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# ASYMPTOTIC FORMULAE OF EIGENVALUES AND FUNDAMENTAL SOLUTIONS OF A FOURTH-ORDER BOUNDARY VALUE PROBLEM

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Abstract. In this work, we study a class of fourth-order boundary value problems with eigenparameter dependent boundary conditions and transmission conditions at a interior point. We obtain asymptotic formulae for its eigenvalues and fundamental solutions.

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## Introduction

It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. In recent years, more and more researches are interested in the discontinuous Sturm-Liouville problems (see $[1-12]$). Various physics applications of this kind problem are found in many literatures, including some boundary value problem with transmission conditions that arise in the theory of heat and mass transfer (see $[5,6]$). The literature on such results is voluminous and we refer to $[1-12]$.

Fourth-order discontinuous boundary value problems with eigen-dependent boundary conditions and with four transmission conditions at the point of discontinuity have been investigated in $[10,11]$.

In this study, we shall consider a fourth-order differential equation

$$Lu := u^{(4)}(x) + q(x)u(x) = \lambda u(x) \tag{1.1}$$

in $I = [-1, 0) \cup (0, 1]$, with boundary conditions at $x = -1$

$$L_1u := \alpha_1 u(-1) + \alpha_2 u'''(-1) = 0, \tag{1.2}$$

$$L_2u := \beta_1 u'(-1) + \beta_2 u''(-1) = 0, \tag{1.3}$$

with the four transmission conditions at the points of discontinuity $x = 0$,

$$L_3u := u(0+) - u(0-) = 0, \quad (1.4)$$
$$L_4u := u'(0+) - u'(0-) = 0, \quad (1.5)$$
$$L_5u := u''(0+) - u''(0-) + \lambda\delta_1 u'(0-) = 0, \quad (1.6)$$
$$L_6u := u'''(0+) - u'''(0-) + \lambda\delta_2 u(0-) = 0, \quad (1.7)$$

and the eigen-dependent boundary conditions at $x = 1$

$$L_7u := \lambda u(1) + u'''(1) = 0, \quad (1.8)$$
$$L_8u := \lambda u'(1) + u''(1) = 0, \quad (1.9)$$

where $q(x)$ is a given real-valued function continuous in $[-1, 0) \cup (0, 1]$ and has a finite limit $q(\pm 0) = \lim_{x\to\pm 0} q(x)$; $\lambda$ is a complex eigenvalue parameter; $\alpha_i, \beta_i, \delta_i$ $(i = 1, 2)$ are real numbers and $|\alpha_1| + |\alpha_2| \neq 0$, $|\beta_1| + |\beta_2| \neq 0$, $|\delta_1| + |\delta_2| \neq 0$.

## 2. Preliminaries

Firstly we define the inner product in $L^2$ for every $f, g \in L^2(I)$ as

$$\langle f, g\rangle_1 = \int_{-1}^{0} f_1\overline{g_1}dx + \int_{0}^{1} f_2\overline{g_2}dx,$$

where $f_1(x) = f(x)\big|_{[-1,0)}$, $f_2(x) = f(x)\big|_{(0,1]}$. It is easy to see that $\left(L^2(I), [\cdot,\cdot]\right)$ is a Hilbert space. Now we define the inner product in the direct sum of spaces $L^2(I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}$ by

$$[F, G] := \langle f, g\rangle_1 + \langle h_1, k_1\rangle + \langle h_2, k_2\rangle + \langle h_3, k_3\rangle + \langle h_4, k_4\rangle$$

for $F := (f, h_1, h_2, h_3, h_4)$, $G := (g, k_1, k_2, k_3, k_4) \in L^2(I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}$. Then $Z := \left(L^2(I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}, [\cdot,\cdot]\right)$ is the direct sum of modified Krein spaces. A fundamental symmetry on the Krein space is given by

$$J := \begin{bmatrix} J_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & sgn\delta_1 & 0 \\ 0 & 0 & 0 & 0 & sgn\delta_2 \end{bmatrix},$$

where $J_0 : L^2(I) \to L^2(I)$ is defined by $(J_0 f)(x) = f(x)$. Let $\langle\cdot,\cdot\rangle = [J\cdot,\cdot]$, then $\langle\cdot,\cdot\rangle$ is a positive definite inner product. It turns $Z$ into a Hilbert space

