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THE VERTEX DOMINATION POLYNOMIAL AND EDGE DOMINATION POLYNOMIAL OF A GRAPH

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Abstract. Let G be a simple graph of order n, the vertex domination polynomial of G is the polynomial $D_0(G,x) = \sum_{i=\gamma_0(G)}^{n} d_0(G,i)x^i$, where $d_0(G,i)$ is the number of vertex dominating sets of G with size i, and $\gamma_0(G)$ is the vertex domination number of G. Similarly, the edge domination polynomial of G is the polynomial $D_1(G,x) = \sum_{i=\gamma_1(G)}^{|E(G)|} d_1(G,i)x^i$, where $d_1(G,i)$ is the number of edge dominating sets of G with size i, and $\gamma_1(G)$ is the edge domination number of G. In this paper, we obtain some properties of the coefficients of the edge domination polynomial of G and show that the edge domination polynomial of G is equal to the vertex domination polynomial of line graph $L(G)$ of G.

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1. Introduction

Let $G = (V, E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$, the open neighborhood of v is the set $\mathrm{N}(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of v is the set $\mathrm{N}[v] = \mathrm{N}(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $\mathrm{N}(S) = \bigcup_{v \in S} \mathrm{N}(v)$ and the closed neighborhood of S is $\mathrm{N}[S] = \mathrm{N}(S) \cup S$. A set $S \subseteq V$ is a vertex dominating set if $\mathrm{N}[S] = V$, or equivalently every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The vertex domination number $\gamma_0(G)$ is the minimum cardinality of a vertex dominating set in G. A dominating set with cardinality $\gamma_0(G)$ is called a γ_0-set. For a detailed treatment of this parameter, the reader is referred to [4]. Similarly, for any edge $e \in E$, the open neighborhood of e is the set $\mathrm{N}(e) = \{e' \in E | e$ and e' have a common vertex and $e \neq e'\}$ and the closed neighborhood of e is the set $\mathrm{N}[e] = \mathrm{N}(e) \cup \{e\}$. For a set $F \subseteq E$, the open neighborhood of F is $\mathrm{N}(F) = \bigcup_{e \in F} \mathrm{N}(e)$ and the closed neighborhood of F is $\mathrm{N}[F] = N(F) \cup F$. A set $F \subseteq E$ is an edge dominating set if $\mathrm{N}[F] = E$, or equivalently, every edge in $E \setminus F$ have a common vertex to at least one edge in F. The edge domination number $\gamma_1(G)$ is the minimum cardinality of an edge dominating set in G.

We denote the family of all edge dominating sets of graph G with cardinality i by $\mathcal{D}_1(G,i)$, and denote the family of all edge dominating sets of G with cardinality i and contain an edge e by $\mathcal{D}_1(G,e,i)$, and $d_1(G,e,i) = |\mathcal{D}_1(G,e,i)|$, where i is a positive integer.

The line graph $L(G)$ of graph G is constructed by taking the edges of G as vertices of $L(G)$, and joining tow vertices in $L(G)$ whenever the corresponding edges in G have a common vertex.

Two graph G and H are called isomorphic, written as $G \cong H$, if there exists a bijective mapping $\psi : V(G) \longrightarrow V(H)$ between the vertex set of two graphs such that $uv \in E(G)$ if and only if $\psi(u)\psi(v) \in E(H)$. An automorphism of the graph G is an isomorphism of G onto itself, and G is edge transitive if for any $u_1u_2, v_1v_2 \in E(G)$ there exist an automorphism ψ of G such that $\psi(u_1u_2) = v_1v_2$. This implies that G is edge transitive if and only if the group of all automorphisms of G, $\mathrm{Aut}(G)$, acting on $E(G)$ produces only one orbit.

2. Edge Domination Polynomial

In this section, we state the definition of edge domination polynomial and some of its properties.

Definition 1. *Let $\mathcal{D}_1(G,i)$ be the family of edge dominating sets of a graph G with cardinality i and let $d_1(G,i) = |\mathcal{D}_1(G,i)|$. Then the edge domination polynomial $D_1(G,x)$ of G is defined as:*

$$D_1(G,x) = \sum_{i=\gamma_1(G)}^{|E(G)|} d_1(G,i)x^i,$$

where $\gamma_1(G)$ is the domination number of G.

For example, the complete graph K_n is the graph with n vertices in which each distinct pair are adjacent. Thus the graph K_4 has the edge domination polynomial:

$$D_1(K_4,x) = \sum_{i=2}^{6} d_1(K_4,i)x^i = 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6.$$

The following theorem is an easy consequence of the definition of the edge domination polynomial of a graph.

Theorem 1. *Let $G = (V, E)$ be a graph. Then:*
1. $d_1(G, i) = 0$ if and only if $i < \gamma_1(G)$ or $i > |E(G)|$, where i is a positive integer.
2. $D_1(G, x)$ has no constant term.
3. $D_1(G, x)$ is a strictly increasing function in $[0, +\infty)$.
4. Let G be a graph and H be any induced subgraph of G. Then $deg(D_1(G, x)) \geq deg(D_1(H, x))$.
5. Zero is a root of $D_1(G, x)$, with multiplicity $\gamma_1(G)$.

