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# INCLUSION PROPERTIES FOR CERTAIN K-UNIFORMLY SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH DZIOK-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce several new $k-$uniformly classes of analytic functions defined by using Dziok-Srivastava operator and investigate various inclusion relationships for these classes. Some interesting applications involving certain classes of integral operators are also considered.

Keywords: Analytic functions, $k-$uniformly starlike functions, $k-$uniformly convex functions, $k-$uniformly close-to-convex functions, $k-$uniformly quasi-convex functions, integral operator, Hadamard product, subordination.

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## 1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $f$ and $g$ are analytic in $\mathbf{U}$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega$, analytic in $\mathbf{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbf{U})$, such that $f(z) = g(\omega(z))$ $(z \in \mathbf{U})$. In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$ (see [15] and [16]). For $0 \le \gamma, \beta < 1$, we denote by $S^*(\gamma)$, $C(\gamma)$, $K(\gamma, \beta)$ and $K^*(\gamma, \beta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike of order $\gamma$, convex of order $\gamma$, close-to-convex of order $\gamma$, and type $\beta$ and quasi-convex of order $\gamma$, and type $\beta$ in $\mathbf{U}$.

Now, we introduce the subclasses $US^*(k;\gamma)$, $UC(k;\gamma)$, $UK(k;\gamma,\beta)$ and $UK^*(k;\gamma,\beta)$ of the class $\mathcal{A}$ for $0 \le \gamma,\beta < 1$, and $k \ge 0$, which are defined by

$$US^*(k;\gamma) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|\right\}, \tag{1.2}$$

$$UC(k;\gamma) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|\right\}, \tag{1.3}$$

$$UK(k;\gamma,\beta) = \left\{ f \in \mathcal{A} : \exists\, g \in US^*(k;\beta), \Re\left(\frac{zf'(z)}{g(z)} - \gamma\right) > k\left|\frac{zf'(z)}{g(z)} - 1\right|\right\}, \tag{1.4}$$

$$UK^*(k;\gamma,\beta) = \left\{ f \in \mathcal{A} : \exists\, g \in UC(k;\gamma), \Re\left(\frac{\left(zf'(z)\right)'}{g'(z)} - \gamma\right) > k\left|\frac{\left(zf'(z)\right)'}{g'(z)} - 1\right|\right\}. \tag{1.5}$$

We note that

$$US^*(0;\gamma) = S^*(k;\gamma),\ UC(0;\gamma) = C(\gamma),$$
$$UK(0;\gamma,\beta) = K(\gamma,\beta),\ UK^*(0;\gamma,\beta) = K^*(\gamma,\beta) \quad (0 \le \gamma,\beta < 1).$$

Corresponding to a conic domain $\Omega_{k,\gamma}$ defined by

$$\Omega_{k,\gamma} = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} + \gamma \right\}, \tag{1.6}$$

we define the function $q_{k,\gamma}(z)$ which maps $\mathbf{U}$ onto the conic domain $\Omega_{k,\gamma}$ such that $1 \in \Omega_{k,\gamma}$ as the following:

$$q_{k,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z} & (k=0), \\ \frac{1-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}\left(\cos^{-1}k\right) i \log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2-\gamma}{1-k^2} & (0<k<1), \\ 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & (k=1), \\ \frac{1-\gamma}{k^2-1}\sin\left\{\frac{\pi}{2\zeta(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right\} + \frac{k^2-\gamma}{k^2-1} & (k>1). \end{cases} \tag{1.7}$$

where $u(z) = \frac{z-\sqrt{k}}{1-\sqrt{kz}}$ and $\zeta(k)$ is such that $k = \cosh\frac{\pi\zeta'(z)}{4\zeta(z)}$. By virture of the properties of the conic domain $\Omega_{k,\gamma}$, we have