$Z_0 := \left(L^2(I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}, [J\cdot,\cdot]\right)$.We define a linear operator $A$ in $Z$ by the domain of definition

$$D(A) := \Big\{(f,h_1,h_2,h_3,h_4) \in Z \mid f_1^{(i)} \in AC_{loc}((-1,0)),\ f_2^{(i)} \in AC_{loc}((0,1)),\ i=\overline{0,3},$$
$$Lf \in L^2(I),\ L_k f = 0,\ k=\overline{1,4},\ h_1 = f(1),\ h_2 = f'(1),\ h_3 = -\delta_1 f'(0),\ h_4 = -\delta_2 f(0)\Big\},$$
$$AF = (Lf,\ -f'''(1),\ -f''(1),\ f''(0+) - f''(0-),\ f'''(0+) - f'''(0-)),$$
$$F = (f,\ f(1),\ f'(1), -\delta_1 f'(0),\ -\delta_2 f(0)) \in D(A).$$

Consequently, the considered problem (1.1)-(1.9) can be rewritten in operator form as

$$AF = \lambda F,$$

i.e., the problem (1.1)-(1.9) can be considered as the eigenvalue problem for the operator $A$. Then, we can write the following conclusions:

**Theorem 2.1.** *The eigenvalues and eigenfunctions of the problem (1.1)-(1.9) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator $A$ respectively.*

**Theorem 2.2.** *The operator $A$ is self-adjoint in Krein space $Z$ (cf. Theorem 2.2 of* [10]).

## 3. Fundamental solutions

**Lemma 3.1.** *Let the real-valued function $q(x)$ be continuous in* $[-1,1]$ and $f_i(\lambda)$ $(i = 1,4)$ *are given entire functions. Then for any $\lambda \in \mathbb{C}$ the equation*

$$u^{(4)}(x) + q(x)u(x) = \lambda u(x), \quad x \in I$$

*has a unique solution $u = u(x,\lambda)$ such that*

$$u(-1) = f_1(\lambda),\ u'(-1) = f_2(\lambda),\ u''(-1) = f_3(\lambda),\ u'''(-1) = f_4(\lambda)$$

$$\Big(\text{or } u(1) = f_1(\lambda),\ u'(1) = f_2(\lambda),\ u''(1) = f_3(\lambda),\ u'''(1) = f_4(\lambda)\Big).$$

*and for each $x \in [-1,1]$, $u(x,\lambda)$ is an entire function of $\lambda$.*

**Proof.** In terms of existence and uniqueness theorem in ordinary differential equation theory, we can conclude this conclusion. ■

Let $\phi_{11}(x,\lambda)$ be the solution of Eq. (1.1) in $[-1,0)$ which satisfies the initial conditions

$$\phi_{11}(-1) = \alpha_2,\ \phi'_{11}(-1) = \phi''_{11}(-1) = 0,\ \phi'''_{11}(-1) = -\alpha_1.$$

By virtue of Lemma 3.1, after defining this solution, we may define the solution $\phi_{12}(x,\lambda)$ of Eq. (1.1) in $(0,1]$ by means of the solution $\phi_{11}(x,\lambda)$ by the initial conditions

$$\phi_{12}(0)=\phi_{11}(0)\,,\ \phi'_{12}(0)=\phi'_{11}(0)\,,\ \phi''_{12}(0)=\phi''_{11}(0)-\lambda\delta_1\phi'_{11}(0)\,,$$
$$\phi'''_{12}(0)=\phi'''_{11}(0)-\lambda\delta_2\phi_{11}(0)\,. \qquad (3.1)$$

After defining this solution, we may define the solution $\phi_{21}(x,\lambda)$ of Eq. (1.1) in $[-1,0)$ which satisfies the initial conditions

$$\phi_{21}(-1)=0,\ \phi'_{21}(-1)=\beta_2,\ \phi''_{21}(-1)=-\beta_1,\ \phi'''_{21}(-1)=0. \qquad (3.2)$$

