalized prequantization map.*

Example. Let $(\mathbb{R}^4, \omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2)$ be the classical symplectic manifold. Then a straightforward calculation shows that for $\delta(q^1)$ and $V(q^1)$ we have:

$$\bar{\delta}_{\delta(q^1)} = \frac{\partial \delta(q^1)}{\partial q^1} \frac{\partial}{\partial p_1} + \frac{1}{i\hbar} \delta(q^1).$$

and also

$$\delta_{V(q^1,q^2)} = -\delta(q^1) \frac{\partial}{\partial p_1} + \frac{1}{i\hbar} V(q^1, q^2).$$

Using the Parker's [2] technique we can also prove:

Proposition 3 *The generalized characteristic curves of $\bar{\delta}_T$ agree with the generalized integral curves of X_T.*

This property of $\bar{\delta}_T$ will be taken to be the assertion that $\bar{\delta}_T$ satisfies the Correspondence Principle for prequantization of a generalized mechanical system.

It is known that for a classical mechanical system the all representations of the Lie algebra $C^\infty(M)$ by first order differential operators with smooth coefficients which satisfy the Correspondence Principle are of the type

$$\nabla_{X_f} + m_f,$$

where $f \in C^\infty(M)$ and $m_f : C^\infty(M) \longrightarrow C^\infty(M)$ is a C-linear map such that:

$$m_f\{\varphi, \psi\} = \{m_f \varphi, \psi\} - \{\varphi, m_f \psi\},$$

for each $\varphi, \psi \in C^\infty(M)$.

Therefore we are lead to consider the space $Diff'(L)$ of first order differential operators with distributions coefficients of the type

$$\bar{\nabla}_{X_T} + m_T,$$

where $m_T : C^\infty_C(M) \longrightarrow \overset{0}{D'}(M)$ is a $\mathbb{C}$-linear map such that

$$m_T\{\varphi, \psi\} = \{m_T \varphi, \psi\} - \{\varphi, m_T \psi\},$$

for all $\varphi, \psi \in C^\infty_C(M)$.

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