$$\Re\{q_{k,\gamma}(z)\} > \frac{k+\gamma}{k+1}. \tag{1.8}$$

Making use of the principal of subordination between analytic functions and the definition of $q_{k,\gamma}(z)$, we may rewrite the subclasses $US^*(k;\gamma)$, $UC(k;\gamma)$, $UK(k;\gamma,\beta)$ and $UK^*(k;\gamma,\beta)$ as the following:

$$US^*(k;\gamma)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec q_{k,\gamma}(z)\right\}, \tag{1.9}$$

$$UC(k;\gamma)=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec q_{k,\gamma}(z)\right\}, \tag{1.10}$$

$$UK(k;\gamma,\beta)=\left\{f\in\mathcal{A}:\exists\ g\in US^*(k;\beta),\frac{zf'(z)}{g(z)}\prec q_{k,\gamma}(z)\right\}, \tag{1.11}$$

$$UK^*(k;\gamma,\beta)=\left\{f\in\mathcal{A}:\exists\ g\in UC(k;\gamma),\frac{\left(zf'(z)\right)'}{g'(z)}\prec q_{k,\gamma}(z)\right\}. \tag{1.12}$$

For two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n$$

their Hadamard product (or convolution) is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n=(g*f)(z).$$

For complex parameters

$$a_1,...,a_q;b_1,...,b_s\left(b_j\notin\mathbb{Z}_0^-=\{0,-1,-2,...\};j=1,...,s\right),$$

the generalized hypergeometric function ${}_qF_s(a_1,...,a_q;b_1,...,b_s;z)$ is given by

$${}_qF_s(a_1,...,a_q;b_1,...,b_s;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_q)_n}{(b_1)_n...(b_s)_n}\frac{z^n}{n!} \tag{1.13}$$

$$\left(q\le s+1;q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,...\};z\in U\right),$$

where $(x)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}1 & (n=0),\\ x(x+1)...(x+n-1) & (n\in\mathbb{N}).\end{cases}$$

Corresponding to a function $h(a_1,...,a_q;b_1,...,b_s;z)$ defined by

$$h(a_1,...,a_q;b_1,...,b_s;z) = z_qF_s(a_1,...,a_q;b_1,...,b_s;z), \tag{1.14}$$

Dziok and Srivastava [4] considered a linear operator $H(a_1,...,a_q;b_1,...,b_s) : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} H(a_1,...,a_q;b_1,...,b_s) f(z) &= h(a_1,...,a_q;b_1,...,b_s;z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_q)_{n-1}}{(b_1)_{n-1} \cdots (b_s)_{n-1}} \frac{z^n}{(n-1)!}, \end{aligned} \tag{1.15}$$

We note that many subclasses of analytic functions, associated with the Dziok-Srivastava operator and many special cases, were investigated recently by Aghalary and Azadi [1], Dziok-Srivastava [5, 6], Liu [11], Liu and Srivastava [12] and others.

Corresponding to the function $h(a_1,...,a_q;b_1,...,b_s;z)$, defined by (1.14), we introduce a function $h_\mu(a_1,...,a_q;b_1,...,b_s;z)$ given by

$$h(a_1,...,a_q;b_1,...,b_s;z) * h_\mu(a_1,...,a_q;b_1,...,b_s;z) = \frac{z}{(1-z)^\mu} \quad (\mu > 0) \tag{1.16}$$

Analogous to $H(a_1,...,a_q;b_1,...,b_s)$, Kwon and Cho [8] introduced the linear operator

$$H^\mu(a_1,...,a_q;b_1,...,b_s) : \mathcal{A} \to \mathcal{A}$$

as follows:

$$H^\mu(a_1,...,a_q;b_1,...,b_s) f(z) = h_\mu(a_1,...,a_q;b_1,...,b_s;z) * f(z) \tag{1.17}$$

$$\left(a_i, b_j \in \mathbb{C} \backslash \mathbb{Z}_0^-, i = 1,...,s, j = 1,...,q; \mu > 0; f \in \mathcal{A}; z \in U\right).$$

