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DIFFERENTIAL SANDWICH THEOREMS FOR P-VALENT FUNCTIONS RELATED TO CERTAIN OPERATOR

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Abstract. In this paper we obtain some subordination and superordination results for p-valent functions by using a certain operator.

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1. Introduction

Let $H(\mathrm{U})$ denotes the class of analytic functions in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a,p]$ denotes the subclass of the functions $f \in H(\mathrm{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, \dots\}).$$

Also, let $A(p)$ be the subclass of the functions $f \in H(\mathrm{U})$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1.1}$$

and set $A_1 \equiv A(1)$.

For $f, g \in H(\mathrm{U})$, we say that the function f is *subordinate* to g, or the function g is *superordinate* to f, if there exists a Schwarz function w, i.e. $w \in H(\mathrm{U})$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathrm{U}$, such that $f(z) = g(w(z))$ for all $z \in \mathrm{U}$. This subordination is usually denoted by $f(z) \prec g(z)$.

It is well-known that, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathrm{U}) \subset g(\mathrm{U})$.

Supposing that h and g are two analytic functions in U, let

$$\varphi(r, s, t; z) : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}.$$

If h and $\varphi(h(z), zh'(z), z^2h''(z); z)$ are univalent functions in U and if h satisfies the second-order superordination

$$g(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z), \tag{1.2}$$

then g is called to be a solution of the differential superordination (1.2). A function $q \in H(\mathrm{U})$ is called a subordinant of (1.2), if $q(z) \prec h(z)$ for all the functions h satisfying (1.2). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (1.2), is said to be the best subordinant.

Recently, Miller and Mocanu [14] obtained sufficient conditions on the functions g, q and φ for which the following implication holds:

$$g(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z) \Rightarrow g(z) \prec h(z).$$

Using the results of Miller and Mocanu [14], Bulboača [6] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [7]. Ali et al. [1], have used the results of Bulboača [6] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U.

Very recently, Shanmugam et al. ([18], [19] and [20]) obtained the such called sandwich results for certain classes of analytic functions. Further subordination results can be found in [17] and [21].

For f given by (1.1) and $g \in A(p)$ defined by $g(z) = z^p + \sum\limits_{k=p+1}^{\infty} b_k z^k$, the Hadamard product or (convolution) is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Using the convolution and for $\lambda \geqslant 0, l \geqslant 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the linear operator $D^m_{p,l,\lambda}(f * g) : A(p) \to A(p)$ by:

$$\begin{aligned}
D^0_{p,l,\lambda}(f*g)(z) &= (f*g)(z); \\
D^1_{p,l,\lambda}(f*g)(z) &= D_{p,l,\lambda}(f*g)(z) = (1-\lambda)(f*g)(z) + \frac{\lambda}{(p+l)z^{l-1}} \left(z^l(f*g)(z)\right)' \\
&= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l+\lambda(k-p)}{p+l}\right) a_k b_k z^k ; \\
D^2_{p,l,\lambda}(f*g)(z) &= (1-\lambda)D_{p,l,\lambda}(f*g)(z) + \frac{\lambda}{(p+l)z^{l-1}} \left(z^l D_{p,l,\lambda}(f*g)(z)\right)' \\
&= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l+\lambda(k-p)}{p+l}\right)^2 a_k b_k z^k
\end{aligned}$$

and (in general)

$$\begin{aligned}
D^m_{p,l,\lambda}(f*g)(z) &= (1-\lambda)D^{m-1}_{p,l,\lambda}(f*g)(z) + \frac{\lambda}{(p+l)z^{l-1}} \left(z^l D^{m-1}_{p,l,\lambda}(f*g)(z)\right)' \\
&= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m a_k b_k z^k . \qquad (1.4)
\end{aligned}$$

From (1.4), we can easily deduce that

$$\lambda z \left(D^m_{p,l,\lambda}(f*g)(z)\right)' = (p+l)D^{m+1}_{p,l,\lambda}(f*g)(z) - [p(1-\lambda)+l]\, D^m_{p,l,\lambda}(f*g)(z) \ (\lambda > 0). \qquad (1.5)$$

We remark that:

(i) For $b_k = 1$ or $g(z) = z^p(1-z)^{-1}$ we have $D^m_{p,l,\lambda}(f*g)(z) = I^m_p(\lambda,l)f(z)$, where the operator $I^m_p(\lambda,l)$ was introduced and studied by Catas [9] which contains intern the operators D^m_p (see [5] and [11]) and D^m_λ (see [2]);

(ii) For $b_k = \dfrac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}(1)_{k-p}}$, the operator $D^m_{p,l,\lambda}(f*g)(z) = I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)$, where the operator $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)$ was introduced and studied by El-Ashwah and Aouf [10], $\alpha_1,\alpha_2,...,\alpha_q$ and $\beta_1,\beta_2,...,\beta_s$ are real or complex numbers $(\beta_j \notin Z_0^- = \{0,-1,-2,...\}; j = 1,2,...,s)(q \le s+1; s,q \in N_0)$ and

$$(d)_k = \begin{cases} 1 & (k=0; d \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ d(d+1)...(d+k-1) & (k \in \mathbb{N}; d \in \mathbb{C}). \end{cases}$$

Also, for many special operators of the operator $I^{m,l}_{p,q,s,\lambda}(\alpha_1,\beta_1)$ see [10];

(iii) For $m = 0$ and $b_k = \dfrac{\Gamma(p+\alpha+\beta)\Gamma(k+\beta)}{\Gamma(p+\beta)\Gamma(k+\alpha+\beta)}$, the operator $D^m_{p,l,\lambda}(f*g)(z) = Q^\alpha_{p,\beta}f(z)$ $(\alpha \ge 0, \beta > -1, p \in \mathbb{N})$, where the operator $Q^\alpha_{p,\beta}$ was introduced by Liu and Owa [12].

2. Definitions and Preliminaries

To prove our results we shall need the following definition and lemmas.

Definition 1 [14]. *Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus E(f)$, where*

$$E(f) = \{\zeta \in \partial \mathrm{U} : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1[13]. *Let q be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(*i*) *Q is a starlike function in U,*

(*ii*) $\mathrm{Re}\, \dfrac{zh'(z)}{Q(z)} > 0,\ z \in \mathrm{U}.$

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2.1).

Lemma 2 [18]. *Let $\mu, \gamma \in C$ with $\gamma \neq 0$, and let q be a convex function in U with*

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\mathrm{Re}\,\frac{\mu}{\gamma}\right\},\ z \in \mathrm{U}.$$

If p is analytic in U and

$$\mu p(z) + \gamma z p'(z) \prec \mu q(z) + \gamma z q'(z), \tag{2.2}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2.2).

Lemma 3 [8]. *Let q be a univalent function in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(*i*) $\mathrm{Re}\, \dfrac{\theta'(q(z))}{\varphi(q(z))} > 0$ *for $z \in U$,*

(*ii*) *$h(z) = zq'(z)\varphi(q(z))$ is starlike in U.*

If $p \in H[q(0), 1] \cap \mathcal{Q}$ with $p(U) \subseteq D$, $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.3}$$

then $q(z) \prec p(z)$, and q is the best subordinant of (2.3).

Note that this result generalize a similar one obtained in [7].

Lemma 4 [14]. *Let q be convex in U and let $\gamma \in C$, with* $\text{Re}\{\gamma\} > 0$. *If $p \in H[q(0),1] \cap \mathcal{Q}$ and $p(z)+\gamma z p'(z)$ is univalent in U, then*

$$q(z)+\gamma z q'(z) \prec p(z)+\gamma z p'(z), \tag{2.4}$$

implies $q(z) \prec p(z)$, and q is the best subordinant (2.4).

This last lemma give us a necessary and sufficient condition for the univalence of a special function which will be used in some particular cases:

Lemma 5 [16]. *The function $q(z)=(1-z)^{-2ab}(a,b \in C^*)$ is univalent in U if and only if $|2ab-1| \le 1$ or $|2ab+1| \le 1$.*

3. Subordination results

Unless otherwise mentioned, we assume throughout this paper that $\lambda > 0, l \geqslant 0, p \in \mathbb{N}$, $m \in \mathbb{N}_0$ and the powers are considered principle values.