After defining this solution, we may define the solution $\phi_{22}(x,\lambda)$ of Eq. (1.1) in $(0,1]$ by means of the solution $\phi_{21}(x,\lambda)$ by the initial conditions

$$\phi_{22}(0)=\phi_{21}(0)\,,\ \phi'_{22}(0)=\phi'_{21}(0)\,,\ \phi''_{22}(0)=\phi''_{21}(0)-\lambda\delta_1\phi'_{21}(0)\,,$$
$$\phi'''_{22}(0)=\phi'''_{21}(0)-\lambda\delta_2\phi_{21}(0)\,. \qquad (3.3)$$

Analogically we shall define the solutions $\chi_{11}(x,\lambda)$ and $\chi_{12}(x,\lambda)$ in the intervals $[-1,0)$ and $(0,1]$ respectively by the initial conditions

$$\chi_{12}(1)=-1,\ \chi'_{12}(1)=\chi''_{12}(1)=0, \chi'''_{12}(1)=\lambda,\ \chi_{11}(0)=\chi_{12}(0)\,,$$
$$\chi'_{11}(0)=\chi'_{12}(0)\,,\ \chi''_{11}(0)=\chi''_{12}(0)+\lambda\delta_1\chi'_{12}(0)\,,\ \chi'''_{11}(0)=\chi'''_{12}(0)+\lambda\delta_2\chi_{12}(0)\,. \qquad (3,4)$$

Moreover, we shall define the solutions $\chi_{21}(x,\lambda)$ and $\chi_{22}(x,\lambda)$ in the intervals $[-1,0)$ and $(0,1]$ respectively by the initial conditions

$$\chi_{22}(1)=0,\ \chi'_{22}(1)=-1,\ \chi''_{22}(1)=\lambda,\ \chi'''_{22}(1)=0,\ \chi_{21}(0)=\chi_{22}(0)\,,$$
$$\chi'_{21}(0)=\chi'_{22}(0)\,,\ \chi''_{21}(0)=\chi''_{22}(0)+\lambda\delta_1\chi'_{22}(0)\,,\ \chi'''_{21}(0)=\chi'''_{22}(0)+\lambda\delta_2\chi_{22}(0)\,. \qquad (3.5)$$

Let us consider the Wronskians

$$W_1(\lambda):=\begin{vmatrix} \phi_{11}(x,\lambda) & \phi_{21}(x,\lambda) & \chi_{11}(x,\lambda) & \chi_{21}(x,\lambda) \\ \phi'_{11}(x,\lambda) & \phi'_{21}(x,\lambda) & \chi'_{11}(x,\lambda) & \chi'_{21}(x,\lambda) \\ \phi''_{11}(x,\lambda) & \phi''_{21}(x,\lambda) & \chi''_{11}(x,\lambda) & \chi''_{21}(x,\lambda) \\ \phi'''_{11}(x,\lambda) & \phi'''_{21}(x,\lambda) & \chi'''_{11}(x,\lambda) & \chi'''_{21}(x,\lambda) \end{vmatrix}$$

and

$$W_2(\lambda):=\begin{vmatrix} \phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda) \\ \phi'_{12}(x,\lambda) & \phi'_{22}(x,\lambda) & \chi'_{12}(x,\lambda) & \chi'_{22}(x,\lambda) \\ \phi''_{12}(x,\lambda) & \phi''_{22}(x,\lambda) & \chi''_{12}(x,\lambda) & \chi''_{22}(x,\lambda) \\ \phi'''_{12}(x,\lambda) & \phi'''_{22}(x,\lambda) & \chi'''_{12}(x,\lambda) & \chi'''_{22}(x,\lambda) \end{vmatrix},$$

which are independent of $x$ and entire functions. This sort of calculation gives $W_1(\lambda) = W_2(\lambda)$.

Now we may introduce in consideration the characteristic function $W(\lambda)$ as $W(\lambda) = W_1(\lambda)$.