For $q = s+1$ and $a_2 = b_1,...,a_q = b_s$, we note that

$$H^\mu(\mu,...,a_q;b_1,...,b_s) f(z) = f(z),$$

and

$$H^2(1,...,a_q;b_1,...,b_s) f(z) = zf'(z).$$

For convenience, we write

$$H^\mu_{q,s}(a_1) = H^\mu(a_1,...,a_q;b_1,...,b_s). \tag{1.18}$$

It is easily verified from the definition (1.17) that

$$z\left(H^\mu_{q,s}(a_1) f(z)\right)' = \mu H^{\mu+1}_{q,s}(a_1) f(z) - (\mu-1) H^\mu_{q,s}(a_1) f(z), \tag{1.19}$$

and

$$z\left(H^{\mu}_{q,s}(a_1+1)f(z)\right)' = a_1 H^{\mu}_{q,s}(a_1)f(z) - (a_1-1)H^{\mu}_{q,s}(a_1+1)f(z). \quad (1.20)$$

In particular, the operator $H^{\mu}(\lambda+1,1;1)\,(\mu>0;\lambda>-1)$ was introduced by Choi et al. [3], who investigated (among other things) several inclusion properties involving various subclasses of analytic and univalent functions. For $\lambda = n\ (n\in\mathbb{N}_0)$ and $\mu=2$, we also note that the Choi-Saigo-Srivastava operator $H^{\mu}(\lambda+1,1;1)$ is the Noor integral operator of $n-$th order of $f$ studied by Liu [10] and Noor [17] and Noor and Noor [18].

Next, by using the operator $H^{\mu}_{q,s}(a_1)$, we introduce the following $k-$uniformly classes of analytic functions for $a_1\in\mathbb{C}\backslash\mathbb{Z}_0^-, s,q\in\mathbb{N}_0, \mu>0, k\geq 0$ and $0\leq\gamma,\beta<1:$

$$US^{*}_{q,s}(\mu;a_1;k;\gamma) = \left\{f\in\mathcal{A}: H^{\mu}_{q,s}(a_1)f(z)\in US^{*}(k;\gamma)\right\}, \quad (1.21)$$

$$UC_{q,s}(\mu;a_1;k;\gamma) = \left\{f\in\mathcal{A}: H^{\mu}_{q,s}(a_1)f(z)\in UC(\mu;a_1;k;\gamma)\right\}, \quad (1.22)$$

$$UK_{q,s}(\mu;a_1;k;\gamma,\beta) = \left\{f\in\mathcal{A}: H^{\mu}_{q,s}(a_1)f(z)\in UK(k;\gamma,\beta)\right\}, \quad (1.23)$$

$$UK^{*}_{q,s}(\mu;a_1;k;\gamma,\beta) = \left\{f\in\mathcal{A}: H^{\mu}_{q,s}(a_1)f(z)\in UK^{*}(k;\gamma,\beta)\right\}. \quad (1.24)$$

We also note that

$$f(z)\in US^{*}_{q,s}(\mu;a_1;k;\gamma)\Leftrightarrow zf'(z)\in UC_{q,s}(\mu;a_1;k;\gamma), \quad (1.25)$$

and

$$f(z)\in UK_{q,s}(\mu;a_1;k;\gamma,\beta)\Leftrightarrow zf'(z)\in UK^{*}_{q,s}(\mu;a_1;k;\gamma,\beta). \quad (1.26)$$

In this paper, we investgate several inclusion properties of the classes $US^{*}_{q,s}(\mu;a_1;k;\gamma)$, $UC_{q,s}(\mu;a_1;k;\gamma)$, $UK_{q,s}(\mu;a_1;k;\gamma,\beta)$, and $UK^{*}_{q,s}(\mu;a_1;k;\gamma,\beta)$ associated with the operator $H^{\mu}_{q,s}(a_1)$. Some applications involving integral operators are also considered.

