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RESULTS ON FINITENESS OF GRADED LOCAL COHOMOLOGY MODULES

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Abstract. Let $R = \bigoplus_{n \in N_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, m_0) and irrelevant ideal R_+, let M be a finitely generated graded $R-$module. In this paper we show that if R_0 is a local ring of dimension one, then $H^i_{R+}(H^1_{m_0R}(M))$ is Artinian for each $i \in N_0$. Let f be the least integer such that $H^i_{m_0R}(M)$ is not finitely generated graded $R-$module. In this case, we prove that $\Gamma_{R+}(H^i_{m_0R}(M))$ is Artinian for all $i \leq f$. Finally let s be the largest positive integer such that $H^i_{m_0R}(M)$ is not Artinian. Then we prove that $H^i_{m_0R}(M)/R_+H^i_{m_0R}(M)$ is Artinian for all $i \geq s$.

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1. Introduction

The local cohomology theory has an important role for the derived category and Grothendieck's duality. For instances, one place to find a relatively simple, concrete exposition in Hartshorne's book [Ha]. The notions local cohomology and derived category have successfully invaded branches of mathematics as remote as mathematical physics and even $C^\star$-Algebra. In this paper we study graded local cohomology.

Let $R = \bigoplus_{n \in N_0} R_n$ ba a Noetherian homogenous ring with local base ring (R_0, m_0). So R_0 is a Noetherian ring and there are finitely many elements $l_1, \cdots, l_r \in R_1$ such that $R = R_0[l_1, \cdots, l_r]$. Let $R_+ := \bigoplus_{n \in N} R_n$ denote the irrelevant ideal of R and let $m := m_0 \bigoplus R_+$ denote the graded maximal ideal of R. Moreover Let $q_0 \subseteq R_0$ be an m_0-primary ideal. Finally let $M = \bigoplus_{n \in Z} M_n$ be a finitely generated graded R-module.

Brodmann, Fumasoli and Tajarod in [2] showed that if the local base ring R_0 s of dimension one, then for all i and for all m_0-primary ideal q_0 the graded R-modules $H^i_{R+}(M)/q_0H^i_{R+}(M)$, $(0,_{H^i_{R+}(M)} q_0)$ are Artinian and hence the length of the components of these graded modules have polynomial growth. Next, the authors in [3] showed that the degrees of these polynomials are independent of the choice of q_0. In the case $\dim(R_0) = 2$, the situation changes drastically. Here, the graded R-modules $(0,_{H^i_{R+}(M)} m_0)$ and $H^i_{R+}(M)/m_0H^i_{R+}(M)$ need be Artinian in general $(cf.[BFT, Examples 4.1, 4.2])$. Moreover the above numerical functions need be polynomial in this case, as shown by examples of Katzman and Sharp.

Let $g = g(M)$ (referred to the *cohomological finite length dimension*) be a the least integer i such that the R_0-module $H^i_{R+}(M)_n$ is of infinite length for infinitely many integer n. Authors in[3] showed that if $i \leq g$ then $\Gamma_{m_0}(H^i_{R+}(M))$ is Artinian. Let $c = c(M)$ (referred to the *cohomological diension of M with respect to* R_+) be the largest integer i such that $H^i_{R+}(M) \neq 0$. Rtthaus and sega in [8] proved that $H^c_{R+}/m_0H^c_{R+}(M)$ is Artinian. Sazeedeh in [3] generalizes this result for invariant $a = a_{R+}(M)$ which is the largest positive number i such that $H^i_{R+}(M)$ is not Artinian. In this paper we will obtain some parallel conclusion to this results when we change the places the irrelevant ideal R_+ and maximal ideal m_0 of R_0. In the following we state some of our results which we have prove in this paper.