Theorem 2. *If a graph G consists of m components $G_1, ..., G_m$, then:*

$$D_1(G, x) = D_1(G_1, x)...D_1(G_m, x).$$

Proof. It suffices to prove when $m = 2$. For $k > \gamma_1(G)$, an edge dominating set of k edge in G arises by choosing an edge dominating set of j edge in G_1 (for some $j \in \{\gamma_1(G_1), \gamma_1(G_1) + 1, ..., |E(G_1)|\}$) and an edge dominating set of $k - j$ edge in G_2. The number of way of doing this over all $j = \gamma_1(G_1), ..., |E(G_1)|$ is exactly the coefficient of x^k in $D_1(G_1, x)D_1(G_2, x)$. Hence both side of the above equation have the same coefficient, so they are identical polynomial.

For example, The edge domination polynomial of the graph G given in Figure 1, is:

$$D_1(G, x) = D_1(K_4, x)D_1(K_4, x) = (15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6)^2 =$$

$$225x^4 + 600x^5 + 850x^6 + 780x^7 + 495x^8 + 220x^9 + 66x^{10} + 12x^{11} + x^{12}.$$

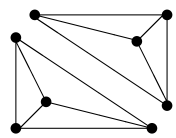

Figure 1

Lemma 1. *If F is an edge domination set of a graph G with size i and $\psi \in Aut(G)$, then $\psi(F)$ is also an edge domination set of G with size i.*
Proof. Since $\mathrm{N}[F] = E(G)$, so $\psi(\mathrm{N}[F]) = \psi(E(G)) = E(G)$. Therefore $\mathrm{N}[\psi(F)] = E(G)$, and the proof is complete.

Theorem 3. *Let $G = (V, E)$ be an edge transitive graph and $e \in E(G)$. Then* *for every* $1 \le i \le |E(G)|$, *we have* $d_1(G,i) = (|E(G)| \times d_1(G,e,i))/i$.

Proof. According to the previous lemma, if F is an edge dominating set of G with size i and $\psi \in \mathrm{Aut}(G)$, then $\psi(F)$ is also an edge domination set of G with size i. Since G is an edge transitive graph, so for every two edges e and e', $d_1(G,e,i) = d_1(G,e',i)$. Now if F is an edge dominating set of size i, then there are exactly i edges $e_{j_1}, ..., e_{j_i}$ such that F counted in $d_1(G, e_{j_r}, i)$, for each $1 \le r \le i$. Hence $d_1(G,i) = (|E(G)| \times d_1(G,e,i))/i$, and the proof is complete.

For example, consider the graph $K_{3,3}$ given in Figure 2, and we shall compute $d_1(K_{3,3}, 3)$:

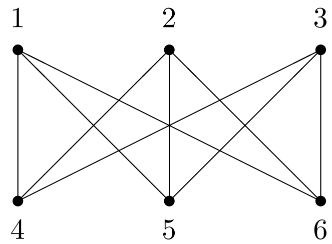

Figure 2

By Theorem 3, it suffices to obtain the edge dominating sets of cardinality 3 containing one edge, say the edge labeled 14. These edge dominating sets are listed below:
$\mathcal{D}_1(K_{3,3}, 14, 3) = \{\{14,25,26\},\{14,25,34\},\{14,25,35\},\{14,25,36\},$
$\{14,26,34\},\{14,26,35\},\{14,26,36\},\{14,35,36\},\{14,15,16\},$
$\{14,15,26\},\{14,15,36\},\{14,16,25\},\{14,16,35\},\{14,24,34\},$
$\{14,24,35\},\{14,24,36\}\}$.
Therefore by Theorem 3, $d_1(K_{3,3}, 3) = (9 \times 16)/3 = 48$.

Theorem 4. *Let* $G = (V,E)$ *be a cubic graph. Then the edge domination polynomial of* G *is:*

$$D_1(G,x) = \sum_{i=\gamma_1(G)}^{|E(G)|-6} d_1(G,i)x^i + (\binom{|E(G)|}{r} - |E(G)|)x^r + \sum_{i=r+1}^{|E(G)|} \binom{|E(G)|}{i} x^i,$$

where $r = |E(G)| - 5$.

Proof. Since G is a cubic graph, so for every $e \in E(G)$, $E(G) \setminus \mathrm{N}[e]$ is not an edge domination set of G with size r. This implies that the number of all subsets of $E(G)$ with size r which are not edge dominating sets is $|E(G)|$. Hence we have $d_1(G,r) = \binom{|E(G)|}{r} - |E(G)|$, and $d_1(G,i) = \binom{|E(G)|}{i}$ for any $r < i \le |E(G)|$.

3. Main Result

In this section, we show that the edge domination polynomial of a simple graph G is equal to the vertex domination polynomial of line graph $L(G)$ of G.

Theorem 5. *Let $G = (V, E)$ be a simple graph of order n. Then:*

$$D_1(G, x) = D_0(L(G), x).$$

Proof. By the definition of $L(G)$, we have a one to one corresponding $\alpha : E(G) \longrightarrow V(L(G))$. Let $F \subseteq E(G)$ be an edge domination set of G. It suffice to show that $\alpha(F)$ is a vertex domination set of $L(G)$.
Let v be a vertex in $V(L(G)) \setminus \alpha(F)$. Since $v \in V(L(G))$ and $v \notin \alpha(F)$, there is $e \in E(G)$ such that $v = \alpha(e) \in V(L(G))$ and $v = \alpha(e) \notin \alpha(F)$, therefore $e \in E(G) \setminus F$. Since F is an edge domination set, there is $e' \in F$ such that e and e' have a common vertex in G. Hence $\alpha(e)\alpha(e') \in E(L(G))$ and $\alpha(e') \in \alpha(F)$. This implies that $\alpha(F)$ is a vertex domination set of $L(G)$, and the proof is complete.

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