## 2. Inclusion Properties Involving the Operator $H^{\mu}_{q,s}(a_1)$

In order to prove the main results, we shall need The following lemmas.

**Lemma 1** [7]. *Let $h(z)$ be convex univalent in* $\mathbf{U}$ *with $h(0)=1$ and $\Re\{\eta h(z)+\gamma\}>0$ $(\eta,\gamma\in\mathbb{C})$. If $p(z)$ is analytic in* $\mathbf{U}$ *with $p(0)=1$, then*

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma}\prec h(z) \quad (2.1)$$

*implies*

$$p(z) \prec h(z). \tag{2.2}$$

**Lemma 2** [14]. *Let $h(z)$ be convex univalent in* $\mathbf{U}$ *and let $w$ be analytic in* $\mathbf{U}$ *with $\Re\{w(z)\} \geq 0$. If $p(z)$ is analytic in* $\mathbf{U}$ *and $p(0) = h(0)$, then*

$$p(z) + w(z)\, zp'(z) \prec h(z) \tag{2.3}$$

*implies*

$$p(z) \prec h(z). \tag{2.4}$$

**Theorem 1.** *Let $a_1, \mu > \dfrac{1-\gamma}{k+1}$. Then,*

$$US^{*}_{q,s}(\mu+1; a_1; k; \gamma) \subset US^{*}_{q,s}(\mu; a_1; k; \gamma) \subset US^{*}_{q,s}(\mu; a_1+1; k; \gamma). \tag{2.5}$$

*Proof.* First of all, we will show that

$$US^{*}_{q,s}(\mu+1; a_1; k; \gamma) \subset US^{*}_{q,s}(\mu; a_1; k; \gamma). \tag{2.6}$$

Let $f \in US^{*}_{q,s}(\mu+1; a_1; k; \gamma)$ and set

$$p(z) = \frac{z\left(H^{\mu}_{q,s}(a_1) f(z)\right)'}{H^{\mu}_{q,s}(a_1) f(z)} \quad (z \in \mathbf{U}), \tag{2.7}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$. Using $(1.19)$, $(2.6)$ and $(2.7)$, we have

$$\frac{z\left(H^{\mu+1}_{q,s}(a_1) f(z)\right)'}{H^{\mu+1}_{q,s}(a_1) f(z)} = p(z) + \frac{zp'(z)}{p(z)+\mu-1} \prec q_{k,\gamma}(z) \qquad (z \in \mathbf{U}). \tag{2.8}$$

Since $\mu > \frac{1-\gamma}{k+1}$ and $\Re\{q_{k,\gamma}(z)\} > \frac{k+\gamma}{k+1}$, we see that

$$\Re\{q_{k,\gamma}(z) + \mu - 1\} > 0 \quad (z \in \mathbf{U}). \tag{2.9}$$

Applying Lemma 1 to (2.8), it follows that $p(z) \prec q_{k,\gamma}(z)$, that is, $f \in US^{*}_{q,s}(\mu; a_1; k; \gamma)$.

To prove the second part, let $f \in US^{*}_{q,s}(\mu; a_1; k; \gamma)$ and put

$$s(z) = \frac{z\left(H^{\mu}_{q,s}(a_1+1) f(z)\right)'}{H^{\mu}_{q,s}(a_1+1) f(z)} \quad (z \in \mathbf{U}), \tag{2.10}$$

where $s(z)$ is analytic function with $s(0) = 1$. Then, by using the arguments similar to those detailed above with (1.20), it follows that $s(z) \prec q_{k,\gamma}(z)$ in $\mathbf{U}$, which implies that $f \in US^{*}_{q,s}(\mu; a_1+1; k; \gamma)$. Therefore, we complete the proof of Theorem 1.