Theorem 1. *Let q be univalent in U, with $q(0)=1$, and suppose that*

$$\text{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\frac{p+l}{\lambda p}\,\text{Re}\,\frac{1}{\delta}\right\},\ z \in \text{U}, \tag{3.1}$$

where $\delta \in C^*$. If $f \in A(p)$ satisfies the subordination

$$\frac{\delta}{p}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}\right)+\frac{p-\delta}{p}\left(\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\right) \prec q(z)+\frac{\delta\lambda p z q'(z)}{p+l}, \tag{3.2}$$

then

$$\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p} \prec q(z),$$

and q is the best dominant of (3.2).

Proof. Let

$$K(z)=\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\ (z \in U), \tag{3.3}$$

then, differentiating (3.3) logarithmically with respect to z, and using the identity (1.5), we have

$$\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}=K(z)+\frac{z\lambda K'(z)}{p+l}.$$

A simple computation shows that

$$\frac{\delta}{p}\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}+\frac{p-\delta}{p}\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}=K(z)+\frac{\delta\lambda z K'(z)}{p(p+l)},$$

hence the subordination (3.2) is equivalent to

$$K(z)+\frac{\delta\lambda zK'(z)}{p(p+l)}\prec q(z)+\frac{\delta\lambda zq'(z)}{p(p+l)}.$$

Now, applying Lemma 2, with $\mu=1$ and $\gamma=\frac{\delta\lambda}{p(p+l)}$, the proof of Theorem 1 is completed.

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ ($-1\leq B<A\leq 1$) in Theorem 1, the condition (3.1) reduces to

$$\operatorname{Re}\frac{1-Bz}{1+Bz}>\max\left\{0;-\frac{p(p+l)}{\lambda}\operatorname{Re}\frac{1}{\delta}\right\},\ z\in\mathrm{U}. \tag{3.4}$$

It is easy to check that the function $\varphi(\zeta)=\dfrac{1-\zeta}{1+\zeta}$, $|\zeta|<|B|$, is convex in U, and since $\varphi(\overline{\zeta})=\overline{\varphi(\zeta)}$ for all $|\zeta|<|B|$, it follows that the image $\varphi(\mathrm{U})$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\operatorname{Re}\frac{1-Bz}{1+Bz}:z\in\mathrm{U}\right\}=\frac{1-|B|}{1+|B|}>0 \tag{3.5}$$

and the inequality (3.3) is equivalent to

$$\frac{p(p+l)}{\lambda}\operatorname{Re}\frac{1}{\delta}\geq\frac{|B|-1}{|B|+1},$$

hence we obtain the following corollary.

Corollary 1. *Let* $-1\leq B<A\leq 1$ *and* $\delta\in C^{*}$ *with*

$$\frac{1-|B|}{1+|B|}\geqslant\max\left\{0;-\frac{p(p+l)}{\lambda}\operatorname{Re}\frac{1}{\delta}\right\}.$$

If $f\in A(p)$, *and*

$$\frac{\delta}{p}\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^{p}}+\frac{p-\delta}{p}\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^{p}}\prec\frac{1+Az}{1+Bz}++\frac{\delta\lambda}{p(p+l)}\frac{(A-B)z}{(1+Bz)^{2}}, \tag{3.6}$$

then

$$\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^{p}}\prec\frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant of* (3.6).

For $p=A=1$ and $B=-1$ in Corollary 1, we have:

Corollary 2. *Let* $\delta \in C^*$ *with* $\frac{p+l}{\lambda}\operatorname{Re}\frac{1}{\delta} \geq 0$. *If* $f \in A_1$, *and*

$$\delta\left(\frac{D_{l,\lambda}^{m+1}(f*g)(z)}{z}\right) + (1-\delta)\left(\frac{D_{l,\lambda}^{m}(f*g)(z)}{z}\right) \prec \frac{1+z}{1-z} + \frac{2\delta\lambda z}{(1+l)(1-z)^2}, \quad (3.7)$$

then

$$\frac{D_{l,\lambda}^{m}(f*g)(z)}{z} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant of* (3.7).

