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LACUNARY STATISTICAL CONVERGENCE OF DIFFERENCE DOUBLE SEQUENCES

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Abstract. In this paper our purpose is to extend some results known in the literature for ordinary difference (single) to difference double sequences of real numbers.Quite recently, Esi [1] defined the statistical analogue for double difference sequences $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be P-statistically $\Delta - convergent$ to L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \{the\ number\ of\ \ (k,l) : k < m, l < n; |\Delta x_{k,l} - L| \geq \varepsilon\} = 0.$$

In this paper we introduce and study lacunary statistical convergence for difference double sequences and we shall also give some inclusion theorems.

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1.Introduction

Before we go into the motivation for this paper and presentation of the main results we give some preliminaries. A double sequence $x = (x_{k,l})$ has a *Pringsheim limit* L (denoted by $P - lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$.We shall describe such an $x = (x_{k,l})$ more briefly as "$P - convergent$" [2].The double sequence $x = (x_{k,l})$ is *bounded* if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l,

$$\|x\| = \sup_{k,l} |x_{k,l}| < \infty.$$

We should note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.The concept of statistical convergence was introduced by Fast [5] in 1951. A complex number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n} \frac{1}{n} |\{\ k \leq n :\ \ |x_k - L| \geq \varepsilon\}| = 0$$

where the vertical bars indicate the number of elements in the enclosed set. Later, Mursaleen and Edely [6] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be $P-$ *statistical convergence to* L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |\{ \ (k,l) : \ k < m, l < n; \ |x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $St_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all $P-$*statistical convergent double sequences* by St_2.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ *where* $k_o = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence space N_θ was defined by Freedman et.al. [7] as follows:

$$N_\theta = \left\{ x = (x_k) : \ \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$k_o = 0, \ h_r = k_r - k_{r-1} \to \infty \ \text{ as } r \to \infty$$

and

$$l_o = 0, \ \overline{h_s} = l_s - l_{s-1} \to \infty \ \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h_s}$ and $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : \ k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q_s} = \frac{l_s}{l_{s-1}} \ \textit{and} \ q_{r,s} = q_r \overline{q_s}. [3]$$

The set of all double lacunary sequences denoted by $N_{\theta_{r,s}}$ and defined by Savaş and Patterson [4] as follows:

$$N_{\theta_{r,s}} = \left\{ x = (x_{k,l}) : \ P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - L| = 0, \text{ for some L} \right\}.$$

2. Definitions and results

We begin with some definitions.

Definition 2.1. The double sequence $x = (x_{k,l})$ is $\Delta -$ *bounded* if there exists a positive number M such that $|\Delta x_{k,l}| < M$ for all k and l,

$$\|x\|_{\Delta} = \sup_{k,l} |\Delta x_{k,l}| < \infty.$$

Where $\Delta x_{k,l} = x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}$. We will denote the set of all bounded double difference sequences by $l_{\infty}^{n}(\Delta)$.

Definition 2.2.[1] A real double sequence $x = (x_{k,l})$ is said to be $P-$*statistical* $\Delta -$ *convergence to* L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |\{ \ (k,l) : \ k < m, l < n; \ |\Delta x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $St_{2,\Delta} - \lim_{k,l} x_{k,l} = L$ and we denote the set of all $P-$ *statistical* $\Delta -$ *convergent double sequences* by $St_{2,\Delta}$.

Definition 2.3. [1] The double sequence $x = (x_{k,l})$ is strong double difference Cesaro summable to L if

$$w_{\Delta}^{n} = \left\{ x = (x_{k,l}) : \ P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} |\Delta x_{k,l} - L| = 0, \text{ for some L} \in \mathbb{C} \right\}.$$

The class of all strongly double difference Cesaro summable sequences is denoted by w_{Δ}^{n}.

Definition 2.4. Let $\theta_{r,s}$ be a double lacunary sequence. The double number sequence $x = (x_{k,l})$ is $N_{\theta_{r,s},\Delta} - P -$ *convergent* to L provided that for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta x_{k,l} - L| = 0.$$

We will denote the set of all $N_{\theta_{r,s},\Delta} - P -$ *convergent* sequences by $N_{\theta_{r,s},\Delta}$.

We now consider the double difference lacunary statistical convergence.