**Theorem 2.** *Let* $a_1, \mu > \frac{1-\gamma}{k+1}$. *Then,*

$$UC_{q,s}(\mu+1; a_1; k; \gamma) \subset UC_{q,s}(\mu; a_1; k; \gamma) \subset UC_{q,s}(\mu; a_1+1; k; \gamma). \tag{2.11}$$

*Proof.* Applying (1.25) and Theorem 1, we observe that

$$\begin{aligned} f(z) \quad &\in \quad UC_{q,s}(\mu+1; a_1; k; \gamma) \Longleftrightarrow zf'(z) \in US^*_{q,s}(\mu+1; a_1; k; \gamma) \\ &\Longrightarrow \quad zf'(z) \in US^*_{q,s}(\mu; a_1; k; \gamma) \quad (by\ Theorem\ 1) \\ &\Longleftrightarrow \quad f(z) \in UC_{q,s}(\mu; a_1; k; \gamma) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} f(z) \quad &\in \quad UC_{q,s}(\mu; a_1; k; \gamma) \Longleftrightarrow zf'(z) \in US^*_{q,s}(\mu; a_1; k; \gamma) \\ &\Longrightarrow \quad zf'(z) \in US^*_{q,s}(\mu; a_1+1; k; \gamma) \quad (by\ Theorem\ 1) \\ &\Longleftrightarrow \quad f(z) \in UC_{q,s}(\mu; a_1+1; k; \gamma). \end{aligned}$$

which evidently proves Theorem 2.

Next, by using Lemma 2, we obtain the following inclusion relation for the class $UK_{q,s}(\mu; a_1; k; \gamma, \beta)$.

**Theorem 3.** *Let* $a_1, \mu > \frac{1-\gamma}{k+1}$. *Then,*

$$UK_{q,s}(\mu+1; a_1; k; \gamma, \beta) \subset UK_{q,s}(\mu; a_1; k; \gamma, \beta) \subset UK_{q,s}(\mu; a_1+1; k; \gamma, \beta). \tag{2.13}$$

*Proof.* We begin by proving that

$$UK_{q,s}(\mu+1; a_1; k; \gamma, \beta) \subset UK_{q,s}(\mu; a_1; k; \gamma, \beta). \tag{2.14}$$

Let $f \in UK_{q,s}(\mu+1; a_1; k; \gamma, \beta)$. Then, from the definition of $UK_{q,s}(\mu+1; a_1; k; \gamma, \beta)$, there exists a function $r(z) \in US^*(k; \gamma)$ such that

$$\frac{z\left(H^{\mu+1}_{q,s}(a_1) f(z)\right)'}{r(z)} \prec q_{k,\gamma}(z) \quad (z \in \mathbf{U}). \tag{2.15}$$

Choose the function $g$ such that $H^{\mu+1}_{q,s}(a_1) g(z) = r(z)$. Then, $g \in US^*_{q,s}(\mu+1; a_1; k; \gamma)$ and

$$\frac{z\left(H^{\mu+1}_{q,s}(a_1) f(z)\right)'}{H^{\mu+1}_{q,s}(a_1) g(z)} \prec q_{k,\gamma}(z) \quad (z \in \mathbf{U}). \tag{2.16}$$

Now let

$$p(z) = \frac{z\left(H^{\mu}_{q,s}(a_1) f(z)\right)'}{H^{\mu}_{q,s}(a_1) g(z)}, \tag{2.17}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$. Since $g \in US^{*}_{q,s}(\mu+1; a_1; k; \gamma)$, by Theorem 1, we know that $g \in US^{*}_{q,s}(\mu; a_1; k; \gamma)$. Let

$$t(z) = \frac{z\left(H^{\mu}_{q,s}(a_1) g(z)\right)'}{H^{\mu}_{q,s}(a_1) g(z)} \quad (z \in \mathbf{U}), \tag{2.18}$$