Theorem 2. *Let* q *be univalent in* U, *with* $q(0) = 1$ *and* $q(z) \neq 0$ *for all* $z \in U$. *Let* $\gamma, \mu \in C^*$ *and* $\nu, \eta \in C$, *with* $\nu + \eta \neq 0$. *Let* $f \in A(p)$ *and suppose that* f *and* q *satisfy the conditions:*

$$\frac{\nu D_{p,l,\lambda}^{m+1}(f*g)(z) + \eta D_{p,l,\lambda}^{m}(f*g)(z)}{(\nu+\eta)z^p} \neq 0, \; z \in \mathrm{U}, \quad (3.8)$$

and

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0. \quad (3.9)$$

If

$$1 + \gamma\mu\left[\frac{\nu z\left[D_{p,l,\lambda}^{m+1}(f*g)(z)\right]' + \eta z\left[D_{p,l,\lambda}^{m}(f*g)(z)\right]'}{\nu D_{p,l,\lambda}^{m+1}(f*g)(z) + \eta D_{p,l,\lambda}^{m}(f*g)(z)} - p\right] \prec 1 + \gamma\frac{zq'(z)}{q(z)}, \quad (3.10)$$

then

$$\left[\frac{\nu D_{p,l,\lambda}^{m+1}(f*g)(z) + \eta D_{p,l,\lambda}^{m}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu} \prec q(z),$$

and q *is the best dominant of* (3.10).

Proof. Let $K(z)$ given by (3.3), then $K(z)$ is analytic in U, differentiating $K(z)$ logarithmically with respect to z, we get

$$\frac{zK'(z)}{K(z)} = \mu\left\{\frac{\nu z[D_{p,l,\lambda}^{m+1}(f*g)(z)]' + \eta z[D_{p,l,\lambda}^{m}(f*g)(z)]'}{\nu D_{p,l,\lambda}^{m+1}(f*g)(z) + \eta D_{p,l,\lambda}^{m}(f*g)(z)} - p\right\}.$$

Now, using Lemma 1 with $\theta(w) = 1$ and $\varphi(w) = \frac{\gamma}{w}$, then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \gamma\frac{zq'(z)}{q(z)},$$

and

$$h(z)=\theta(q(z))+Q(z)=1+\gamma\frac{zq'(z)}{q(z)},$$

then, $Q(0)=0$ and $Q'(0)\neq 0$, and the assumption (3.9) yields that Q is a starlike function in U and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0\ (z\in U),$$

and then, by using Lemma 1, we deduce that the assumption (3.10) implies $K(z)\prec q(z)$ and the function q is the best dominant of (3.10).

Taking $\nu=0$, $\eta=1$, $\gamma=1$ and $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 2, it is easy to check that the assumption (3.9) holds whenever $-1\leq A<B\leq 1$, hence we obtain the next result:

Corollary 3. *Let* $-1\leq A<B\leq 1$ *and* $\mu\in C^*$. *Let* $f\in A(p)$ *and suppose that*

$$\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\neq 0,\ z\in \mathrm{U}.$$

If

$$1+\mu\left[\frac{z\left[D^m_{p,l,\lambda}(f*g)(z)\right]'}{D^m_{p,l,\lambda}(f*g)(z)}-p\right]\prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)}, \qquad (3.11)$$

then

$$\left[\frac{D^m_{p,l,\lambda}(f*g)(z)}{z^p}\right]^{\mu}\prec\frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant of* (3.11).

Putting $\nu=0$, $\eta=\lambda=p=1$, $m=l=0$, $\gamma=\dfrac{1}{ab}$ $(a,b\in\mathbb{C}^*)$, $\mu=a$, and $q(z)=(1-z)^{-2ab}$ in Theorem 2 and combining this together with Lemma 5 we obtain the result due to Obradović et al. [15, Theorem 1].