**Theorem 3.2.** *The eigenvalues of the problem (1.1)-(1.9) are the zeros of the function* $W(\lambda)$.

**Proof.** Let $W(\lambda) = 0$. Then the functions $\phi_{11}(x,\lambda)$, $\phi_{21}(x,\lambda)$ and $\chi_{11}(x,\lambda)$, $\chi_{21}(x,\lambda)$ are linearly dependent, i.e.,

$$k_1\phi_{11}(x,\lambda) + k_2\phi_{21}(x,\lambda) + k_3\chi_{11}(x,\lambda) + k_4\chi_{21}(x,\lambda) = 0$$

for some $k_1 \neq 0$ or $k_2 \neq 0$ or $k_3 \neq 0$ or $k_4 \neq 0$. From this, it follows that $k_3\chi_{11}(x,\lambda) + k_4\chi_{21}(x,\lambda)$ satisfies the boundary conditions (1.2)-(1.3). Therefore

$$\begin{cases} k_3\chi_{11}(x,\lambda) + k_4\chi_{21}(x,\lambda), & x \in [-1,0), \\ k_3\chi_{12}(x,\lambda) + k_4\chi_{22}(x,\lambda), & x \in (0,1] \end{cases}$$

is an eigenfunction of the problem (1.1)-(1.9) corresponding to eigenvalue $\lambda$.

Now we let $u(x)$ be any eigenfunction corresponding to eigenvalue $\lambda$, but $W(\lambda) \neq 0$. Then the functions $\phi_{11}$, $\phi_{21}$, $\chi_{11}$, $\chi_{21}$ would be linearly independent on $(0,1]$. Therefore $u(x)$ may be represented as

$$u(x) = \begin{cases} c_1\phi_{11}(x,\lambda) + c_2\phi_{21}(x,\lambda) + c_3\chi_{11}(x,\lambda) + c_4\chi_{21}(x,\lambda), & x \in [-1,0); \\ c_5\phi_{12}(x,\lambda) + c_6\phi_{22}(x,\lambda) + c_7\chi_{12}(x,\lambda) + c_8\chi_{22}(x,\lambda), & x \in (0,1], \end{cases}$$

where at least one of the constants $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$, $c_7$ and $c_8$ is not zero. Considering the equations

$$L_\upsilon(u(x)) = 0, \quad \upsilon = \overline{1,8} \tag{3.6}$$

as a system of linear equations of the variables $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$, $c_7$, $c_8$ and taking (3.1)-(3.5) into account, it follows that the determinant of this system is

$$\begin{vmatrix} 0 & 0 & L_1\chi_{11} & L_1\chi_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & L_2\chi_{11} & L_2\chi_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_3\phi_{12} & L_3\phi_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_4\phi_{12} & L_4\phi_{22} & 0 & 0 \\ -\phi_{12}(0) & -\phi_{22}(0) & -\chi_{12}(0) & -\chi_{22}(0) & \phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \chi_{22}(0) \\ -\phi'_{12}(0) & -\phi'_{22}(0) & -\chi'_{12}(0) & -\chi'_{22}(0) & \phi'_{12}(0) & \phi'_{22}(0) & \chi'_{12}(0) & \chi'_{22}(0) \\ -\phi''_{12}(0) & -\phi''_{22}(0) & -\chi''_{12}(0) & -\chi''_{22}(0) & \phi''_{12}(0) & \phi''_{22}(0) & \chi''_{12}(0) & \chi''_{22}(0) \\ -\phi'''_{12}(0) & -\phi'''_{22}(0) & -\chi'''_{12}(0) & -\chi'''_{22}(0) & \phi'''_{12}(0) & \phi'''_{22}(0) & \chi'''_{12}(0) & \chi'''_{22}(0) \end{vmatrix}$$
$$= -W(\lambda)^3 \neq 0.$$

Therefore, the system (3.6) has only the trivial solution $c_i = 0$ $\left(i = \overline{1,8}\right)$. Thus we get a contradiction, which completes the proof. ■

## 4. Asymptotic formulae for eigenvalues and fundamental solutions

We start by proving some lemmas.