Definition 2.5. Let $\theta_{r,s}$ be a double lacunary sequence. The double number sequence $x = (x_{k,l})$ is $S_{\theta_{r,s},\Delta} - P -$ *convergent* to L provided that for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \ |\Delta x_{k,l} - L| \geq \varepsilon\}| = 0.$$

We will denote the set of all $S_{\theta_{r,s},\Delta} - P -$ *convergent* sequences by $S_{\theta_{r,s},\Delta}$.

Theorem 2.1. Let $\theta_{r,s}$ be a double lacunary sequence.Then
(i) $N_{\theta_{r,s},\Delta} \subset S_{\theta_{r,s},\Delta}$ and the inclusion is strict,
(ii) If $x = (x_{k,l}) \in l_{\infty}^{n}(\Delta) \cap S_{\theta_{r,s},\Delta}$ then $x = (x_{k,l}) \in N_{\theta_{r,s},\Delta}$,
(iii) $l_{\infty}^{n}(\Delta) \cap S_{\theta_{r,s},\Delta} = l_{\infty}^{n}(\Delta) \cap N_{\theta_{r,s},\Delta}$.
Proof. *(i)* Since

$$\left|\{(k,l) \in I_{r,s} : \ |\Delta x_{k,l} - L| \geq \varepsilon\}\right| \leq \sum_{(k,l)\in I_{r,s} \ \& \ |\Delta x_{k,l} - L| \geq \varepsilon} |\Delta x_{k,l} - L|$$

$$\leq \sum_{(k,l)\in I_{r,s}} |\Delta x_{k,l} - L|$$

and so if $x = (x_{k,l}) \in N_{\theta_{r,s},\Delta}$ then we have $x = (x_{k,l}) \in S_{\theta_{r,s},\Delta}$. To show the inclusion is strict, we define $x = (x_{k,l})$ as follows:

$$\Delta x_{k,l} = \begin{pmatrix} 1 & 2 & 3 & ... & \left[\sqrt[3]{h_{r,s}}\right] & 0 & 0 & ... \\ 2 & 2 & 3 & ... & \left[\sqrt[3]{h_{r,s}}\right] & 0 & 0 & ... \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 2 & \left[\sqrt[3]{h_{r,s}}\right] & \left[\sqrt[3]{h_{r,s}}\right] & ... & \left[\sqrt[3]{h_{r,s}}\right] & 0 & 0 & ... \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & ... \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}.$$

It is clear that $x = (x_{k,l})$ is not Δ-bounded double sequence and for $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\{(k,l) \in I_{r,s} : \ |\Delta x_{k,l} - L| \geq \varepsilon\}\right| = P - \lim_{r,s} \frac{\left[\sqrt[3]{h_{r,s}}\right]}{h_{r,s}} = 0.$$

So $x = (x_{k,l}) \in S_{\theta_{r,s},\Delta}$. But

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta x_{k,l} - L| = P - \lim_{r,s} \frac{\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right] + 1\right)\right)}{2h_{r,s}} = \frac{1}{2}.$$

Therefore $x = (x_{k,l}) \notin N_{\theta_{r,s},\Delta}$. This completes the prof of (i).

(ii) Suppose that $x = (x_{k,l}) \in l_{\infty}^{n}(\Delta) \cap S_{\theta_{r,s},\Delta}$.Then $|\Delta x_{k,l}| < M$ for all k and l, also for given $\varepsilon > 0$ and sufficiently large r and s, we obtain the following

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta x_{k,l} - L| = \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s} \ \& \ |\Delta x_{k,l} - L| \geq \varepsilon} |\Delta x_{k,l} - L|$$

$$+\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}\ \&\ \left|\Delta x_{k,l}-L\right|<\varepsilon}\left|\Delta x_{k,l}-L\right|$$

$$\leq \frac{M}{h_{r,s}}\left|\{(k,l)\in I_{r,s}:\ \ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|+\varepsilon.$$

Therefore $x=(x_{k,l})\in l_{\infty}^{n}(\Delta)\cap S_{\theta_{r,s},\Delta}$ implies $x=(x_{k,l})\in N_{\theta_{r,s},\Delta}$.

(iii) It follows from (i) and (ii).