Let (R_0, m_0) be a local ring of dimension one. Then we show that $H^i_{R+}/(H^1_{m_0R}(M))$ is Artinian for each $i \in N_0$. Moreover, we prove that $H^i_{m_0}(M)/R_+H^i_{m_0}(M)$ are Artinian for all $i \in N_0$. Let f be the least integer such that $H^f_{m_0R}(M)$ is not finitely generated graded R-module. We will show that $\Gamma_{R+}(H^i_{m_0R}(M))$ is Artinian for all $i \leq f$. Lastly if s is the largest positive integer such that $H^s_{m_0R}(M)$ is not Artinian. Then we prove that $H^i_{m_0R}(M)/R_+H^i_{m_0R}(M)$ is Artinian for all $i \geq s$.

2.The results

Theorem 1 *Let* R_0 *be a local ring of dimension one. Then* $H^i_{R+}(H^1_{m_0R}(M))$ *is Artinian for each* $i \in N_0$.

*Proof.*As R_0 is of dimension one, there exists an element $x \in m_0$ such that x is a system of parameter of R_0. Moreover, since since there is a trivial isomorphism $H^1_{m_0R}(M) \cong H^1_{m_0}(M/\Gamma_{m_0}(M))$, we may assume that $\Gamma_{m_0}(M) = 0$; and hence there exists an element $x_0 \in m_0$ such that it is an M-sequence. By using the usual short exact sequence constructed by x_0, there exists an exact sequence

$$0 \to \Gamma_{m_0R}(M/x_0M) \to H^1_{m_0R}(M) \xrightarrow{x_0\cdot} H^1_{m_0R}(M) \to 0.$$

Application of the functor $H^i_{R_+}(-),.$ to this exact sequence induces the following exact sequence $H^i_{R_+}(\Gamma_{m_0R}(M/x_0M)) \to H^i_{R_+}(H^1_{m_0R}(M)) \xrightarrow{x_0\cdot} H^i_{R_+}(H^1_{m_0R}(M)) \to H^{i+1}_{R_+}(\Gamma_{m_0R}(M/x_0M))$. We note that $H^i_{R+}(\Gamma_{m_0R}(M/x_0M))$ is Artinian, and hence

$(0 :_{H_{R_+}(H^1_{m_0R}(M))} x)$ is Artinian. On the other hand, since $H^i_{R_+}(\Gamma_{m_0R}(M/x_0M))$ is x-torsion, using Melkersson Lemma, we get our assertion.

Theorem 2 *Let f be the least integer such that $H^f_{m_0R}(M)$ is not finitely generated graded R-module. Then $\Gamma_{R_+}(H^i_{m_0R}(M))$ is Artinian for all $i \leq f$.*

Proof. At first, we assume that $i < f$. In this case, since $H^i_{m_0R}(M)$ is finitely generated and m_0-torsion, $\Gamma_{R_+}(H^i_{m_0R}(M))$ is finitely generated and m-torsion and so there exists an positive integer t such that $m^t\Gamma_{R_+}(H^i_{m_0R}(M)) = 0$ and this implies that $\Gamma_{R_+}(H^i_{m_0R}(M))$ is Artinian. Let $i = f$. As there is an isomorphism $H^i_{m_0R}(M) \cong H^i_{m_0R}(M/\Gamma_{m_0R}(M))$ for each i, we may assume that $\Gamma_{m_0R}(M) = 0$ and so there exists an element $x \in m_0$ such that it is an M-sequence. In view of the usual short exact sequence constructed by x, we have the following exact sequence

$$H^{f-1}_{m_0R}(M) \to H^{f-1}_{m_0R}(M/xM) \to H_{m_0R}(M) \xrightarrow{x_0\cdot} H_{m_0R}(M).$$

Consider $U = Im(H^{f-1}_{m_0R}(M) \to H^{f-1}_{m_0R}(M/xM))$ and $V = (0 :_{H^f_{m_0R}(M)} x)$. We note that $H^{f-1}_{m_0R}(M)$ is finitely generated and m_0-torsion and so is V. This implies that $H^i_{R_+}(V)$ and $H^i_{R_+}(H^{f-1}_{m_0R}(M))$ are Artinian for all i and hence $\Gamma_{R_+}(V) = \Gamma_{R_+}((0 :_{H^f_{m_0R}(M)} x)) = (0 :_{\Gamma_{R_+}(H^f_{m_0R}(M))} x)$ is Artinaian. Now. since $\Gamma_{R_+}(H^f_{m_0R}(M))$ is x-torsion, by using Melkersson Lemma the result follows.