Putting $\nu=0$, $p=\eta=\lambda=\gamma=1$, $m=l=0$, and $q(z)=(1+Bz)^{\frac{\mu(A-B)}{B}}$ $(-1\leq B<A\leq 1,\ B\neq 0)$ in Theorem 2, and using Lemma 5, we get the next corollary:

Corollary 4. *Let* $-1\leq B<A\leq 1$, *with* $B\neq 0$, *and suppose that* $\left|\dfrac{\mu(A-B)}{B}-1\right|\leq 1$ *or* $\left|\dfrac{\mu(A-B)}{B}+1\right|\leq 1$. *Let* $f\in A_1$ *such that* $\dfrac{f(z)}{z}\neq 0$ *for all* $z\in U$, *and let*

$\mu \in C^*$. *If*

$$1+\mu\left(\frac{zf'(z)}{f(z)}-1\right) \prec \frac{1+[B+\mu(A-B)]\,z}{1+Bz}, \tag{3.12}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}},$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ *is the best dominant of* (3.12).

Putting $\nu=0$, $\eta=\lambda=p=1$, $m=l=0$, $\gamma=\dfrac{e^{i\zeta}}{ab\cos\zeta}$ $\left(a,b\in\mathbb{C}^*;\ |\zeta|<\dfrac{\pi}{2}\right)$, $\mu=a$ and $q(z)=(1-z)^{-2ab\cos\zeta e^{-i\zeta}}$ in Theorem 2, we obtain the result due to Aouf et al. [3].

Theorem 3. *Let q be univalent in U with $q(0)=1$, let $\mu,\gamma\in C^*$, and let $\sigma,\Omega,\nu,\eta\in C$ with $\nu+\eta\neq 0$. Let $f\in A(p)$ and suppose that f and q satisfy the next two conditions:*

$$\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\neq 0,\ z\in \mathrm{U}, \tag{3.13}$$

and

$$\mathrm{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\mathrm{Re}\,\frac{\sigma}{\gamma}\right\},\ z\in \mathrm{U}. \tag{3.14}$$

If

$$\psi(z)\equiv\left[\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu}.$$

$$\left[\sigma+\gamma\mu\left(\frac{\nu z\left[D^{m+1}_{p,l,\lambda}(f*g)(z)\right]'+\eta z\left[D^{m}_{p,l,\lambda}(f*g)(z)\right]'}{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}-p\right)\right]+\Omega \tag{3.15}$$

and

$$\psi(z)\prec\sigma q(z)+\gamma zq'(z)+\Omega, \tag{3.16}$$

then

$$\left[\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q(z),$$

and q is the best dominant of (3.16).

Proof. Let

$$G(z)=\left[\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu} \tag{3.17}$$

Then $G(z)$ is analytic in U, differentiating (3.17) logarithmically with respect to z, we have

$$\frac{zG'(z)}{G(z)} = \mu\left\{\frac{\nu z\left[D_{p,l,\lambda}^{m+1}(f*g)(z)\right]' + \eta z\left[D_{p,l,\lambda}^{m}(f*g)(z)\right]'}{\nu D_{p,l,\lambda}^{m+1}(f*g)(z) + \eta D_{p,l,\lambda}^{m}(f*g)(z)} - p\right\},$$

hence

$$zG'(z) = \mu G(z)\left\{\frac{\nu z\left[D_{p,l,\lambda}^{m+1}(f*g)(z)\right]' + \eta z\left[D_{p,l,\lambda}^{m}(f*g)(z)\right]'}{\nu D_{p,l,\lambda}^{m+1}(f*g)(z) + \eta D_{p,l,\lambda}^{m}(f*g)(z)} - p\right\}.$$

Now, let

$$\theta(w) = \sigma w + \Omega, \quad \varphi(w) = \gamma,\ w \in \mathbb{C},$$
$$Q(z) = zq'(z)\varphi(q(z)) = \gamma zq'(z)\ (z \in U)$$

and

$$h(z) = \theta(q(z)) + Q(z) = \sigma q(z) + \gamma zq'(z) + \Omega\ (z \in U).$$

Using (3.14), we see that Q is starlike in U and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left\{\frac{\sigma}{\gamma} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0,$$

hence, by applying Lemma 1, the proof of Theorem 3 is completed.