where $t(z)$ is analytic in $\mathbf{U}$ with $\Re\{t(z)\} > \frac{k+\beta}{k+1}$. Also, from (2.17), we note that

$$H^{\mu}_{q,s}(a_1) z f'(z) = H^{\mu}_{q,s}(a_1) g(z)\ p(z). \tag{2.19}$$

Differentiating both sides of (2.19) with respect to $z$, we obtain

$$\begin{aligned} \frac{z\left(H^{\mu}_{q,s}(a_1) z f'(z)\right)'}{H^{\mu}_{q,s}(a_1) g(z)} &= \frac{z\left(H^{\mu}_{q,s}(a_1) g(z)\right)'}{H^{\mu}_{q,s}(a_1) g(z)} p(z) + z p'(z) \\ &= t(z) p(z) + z p'(z). \end{aligned} \tag{2.20}$$

Now using the identity (1.19) and (2.22), we obtain

$$\begin{aligned} \frac{z\left(H^{\mu+1}_{q,s}(a_1) f(z)\right)'}{H^{\mu+1}_{q,s}(a_1) g(z)} &= \frac{H^{\mu+1}_{q,s}(a_1) z f'(z)}{H^{\mu+1}_{q,s}(a_1) g(z)} \\ &= \frac{z\left(H^{\mu}_{q,s}(a_1) z f'(z)\right)' + (\mu-1) H^{\mu}_{q,s}(a_1) z f'(z)}{z\left(H^{\mu}_{q,s}(a_1) g(z)\right)' + (\mu-1) H^{\mu}_{q,s}(a_1) g(z)} \\ &= \frac{\frac{z\left(H^{\mu}_{q,s}(a_1) z f'(z)\right)'}{H^{\mu}_{q,s}(a_1) g(z)} + (\mu-1) \frac{z\left(H^{\mu}_{q,s}(a_1) f(z)\right)'}{H^{\mu}_{q,s}(a_1) g(z)}}{\frac{z\left(H^{\mu}_{q,s}(a_1) g(z)\right)'}{H^{\mu}_{q,s}(a_1) g(z)} + \mu - 1} \\ &= \frac{t(z) p(z) + z p'(z) + (\mu-1) p(z)}{t(z) + \mu - 1} \\ &= p(z) + \frac{z p'(z)}{t(z) + \mu - 1}. \end{aligned} \tag{2.21}$$

Since $\mu > \frac{1-\gamma}{k+1}$ and $\Re\{t(z)\} > \frac{k+\gamma}{k+1}$, we see that

$$\Re\{t(z) + \mu - 1\} > 0 \quad (z \in \mathbf{U}). \tag{2.22}$$

Hence, applying Lemma 2, we can show that $p(z) \prec q_{k,\gamma}(z)$ so that $f \in UK_{q,s}(\mu+1; a_1; k; \gamma, \beta)$. For the second part, by using the arguments similar to those detailed above with (1.20), we obtain

$$UK_{q,s}(\mu; a_1; k; \gamma, \beta) \subset UK_{q,s}(\mu; a_1+1; k; \gamma, \beta). \tag{2.23}$$

Therefore, we complete the proof of Theorem 3.

**Theorem 4.** *Let* $a_1, \mu > \dfrac{1-\gamma}{k+1}$. *Then,*

$$UK_{q,s}(\mu+1; a_1; k; \gamma, \beta) \subset UK_{q,s}(\mu; a_1; k; \gamma, \beta) \subset UK_{q,s}(\mu; a_1+1; k; \gamma, \beta). \tag{2.24}$$

*Proof.* Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (1.25), we can also prove Theorem 4 by using Theorem 3 and the equivalence (1.26).