Proposition 3 *Let (R_0, m_0) be a local ring of dimension one. Then*

$$H^i_{m_0}(M)/R_+H^i_{m_0}(M)$$

are Artinian for all $i \in \mathbb{N}_0$.

Proof. Since M is finitely generated, there exists a free resolution $\cdots \to R^{n_1} \to R^{n_0} \to M \to 0$ of R-modules. Consider $K_i = Ker(R^{n_i} \to R^{n_{i+1}})$. By applying the functor $V \otimes_R -$ to the above exact sequence, we get the following exact sequence $0 \to Tor^R_1(M, V) \to K \otimes_R M \to V \otimes_R R^{n_0} \to V \otimes_R M \to 0$. The above exact sequence implies that $V \otimes_R M$ is Artinian. Now, if we replace K by M, then we conclude that $K \otimes_R M$ is Artinian too and hence $Tor^R_1(M, V)$ is Artinian. Now, by using an easy induction we can get our claim.

Theorem 4 *Let s be the largest positive integer such that $H^s_{m_0R}(M)$ is not Artinian. Then $H^i_{m_0R}(M)/R_+H^i_{m_0R}(M)$ is Artinian for all $i \geq s$.*

Proof. At first consider $i > s$. In this case $H^i_{m_0R}(M)$ is Artinian and then the graded module $H^i_{m_0R}(M)/R_+H^i_{m_0R}(M)$ is Artinian by the previous lemma. Now, assume that $i = s$. By a usual proof mentioned in Proposition (3), we may assume that the residue field R_0/m_0 is infinite. Let $d = \dim M$. We proceed by induction on d. Set $d = 1$. Since, for each $i \in \mathbb{N}_0$, there is an isomorphism $H^i_{m_0R}(M) \cong H^i_{m_0R}(M/\Gamma_{m_0R}(M))$, we may assume that $\Gamma_{m_0R}(M) = 0$. Now, by a similar proof that mentioned in Proposition (3), we can get our assertion. Suppose, inductively that the result has been proved for all values smaller than d and we prove it for d. Consider the exact sequence $0 \to \Gamma_{R_+}(M) \to M \to M/\Gamma_{R_+}(M) \to 0$. Application of the functor $H^i_{m_0R}(-)$ induces the following exact sequence

$$H^i_{m_0R}(\Gamma_{R_+}(M)) \to H^i_{m_0R}(M) \xrightarrow{\alpha} H^i_{m_0R}(M/\Gamma_{R_+}(M)) \to H^{i+1}_{m_0R}(\Gamma_{R_+}(M)).$$

Set $U := Ker(\alpha), V := Im(\alpha)$ and $W := Coker(\alpha)$. It should be noted that $H^i_{m_0R}(\Gamma_{R_+}(M))$ is Artinian for each $i \in \mathbb{N}_0$ and so are U and W. Now, consider the following exact sequences

$$0 \to U \to H^i_{m_0R}(M) \to V \to 0,$$

$$0 \to V \to H^i_{m_0R}(M/\Gamma_{R_+}(M)) \to W \to 0.$$

By effecting the functor $R/R_+ \otimes_R$- to the exact sequence above and using Lemma 1.4, we can conclude that $R/R_+ \otimes_R H^i_{m_0R}(M)$ is Artinian if and only if $R/R_+ \otimes_R H^i_{m_0R}(M/\Gamma_{R_+}(M))$ is Artinian; and hence we may assume that $\Gamma_{R_+}(M) = 0$. In view of the previous argument, we can choose an element $x \in R_1$ which is an M-sequence and then there is an exact sequence $0 \to M \xrightarrow{x.} M \to M/xM \to 0$. Applying the functor $H^i_{m_0R}(-)$ to the exact sequence above induces the following exact sequence

$$H^{i-1}_{m_0R}(M/xM) \to H^i_{m_0R}(M) \xrightarrow{x.} H^i_{m_0R}(M) \to H^i_{m_0R}(M/xM).$$