**Lemma 4.1.** *Let $\phi(x,\lambda)$ be the solution of Eq. (1.1), and let $\lambda = s^4$, $s = \sigma + it$. Then the following integral equations hold for $k = \overline{0,3}$ in $[-1,0)\cup(0,1]$ :*

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{11}(x,\lambda) &= \frac{\alpha_2}{2}\frac{d^k}{dx^k}\cos s(x+1) + \frac{\alpha_1}{2s^3}\frac{d^k}{dx^k}\sin s(x+1)\\ &+ \left(\frac{\alpha_2}{4} - \frac{\alpha_1}{4s^3}\right)\frac{d^k}{dx^k}e^{s(x+1)} + \left(\frac{\alpha_2}{4} + \frac{\alpha_1}{4s^3}\right)\frac{d^k}{dx^k}e^{-s(x+1)}\\ &+ \frac{1}{2s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin s(x-y) - e^{s(x-y)} + e^{-s(x-y)}\right)q(y)\,\phi_{11}(y,\lambda)\,dy.\end{aligned} \tag{4.1}$$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{12}(x,\lambda) &= \left(\frac{\phi_{12}(0)}{2} - \frac{\phi''_{12}(0)}{2s^2}\right)\frac{d^k}{dx^k}\cos sx + \left(\frac{\phi'_{12}(0)}{2s} - \frac{\phi'''_{12}(0)}{2s^3}\right)\\ &\times \frac{d^k}{dx^k}\sin sx + \left(\frac{\phi_{12}(0)}{4} + \frac{\phi'_{12}(0)}{4s} + \frac{\phi''_{12}(0)}{4s^2} + \frac{\phi'''_{12}(0)}{4s^3}\right)\\ &\times \frac{d^k}{dx^k}e^{sx} + \left(\frac{\phi_{12}(0)}{4} - \frac{\phi'_{12}(0)}{4s} + \frac{\phi''_{12}(0)}{4s^2} - \frac{\phi'''_{12}(0)}{4s^3}\right)\frac{d^k}{dx^k}e^{-sx}\\ &+ \frac{1}{2s^3}\int_{0}^{x}\frac{d^k}{dx^k}\left(\sin s(x-y) - e^{s(x-y)} + e^{-s(x-y)}\right)q(y)\,\phi_{12}(y,\lambda)\,dy.\end{aligned} \tag{4.2}$$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{21}(x,\lambda) &= \frac{\beta_1}{2s^2}\frac{d^k}{dx^k}\cos s(x+1) + \frac{\beta_2}{2s}\frac{d^k}{dx^k}\sin s(x+1)\\ &+ \left(\frac{\beta_2}{4s} - \frac{\beta_1}{4s^2}\right)\frac{d^k}{dx^k}e^{s(x+1)} - \left(\frac{\beta_2}{4s} + \frac{\beta_1}{4s^2}\right)\frac{d^k}{dx^k}e^{-s(x+1)}\\ &+ \frac{1}{2s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin s(x-y) - e^{s(x-y)} + e^{-s(x-y)}\right)q(y)\,\phi_{21}(y,\lambda)\,dy.\end{aligned} \tag{4.3}$$

$$\frac{d^k}{dx^k}\phi_{22}(x,\lambda)=\left(\frac{\phi_{22}(0)}{2}-\frac{\phi''_{22}(0)}{2s^2}\right)\frac{d^k}{dx^k}\cos sx+\left(\frac{\phi'_{22}(0)}{2s}-\frac{\phi'''_{22}(0)}{2s^3}\right)$$
$$\times\frac{d^k}{dx^k}\sin sx+\left(\frac{\phi_{22}(0)}{4}+\frac{\phi'_{22}(0)}{4s}+\frac{\phi''_{22}(0)}{4s^2}+\frac{\phi'''_{22}(0)}{4s^3}\right)$$
$$\times\frac{d^k}{dx^k}e^{sx}+\left(\frac{\phi_{22}(0)}{4}-\frac{\phi'_{22}(0)}{4s}+\frac{\phi''_{22}(0)}{4s^2}-\frac{\phi'''_{22}(0)}{4s^3}\right)\frac{d^k}{dx^k}e^{-sx}$$
$$+\frac{1}{2s^3}\int_0^x\frac{d^k}{dx^k}\left(\sin s(x-y)-e^{s(x-y)}+e^{-s(x-y)}\right)q(y)\phi_{22}(y,\lambda)\,dy. \tag{4.4}$$