## 3. Inclusion Properties Involving the Integral Operator $F_c$

In this section, we consider the generalized Libera integral operator $F_c$ (see [2] , [9] and [13] ) defined by

$$F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \quad (f \in \mathcal{A}; c > -1). \tag{3.1}$$

**Theorem 5.** Let $c \geq -\dfrac{k+\gamma}{k+1}$. If $f \in US^*_{q,s}(\mu; a_1; k; \gamma)$, then

$$F_c(f) \in US^*_{q,s}(\mu; a_1; k; \gamma).$$

*Proof.* Let $f \in US^*_{q,s}(\mu; a_1; k; \gamma)$ and set

$$p(z) = \frac{z\left(H^{\mu}_{q,s}(a_1) F_c(f)(z)\right)'}{H^{\mu}_{q,s}(a_1) F_c(f)(z)} \quad (z \in \mathbf{U}), \tag{3.2}$$

where $p(z)$ is analytic in $U$ with $p(0) = 1$. From (3.1), we have

$$z\left(H^{\mu}_{q,s}(a_1) F_c(f)(z)\right)' = (c+1) H^{\mu}_{q,s}(a_1) f(z) - cH^{\mu}_{q,s}(a_1) F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(c+1)\frac{H^{\mu}_{q,s}(a_1) f(z)}{H^{\mu}_{q,s}(a_1) F_c(f)(z)} = p(z) + c. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by $z$, we have

$$\frac{z\left(H_{q,s}^{\mu}(a_1) f(z)\right)^{'}}{H_{q,s}^{\mu}(a_1) f(z)} = p(z) + \frac{zp^{'}(z)}{p(z)+c} \prec q_{k,\gamma}(z) \quad (z \in \mathbf{U}). \tag{3.5}$$

Hence, by virtue of Lemma 1, we conclude that $p(z) \prec q_{k,\gamma}(z)$ in $\mathbf{U}$, which implies that $F_c(f) \in US_{q,s}^{*}(\mu; a_1; k; \gamma)$.

Next, we derive an inclusion property involving $F_{p,c}(f)$, which is given by the following.

**Theorem 6.** Let $c \geq -\dfrac{k+\gamma}{k+1}$. If $f \in UC_{q,s}(\mu; a_1; k; \gamma)$, then $F_c(f) \in UC_{q,s}(\mu; a_1; k; \gamma)$.

*Proof.* By applying Theorem 5, it follows that

$$\begin{aligned} f(z) &\in UC_{q,s}(\mu; a_1; k; \gamma) \Longleftrightarrow zf^{'}(z) \in US_{q,s}^{*}(\mu; a_1; k; \gamma) \\ &\Rightarrow F_c\left(zf^{'}(z)\right) \in US_{q,s}^{*}(\mu; a_1; k; \gamma) \quad (\textit{by Theorem } 5) \\ &\Leftrightarrow z\left(F_c(f)(z)\right)^{'} \in US_{q,s}^{*}(\mu; a_1; k; \gamma) \\ &\Leftrightarrow F_c(f)(z) \in UC_{q,s}(\mu; a_1; k; \gamma), \end{aligned} \tag{3.6}$$

which proves Theorem 6.

**Theorem 7.** Let $c \geq -\dfrac{k+\gamma}{k+1}$. If $f \in UK_{q,s}(\mu; a_1; k; \gamma, \beta)$, then $F_c(f) \in UK_{q,s}(\mu; a_1; k; \gamma, \beta)$.

*Proof.* Let $f \in UK_{q,s}(\mu; a_1; k; \gamma, \beta)$. Then, in view of the definition of the class $UK_{q,s}(\mu; a_1; k; \gamma, \beta)$, there exists a function $g \in US_{q,s}^{*}(\mu; a_1; k; \gamma)$ such that

$$\frac{z\left(H_{q,s}^{\mu}(a_1) f(z)\right)^{'}}{H_{q,s}^{\mu}(a_1) g(z)} \prec q_{k,\gamma}(z) \quad (z \in \mathbf{U}). \tag{3.7}$$