Taking $q(z) = \dfrac{1+Az}{1+Bz}(\,-1 \le B < A \le 1)$, in Theorem 3 and according to (3.5), the condition (3.14) reduces to

$$\max\left\{0; -\operatorname{Re}\frac{\sigma}{\gamma}\right\} \le \frac{1-|B|}{1+|B|}.$$

Hence, for the special case $\nu = \gamma = 1$, $\eta = 0$, we obtain the following result:
Corollary 5. *Let $-1 \le B < A \le 1$ and let $\sigma \in C$ with*

$$\max\{0; -\operatorname{Re}\sigma\} \le \frac{1-|B|}{1+|B|}.$$

Let $f, g \in A(p)$ and suppose that $\frac{D_{p,l,\lambda}^{m+1}(f*g)(z)}{z^p} \neq 0,\ z \in U$, and let $\mu \in C^*$. If

$$\left[\frac{D_{p,l,\lambda}^{m+1}(f*g)(z)}{z^p}\right]^{\mu}\left[\sigma\zeta + \mu\left(\frac{z\left[D_{p,l,\lambda}^{m}(f*g)(z)\right]'}{D_{p,l,\lambda}^{m}(f*g)(z)} - p\right)\right] + \Omega$$

$$\prec \sigma \frac{1+Az}{1+Bz} + \Omega + z\frac{(A-B)}{(1+Bz)^2}, \tag{3.18}$$

then

$$\left[\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}\right]^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of* (3.18).

Taking $\eta = \gamma = \lambda = p = 1$, $\nu = m = l = 0, g(z) = z(1-z)^{-1}$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 3, we obtain the next corollary:

Corollary 6. *Let* $f \in A_1$ *such that* $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$, *and let* $\mu \in C^*$. *If*

$$\left[\frac{f(z)}{z}\right]^{\mu}\left[\sigma + \mu\left(\frac{zf'(z)}{f(z)} - 1\right)\right] + \Omega \prec \sigma\frac{1+z}{1-z} + \Omega + \frac{2z}{(1-z)^2}, \tag{3.19}$$

then

$$\left[\frac{f(z)}{z}\right]^{\mu} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ *is the best dominant of* (3.19).

4. Superordination and sandwich results

Theorem 4. *Let* q *be convex in* U *with* $q(0) = 1$ *and* $\delta \in C^*$ *with* $\frac{\lambda}{p(p+l)}\operatorname{Re}\{\delta\} > 0$. *Let* $f, g \in A(p)$ *and suppose that* $\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p} \in H[q(0), 1] \cap \mathcal{Q}$. *If the function*

$$\frac{\delta}{p}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}\right) + \frac{p-\delta}{p}\left(\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\right)$$

is univalent in the unit disc U, *and*

$$q(z) + \frac{\delta\lambda z q'(z)}{p(p+l)} \prec \frac{\delta}{p}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}\right) + \frac{p-\delta}{p}\left(\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\right), \tag{4.1}$$

then

$$q(z) \prec \frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p},$$

and q is the best subordinant of (4.1)*.*

Proof. Let $K(z)$ be given by (3.3), then, from the assumption of the theorem it is analytic in U. Differentiating $K(z)$ logarithmically with respect to z, and using (1.5), we have

$$K(z)+\frac{\delta\lambda zK'(z)}{p(p+l)}=\frac{\delta}{p}\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}+\frac{p-\delta}{p}\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}.$$

Using Lemma 4, the proof of Theorem 4 is completed.

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 4, where $-1\leq B<A\leq 1$, we obtain the next corollary:

Corollary 7. *Let q be convex in U with* $q(0)=1$*, let* $\delta\in C^*$ *and with* $\frac{\lambda}{p(p+l)}\operatorname{Re}\{\delta\}>0$*. Let* $f,g\in A(p)$ *suppose that* $\dfrac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\in H[q(0),1]\cap\mathcal{Q}$*. If the function*

$$\frac{\delta}{p}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}\right)+\frac{p-\delta}{p}\left(\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\right)$$

is univalent in U*, and*

$$\frac{1+Az}{1+Bz}+\frac{\delta\lambda(A-B)z}{p(p+l)(1+Bz)^2}\prec\frac{\delta}{p}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}\right)+$$

$$+\frac{p-\delta}{p}\left(\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\right), \tag{4.2}$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant of* (4.2)*.*

Using the same tequnique of the proof of Theorem 3, and applying Lemma 3, we obtain the following result.