**Proof.** Regard $\phi_{11}(x,\lambda)$ as the solution of the following non-homogeneous Cauchy problem:

$$\begin{cases} -\phi_{11}^{(4)}(x)+s^4\phi_{11}(x)=q(x)\phi_{11}(x,\lambda), \\ \phi_{11}(-1,\lambda)=\alpha_2,\ \phi'_{11}(-1,\lambda)=0, \\ \phi''_{11}(-1,\lambda)=0,\ \phi'''_{11}(-1,\lambda)=-\alpha_1. \end{cases}$$

Using the method of variation of parameters, $\phi_{11}(x,\lambda)$ satisfies

$$\phi_{11}(x,\lambda)=\frac{\alpha_2}{2}\cos s(x+1)+\frac{\alpha_1}{2s^3}\sin s(x+1)+\left(\frac{\alpha_2}{4}-\frac{\alpha_1}{4s^3}\right)e^{s(x+1)}$$
$$+\left(\frac{\alpha_2}{4}+\frac{\alpha_1}{4s^3}\right)e^{-s(x+1)}+\frac{1}{2s^3}\int_{-1}^x\left(\sin s(x-y)-e^{s(x-y)}+e^{-s(x-y)}\right)q(y)\phi_{11}(y,\lambda)\,dy.$$

Then differentiating it with respect to $x$, we have $(4.1)$. The proof for $(4.2)$, $(4.3)$ and $(4.4)$ is similar. ■

**Lemma 4.2.** *Let $\lambda=s^4$, $s=\sigma+it$. Then the following asymptotic formulae hold for $k=\overline{0,3}$ :*

$$\frac{d^k}{dx^k}\phi_{11}(x,\lambda)=\frac{\alpha_2}{2}\frac{d^k}{dx^k}\cos s(x+1)+\frac{\alpha_2}{4}\frac{d^k}{dx^k}\left(e^{s(x+1)}+e^{-s(x+1)}\right)+O\left(|s|^{k-1}e^{|s|(x+1)}\right). \tag{4.5}$$

$$\frac{d^k}{dx^k}\phi_{12}(x,\lambda)=\frac{s^2\delta_1\phi'_{11}(0)}{2}\frac{d^k}{dx^k}\cos sx+\frac{s\delta_2\phi_{11}(0)}{2}\frac{d^k}{dx^k}\sin sx$$
$$-\frac{s^2\delta_1\phi'_{11}(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}+e^{-sx}\right)-\frac{s\delta_2\phi_{11}(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}-e^{-sx}\right) \tag{4.6}$$
$$+O\left(e^{|s|^k(x+1)}\right).$$

$$\frac{d^k}{dx^k}\phi_{21}(x,\lambda) = \frac{\beta_2}{2s}\frac{d^k}{dx^k}\sin s(x+1) + \frac{\beta_2}{4s}\frac{d^k}{dx^k}\left(e^{s(x+1)} - e^{-s(x+1)}\right) + O\left(|s|^{k-2}e^{|s|(x+1)}\right).$$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{22}(x,\lambda) &= \frac{s^2\delta_1\phi'_{21}(0)}{2}\frac{d^k}{dx^k}\cos sx + \frac{s\delta_2\phi_{21}(0)}{2}\frac{d^k}{dx^k}\sin sx \\ &- \frac{s^2\delta_1\phi'_{21}(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}+e^{-sx}\right) - \frac{s\delta_2\phi_{21}(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}-e^{-sx}\right) \\ &+ O\left(e^{|s|^{k-1}(x+1)}\right).\end{aligned}$$

*Each of these asymptotic formulae holds uniformly for $x$ as $|\lambda| \to \infty$.*

**Proof.** Let $F_{11}(x,\lambda) = e^{-|s|(x+1)}\phi_{11}(x,\lambda)$. It is easy to see that $F_{11}(x,\lambda)$ is bounded. Therefore $\phi_{11}(x,\lambda) = O\left(e^{|s|(x+1)}\right)$. Substituting it into (4.1) and differentiating it with respect to $x$ for $k=\overline{0,3}$, we obtain (4.5). According to transmission conditions (1.4)-(1.7) as $|\lambda| \to \infty$, we get

$$\phi_{12}(0) = \phi_{11}(0),\ \phi'_{12}(0) = \phi'_{11}(0),\ \phi''_{12}(0) = -s^4\delta_1\phi'_{11}(0),\ \phi'''_{12}(0) = -s^4\delta_2\phi_{11}(0).$$