Thus, we set

$$p(z) = \frac{z\left(H_{q,s}^{\mu}(a_1) F_c(f)(z)\right)^{'}}{H_{q,s}^{\mu}(a_1) F_c(g)(z)} \quad (z \in \mathbf{U}), \tag{3.8}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$. Since $g \in US_{q,s}^{*}(\mu; a_1; k; \gamma)$, we see fromTheorem 5 that $F_c(f) \in US_{q,s}^{*}(\mu; a_1; k; \gamma)$. Let

$$t(z) = \frac{z\left(H_{q,s}^{\mu}(a_1) F_c(g)(z)\right)^{'}}{H_{q,s}^{\mu}(a_1) F_c(g)(z)} \quad (z \in \mathbf{U}), \tag{3.9}$$

where $t(z)$ is analytic in $\mathbf{U}$ with $\Re\{t(z)\} > \frac{k+\beta}{k+1}$. Also, from (3.8), we note that

$$H_{q,s}^{\mu}(a_1) zF_c'(f)(z) = H_{q,s}^{\mu}(a_1) F_c(g)(z) \; p(z). \tag{3.10}$$

Differentiating both sides of (3.10) with respect to $z$, we obtain

$$\begin{aligned} \frac{z\left(H_{q,s}^{\mu}(a_1) zF_c'(f)(z)\right)'}{H_{q,s}^{\mu}(a_1) F_c(g)(z)} &= \frac{z\left(H_{q,s}^{\mu}(a_1) F_c(g)(z)\right)'}{H_{q,s}^{\mu}(a_1) F_c(g)(z)} p(z) + zp'(z) \\ &= t(z) p(z) + zp'(z). \end{aligned} \tag{3.11}$$

Now using the identity (3.3) and (3.11), we obtain

$$\begin{aligned} \frac{z\left(H_{q,s}^{\mu}(a_1) f(z)\right)'}{H_{q,s}^{\mu}(a_1) g(z)} &= \frac{z\left(H_{q,s}^{\mu}(a_1) zF_c'(f)(z)\right)' + cH_{q,s}^{\mu}(a_1) zF_c'(f)(z)}{z\left(H_{q,s}^{\mu}(a_1) F_c(g)(z)\right)' + cH_{q,s}^{\mu}(a_1) F_c(g)(z)} \\ &= \frac{\frac{z\left(H_{q,s}^{\mu}(a_1) zF_c'(f)(z)\right)'}{H_{q,s}^{\mu}(a_1)F_c(g)(z)} + c\frac{z\left(H_{q,s}^{\mu}(a_1)F_c(f)(z)\right)'}{H_{q,s}^{\mu}(a_1)F_c(g)(z)}}{\frac{z\left(H_{q,s}^{\mu}(a_1)F_c(g)(z)\right)'}{H_{q,s}^{\mu}(a_1)F_c(g)(z)} + c} \\ &= \frac{t(z) p(z) + zp'(z) + cp(z)}{t(z) + c} \\ &= p(z) + \frac{zp'(z)}{t(z) + c}. \end{aligned} \tag{3.12}$$

Since $c \geq -\frac{k+\gamma}{k+1}$ and $\Re\{t(z)\} > \frac{k+\gamma}{k+1}$, we see that

$$\Re\{t(z) + c\} > 0 \quad (z \in \mathbf{U}). \tag{3.13}$$

Hence, applying Lemma 2 to (3.12), we can show that $p(z) \prec q_{k,\gamma}(z)$ so that $f \in UK_{q,s}(\mu; a_1; k; \gamma, \beta)$.

**Theorem 8.** *Let* $c \geq -\dfrac{k+\gamma}{k+1}$. *If* $f \in UK_{q,s}^{*}(\mu; a_1; k; \gamma, \beta)$, *then* $F_c(f) \in UK_{q,s}^{*}(\mu; a_1; k; \gamma, \beta)$.

*Proof.* Just as we derived Theorem 6 as consequence of Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7.

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