Theorem 5. *Let q be convex in U with* $q(0)=1$*, let* $\mu,\gamma\in C^*$*, and let* $\sigma,\Omega,\nu,\eta\in C$ *with* $\nu+\eta\neq 0$ *and* $\operatorname{Re}\dfrac{\sigma}{\gamma}>0$*. Let* $f,g\in A(p)$ *and suppose that f satisfies the next conditions:*

$$\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\neq 0,\ z\in \mathrm{U},$$

and

$$\left[\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu}\in H[q(0),1]\cap\mathcal{Q}.$$

If the function ψ *given by* (3.15) *is univalent in* U*, and*

$$\sigma q(z)+\gamma zq'(z)+\Omega\prec\psi(z), \tag{4.3}$$

then

$$q(z)\prec\left[\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu},$$

and q *is the best subordinant of* (4.3).

Combining Theorem 1 with Theorem 4 and Theorem 3 with Theorem 5 , we obtain respectively the following *sandwich results*:

Theorem 6. *Let* q_1 *and* q_2 *be two convex functions in* U *with* $q_1(0)=q_2(0)=1$*, let* $\delta\in C^*$ *with* $\frac{\lambda}{p(p+l)}\operatorname{Re}\{\delta\}>0$*. Let* $f,g\in A(p)$ *and suppose that* $\dfrac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\in H[q(0),1]\cap\mathcal{Q}$*. If the function*

$$\frac{\delta}{p}\left(\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}\right)+\frac{p-\delta}{p}\left(\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\right)$$

is univalent in the unit disc U*, and*

$$q_1(z)+\frac{\delta\lambda zq_1'(z)}{p(p+l)}\prec\frac{\delta}{p}\frac{D^{m+1}_{p,l,\lambda}(f*g)(z)}{z^p}+\frac{p-\delta}{p}\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}$$

$$\prec q_2(z)+\frac{\delta\lambda zq'(z)}{p(p+l)}, \tag{4.4}$$

then

$$q_1(z)\prec\frac{D^{m}_{p,l,\lambda}(f*g)(z)}{z^p}\prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant of* (4.4).

Theorem 7. *Let* q_1 *and* q_2 *be two convex functions in* U *with* $q_1(0)=q_2(0)=1$*, let* $\mu,\gamma\in C^*$*, and let* $\sigma,\Omega,\nu,\eta\in C$ *with* $\nu+\eta\neq0$ *and* $\operatorname{Re}\dfrac{\sigma}{\gamma}>0$*. Let* $f,g\in A(p)$ *and suppose that* f *satisfies the next conditions:*

$$\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\neq0,\ z\in\mathrm{U},$$

and

$$\left[\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu}\in H[q(0),1]\cap\mathcal{Q}.$$

If the function ψ *given by* (3.15) *is univalent in* U, *and*

$$\sigma q_1(z)+\gamma z q_1'(z)+\Omega\prec\psi(z)\prec\sigma q_2(z)+\gamma z q_2'(z)+\Omega, \tag{4.5}$$

then

$$q_1(z)\prec\left[\frac{\nu D^{m+1}_{p,l,\lambda}(f*g)(z)+\eta D^{m}_{p,l,\lambda}(f*g)(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant of* (4.5).
Remark. (i) Taking $b_k=1$ or $g(z)=z^p(1-z)^{-1}$ in the above results, we obtain results corresponding to the operator $I_p^m(\lambda,l)$;

(ii) Taking $b_k=\dfrac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}(1)_{k-p}}$, in the above results, we obtain the results obtained by El-Ashwah and Aouf [10];

(iii) Taking $m=0$ and $b_k=\dfrac{\Gamma(p+\alpha+\beta)\Gamma(k+\beta)}{\Gamma(p+\beta)\Gamma(k+\alpha+\beta)}$, in the above results, we obtain the results obtained by Aouf and Bulboaca [4].

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