Substituting these asymptotic formulae into (4.2) for $k=0$, we obtain

$$\begin{aligned}\phi_{12}(x,\lambda) &= \frac{s^2\delta_1\phi'_{11}(0)}{2}\cos sx + \frac{s\delta_2\phi_{11}(0)}{2}\sin sx \\ &- \frac{s^2\delta_1\phi'_{11}(0)}{4}\left(e^{sx}+e^{-sx}\right) - \frac{s\delta_2\phi_{11}(0)}{4}\left(e^{sx}-e^{-sx}\right) \\ &+ \frac{1}{2s^3}\int_0^x\left(\sin s(x-y) - e^{s(x-y)} + e^{-s(x-y)}\right)q(y)\phi_{12}(y,\lambda)\,dy \\ &+ O\left(e^{|s|(x+1)}\right).\end{aligned}\tag{4.7}$$

Multiplying through by $|s|^{-3}e^{-|s|(x+1)}$, and denoting

$$F_{12}(x,\lambda) := O\left(|s|^{-3}e^{-|s|(x+1)}\right)\phi_{12}(x,\lambda).$$

Denoting $M := \max_{x\in[0,1]}|F_{12}(x,\lambda)|$ from the last formula, it follows that

$$M(\lambda) \le \frac{3|\alpha_2\delta_1|}{4} + \frac{|\alpha_2\delta_2|}{4|s|^2} + \frac{M(\lambda)}{2|s|^3}\int_0^x q(y)\,dy + M_0$$

for some $M_0 > 0$. From this, it follows that $M(\lambda) = O(1)$ as $|\lambda| \to \infty$, so

$$\phi_{12}(x,\lambda) = O\left(|s|^3 e^{|s|(x+1)}\right).$$

Substituting this back into the integral on the right side of (4.7) yields (4.6) for $k=0$. The other cases may be considered analogically. ■

Similarly one can establish the following lemma. for $\chi_{ij}(x,\lambda)$ $(i=1,2,\ j=1,2)$.

**Lemma 4.3.** *Let* $\lambda=s^4$, $s=\sigma+it$. *Then the following asymptotic formulae hold for* $k=\overline{0,3}$ :

$$\begin{aligned}\frac{d^k}{dx^k}\chi_{11}(x,\lambda)&=-\frac{s^2\delta_1\chi_{12}'(0)}{2}\frac{d^k}{dx^k}\cos sx+\frac{s\delta_2\chi_{12}(0)}{2}\frac{d^k}{dx^k}\sin sx\\&+\frac{s^2\delta_1\chi_{12}'(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}+e^{-sx}\right)+\frac{s\delta_2\chi_{12}(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}-e^{-sx}\right)\\&+O\left(|s|^{k+1}e^{|s|(1-x)}\right).\end{aligned}$$

$$\frac{d^k}{dx^k}\chi_{12}(x,\lambda)=-\frac{s}{2}\frac{d^k}{dx^k}\sin s(x-1)+\frac{s\delta_2}{4}\frac{d^k}{dx^k}\left(e^{s(x-1)}-e^{-s(x-1)}\right)+O\left(|s|^{k+1}e^{|s|(1-x)}\right).$$

$$\begin{aligned}\frac{d^k}{dx^k}\chi_{21}(x,\lambda)&=-\frac{s^2\delta_1\chi_{22}'(0)}{2}\frac{d^k}{dx^k}\cos sx+\frac{s\delta_2\chi_{22}(0)}{2}\frac{d^k}{dx^k}\sin sx\\&+\frac{s^2\delta_1\chi_{22}'(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}+e^{-sx}\right)+\frac{s\delta_2\chi_{22}(0)}{4}\frac{d^k}{dx^k}\left(e^{sx}-e^{-sx}\right)\\&+O\left(|s|^{k+2}e^{|s|(1-x)}\right).\end{aligned}$$

$$\frac{d^k}{dx^k}\chi_{22}(x,\lambda)=-\frac{s^2}{2}\frac{d^k}{dx^k}\cos s(x-1)+\frac{s^2}{4}\frac{d^k}{dx^k}\left(e^{s(x-1)}-e^{-s(x-1)}\right)+O\left(|s|^{k+1}e^{|s|(1-x)}\right),$$

*where* $k=\overline{0,3}$. *Each of these asymptotic formulae holds uniformly for* $x$.