Theorem 2.2. Let $\theta_{r,s}$ be a double lacunary sequence.Then

(i) $St_{2,\Delta}\subset S_{\theta_{r,s},\Delta}$ if $\liminf q_r>1$ and $\liminf \overline{q_s}>1$.

(ii) $S_{\theta_{r,s},\Delta}\subset St_{2,\Delta}$ if $\liminf q_r<\infty$ and $\liminf \overline{q_s}<\infty$,

(iii) $St_{2,\Delta}=S_{\theta_{r,s},\Delta}$ if $1<\liminf q_r<\infty$ and $1<\liminf \overline{q_s}<\infty$

*Proof (i)***.** Suppose that $\liminf q_r>1$ and $\liminf \overline{q_s}>1$.Then there exists $\delta>0$ such that both $q_r>1+\delta$ and $\overline{q_s}>1+\delta$. This implies $\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}$ and $\frac{\overline{h_s}}{l_s}\geq\frac{\delta}{1+\delta}$. If $x=(x_{k,l})\in St_{2,\Delta}$ then for each $\varepsilon>0$ and for sufficiently large r and s we obtain the following:

$$\frac{1}{k_{r,s}}\left|\{(k,l)\in I_{r,s}:k\leq k_r \text{ and } l\leq l_s\ ;\ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|$$

$$\geq\frac{1}{k_{r,s}}\left|\{(k,l)\in I_{r,s}:\ \ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|=\frac{h_{r,s}}{k_{r,s}}\frac{1}{h_{r,s}}\left|\{(k,l)\in I_{r,s}:\ \ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|$$

$$\geq\left(\frac{\delta}{1+\delta}\right)^{2}\frac{1}{h_{r,s}}\left|\{(k,l)\in I_{r,s}:\ \ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|.$$

Therefore $x=(x_{k,l})\in S_{\theta_{r,s},\Delta}$.

(ii) Suppose that $\liminf q_r<\infty$ and $\liminf \overline{q_s}<\infty$, then there exists $K>0$ such that $q_r\leq K$, $\overline{q_s}\leq K$ for all r and s. Let $x=(x_{k,l})\in S_{\theta_{r,s},\Delta}$ and $N_{r,s}=\left|\{(k,l)\in I_{r,s}:\ \ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|$. So, given $\varepsilon>0$ there exists a positive integer r_o such that $\frac{N_{r,s}}{h_{r,s}}<\varepsilon$ for all $r,s>r_o$. Let $M=max\left\{N_{r,s}:\ 1\leq r,s\leq r_o\right\}$. Let m and n be such that $k_{r-1}<m\leq k_r$ and $l_{s-1}<n\leq l_s$. Therefore we obtain

$$\frac{1}{mn}\left|\{k\leq m \text{ and } l\leq n:\ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|$$

$$\leq\frac{1}{k_{r-1}l_{s-1}}\left|\{(k,l)\in I_{r,s}:k\leq k_r \text{ and } l\leq l_s\ ;\ |\Delta x_{k,l}-L|\geq\varepsilon\}\right|$$

$$=\frac{1}{k_{r-1}l_{s-1}}\sum_{i,j=1,1}^{r,s}N_{i,j}$$

$$\leq\frac{Mr_o^2}{k_{r-1}l_{s-1}}+\frac{1}{k_{r-1}l_{s-1}}\sum_{i,j=r_o+1,r_o+1}^{r,s}N_{i,j}$$

$$= \frac{Mr_o^2}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \sum_{i,j=r_o+1,r_o+1}^{r,s} N_{i,j} \frac{h_{i,j}}{h_{i,j}}$$

$$\leq \frac{Mr_o^2}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \left(\sup_{i,j \geq r_o,r_o} \frac{N_{i,j}}{h_{i,j}} \right) \left(\sum_{i,j=r_o+1,r_o+1}^{r,s} h_{i,j} \right)$$

$$\leq \frac{Mr_o^2}{k_{r-1}l_{s-1}} + \varepsilon \sum_{i,j=r_o+1,r_o+1}^{r,s} h_{i,j} \leq \frac{Mr_o^2}{k_{r-1}l_{s-1}} + \varepsilon K^2.$$

The result follows immediately.

(iii) Combining (i) and (ii) we have the proof of (iii).

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