By a similar proof that mentioned in [S,Lemma 2.2], we can deduce that $a_{m_0R}(M/xM) \leq a_{m_0R}(M)$. As $\dim M/xM = d - 1$, induction hypotheses implies that $R/R_+ \otimes_R H^s_{m_0R}(M/xM)$ is Artinian. Now set $A := Coker(x.)$. Applying the functor $R/R_+ \otimes_R -$ induces the exact sequence

$$R/R_+ \otimes_R H^s_{m_0R}(M) \xrightarrow{id_{R/R_+} \otimes x.} R/R_+ \otimes_R H^s_{m_0R}(M) \to R/R_+ \otimes_R A \to 0.$$

Since $x \in R_1$ the map $id_{R/R_+} \otimes x.$ is zero and hence there is an isomorphism $R/R_+ \otimes_R H^s_{m_0R}(M) \cong R/R_+ \otimes_R A$. On the other hand there is an exact sequence $0 \to A \to H^s_{m_0R}(M/xM) \to B \to 0$ in which B is a submodule of the Artinian module $H^{s+1}_{m_0R}(M)$. Now, by effecting the functor $R/R_+ \otimes_R -$ to the above exact sequence and using Lemma 2.4, we can conclude that $R/R_+ \otimes_R A$ is Artinian and so is $H^s_{m_0R}(M)/R_+H^s_{m_0R}(M)$.

Proposition 5 *Let m_0 be a principal ideal of R_0. Then $H^i_{R_+}(H^j_{m_0R}(M))$ is Artinian for all $i, j \in \mathbb{N}_0$.*

Proof. Since m_0 is principal. there is $H^j_{m_0}(M) = 0$ for all $j > 1$. If $j = 0$, then $\Gamma_{m_0}(M)$ is a finitely generated graded m_0-torsion R-module, and hence for each i, there is an isomorphism $H^i_{R_+}(\Gamma_{m_0}(M)) \cong H^i_m(\Gamma_{m_0}(M))$. We note that the last term is Artinian and the result is clear in this case. Now, consider $j = 1$. Since there is an isomorphism $H^1_{m_0}(M) \cong H^1_{m_0}(M/\Gamma_{m_0}(M))$, we may assume that $\Gamma_{m_0}(M) = 0$. thus there exists an element $x \in m_0$ which is a non-zerodivisor with respect to M and so there is the following exact sequence $0 \to M \xrightarrow{x.} M \to M/xM \to 0$. Application of the functor $H^j_{m_0}(-)$ to this exact sequence induces the following exact sequence $0 \to \Gamma_{m_0}(M/xM) \to H^1_{m_0}(M) \xrightarrow{x.} H^1_{m_0}(M) \to 0$. Now, if we apply the functor $H^i_{R_+}(-)$ to the last exact sequence, we get the following exact sequence $H^{i-1}_{R_+}(\Gamma_{m_0}(M/xM)) \to H^i_{R_+}(H^1_{m_0}(M)) \xrightarrow{x.} H^i_{R_+}(H^1_{m_0}(M)) \to 0$. It should be noted that $H^{i-1}_{R_+}(\Gamma_{m_0}(M/xM)) \cong H^{i-1}_m(\Gamma_{m_0}(M/xM))$ is Artinian. So this fact implies that $(0 :_{H^{i-1}_{R_+}(H^1_{m_0}(M))} x)$ is Artinian. Now, since $H^i_{R_+}(H^1_{m_0}(M))$ is x-torsion, using Melkersson's Lemma this module is Artinian.

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