**Theorem 4.4.** *Let* $\lambda=s^4$, $s=\sigma+it$. *Then the characteristic functions* $W_i(\lambda)$ $(i=1,2)$ *have the following asymptotic formulae:*

$$W_1(\lambda)=W_2(\lambda)=-\frac{\delta_1\delta_2\alpha_2\beta_2 s^{12}}{16}\left(2+\cos s\left(e^{-s}+e^{s}\right)\right)\left(e^{-s}+e^{s}\right)\cos s+O\left(|s|^{11}e^{4|s|}\right).$$

**Proof.** Substituting the asymptotic equalities $\frac{d^k}{dx^k}\chi_{11}(-1,\lambda)$ and $\frac{d^k}{dx^k}\chi_{21}(-1,\lambda)$ into the representation of $W_1(\lambda)$, we get

$$W_1(\lambda)=\begin{vmatrix}\alpha_2&0&\chi_{11}(-1,\lambda)&\chi_{21}(-1,\lambda)\\0&\beta_2&\chi_{11}'(-1,\lambda)&\chi_{21}'(-1,\lambda)\\0&-\beta_1&\chi_{11}''(-1,\lambda)&\chi_{21}''(-1,\lambda)\\-\alpha_1&0&\chi_{11}'''(-1,\lambda)&\chi_{21}'''(-1,\lambda)\end{vmatrix}=\frac{\delta_1\delta_2 s^3}{8}\left(\chi_{12}'(0)\chi_{22}(0)-\chi_{12}(0)\chi_{22}'(0)\right)$$

$$\times\left(\begin{vmatrix}\alpha_2&0&\cos s&e^{-s}-e^{s}\\0&\beta_2&-s\sin s&s\left(-e^{-s}-e^{s}\right)\\0&-\beta_1&-s^2\cos s&s^2\left(e^{s}-e^{s}\right)\\-\alpha_1&0&-s^3\sin s&s^3\left(-e^{-s}-e^{s}\right)\end{vmatrix}\right)$$

$$+\begin{vmatrix}1&0&\sin s&e^{-s}+e^{s}\\0&0&s\cos s&s\left(-e^{-s}+e^{s}\right)\\0&-1&-s^2\sin s&s^2\left(e^{s}+e^{s}\right)\\0&0&-s^3\sin s&s^3\left(-e^{-s}+e^{s}\right)\end{vmatrix}+O\left(|s|^{15}e^{4|s|}\right)=0.$$

Analogically, we can obtain the asymptotic formulae of $W_2(\lambda)$. ■

**Corollary 4.5.** *The real eigenvalues of the problem (1.1)-(1.9) are bounded below.*

**Proof.** Putting $s^2 = it^2$ $(t > 0)$ in the above formulas, it follows that $W\left(-t^2\right) \to \infty$ as $t \to \infty$. Therefore, $W(\lambda) \neq 0$ for $\lambda$ negative and sufficiently large in modulus. ■

Now we can obtain the asymptotic approximation formulae for the eigenvalues of the considered problem (1.1)-(1.9).

Since the eigenvalues coincide with the zeros of the entire function $W(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollary 4.5 that all real eigenvalues are bounded below. Hence, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ...$, listed according to their multiplicity.

**Theorem 4.6.** *The eigenvalues $\lambda_n = s_n^4$, $n = 0, 1, 2, ...$ of the problem (1.1)-(1.9) have the following asymptotic formulae for $n \to \infty$ :*

$$\sqrt[4]{\lambda_n'} = \frac{\pi(2n-1)}{2} + O\left(\frac{1}{n}\right), \quad \sqrt[4]{\lambda_n''} = \frac{\pi(2n+1)}{2} + O\left(\frac{1}{n}\right).$$

**Proof.** By applying the well-known Rouché's theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour $C$, and $|g(s)| < |f(s)|$ on $C$, then $f(s)$ and $f(s)+g(s)$ have the same number zeros inside $C$ provided that each zero is counted according to their multiplicity, we can obtain these conclusions. ■

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