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HYERS-ULAM-RASSIAS STABILITY OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS VIA A FIXED POINT APPROACH

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Abstract. By using a fixed point method, we establish the Hyers–Ulam stability and the Hyers–Ulam–Rassias stability for a general class of nonlinear Volterra integral equations in Banach spaces.

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1. Introduction

1.1. The stability theory for functional equations started with a problem related to the stability of group homomorphisms that was considered by S.M. Ulam in 1940 (see [33] and [34]).

Ulam considered the following question:

Let G_1 be a group and let G_2 be a group endowed with a metric d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta,$$

for all $x, y \in G_1$, then we can find a homomorphism $\theta : G_1 \to G_2$ such that

$$d(h(x), \theta(x)) < \epsilon,$$

for all $x \in G_1$?

An affirmative answer to this equation was given by D. H. Hyers (see [9]) for the case of Banach spaces. This answer, in this case, says that the Cauchy functional equation is stable in the sens of Heyers-Ulam.

In 1950, T. Aoki (see [4]) was the second author to treat this problem for additive mappings (see also [5]).

In 1978, Th. M. Rassias [25] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. In [25], Th. M. Rassias has introduced a new type of stability which is called the Hyers–Ulam–Rassias stability.
The result obtained by Th. M. Rassias (see [25]) reads as follows.

Theorem 1.1. *Consider E, F to be two Banach spaces, and let $f : E \to F$ be a mapping such that the function $t \mapsto f(tx)$ from $\mathbf{R}$ into F is continuous for each fixed $x \in E$. Assume that there exists $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \quad \forall x, y \in E. \tag{1.1}$$

Then there exists a unique additive mapping $T : E \longrightarrow F$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p, \tag{1.2}$$

for any $x \in E$.

We point out that the results of D. H. Hyers and Th. M. Rassias have been generalized in several ways to other settings. For instance, several authors have studied the stability for differential equations (see [3], [14], [15], [16], [17], [19], [20], [21], [22], [32] and other papers).

In [1] and [2], M. Akkouchi and Elqorachi have studied the stability of the Cauchy and Wilson equations and the generalized Cauchy and Wilson equations by using tools from Harmonic analysis.

Research in Stability theory is now very extensive and many papers and books have been published (for more details, see [11], [26], [27], [28], [29]).

1.2. The purpose of this paper is to investigate the stability for a class of nonlinear Volterra integral equations under some natural conditions.

Let X be a (real or complex) normed space over the (real or complex) field $\mathbf{K}$ and let $I = [a, b]$ be a closed and bounded interval.
Let $G : I \times I \times X \to X$ be a continuous mapping. Let $\lambda \in \mathbf{K}$ and let $h : I \to X$ be a mapping. We consider the nonlinear Volterra integral equation (of second kind) given by

$$f(x) = h(x) + \lambda \int_a^x G(x, y, f(y))dy, \quad \forall x \in I, \tag{1.3}$$

where $f : I \to X$ is unknown function. The set of solutions will be the Banach space $\mathcal{C}(I, X)$ of continuous functions from I to X.

We say that the integral equation (1.1) has the Hyers-Ulam stability, if for all $\epsilon > 0$ and all function $f : I \to X$ satisfying the inequality

$$\|f(x) - h(x) - \lambda \int_a^x G(x, y, f(y))dy\| \le \epsilon \quad \forall x \in I, \tag{1.4}$$

there exists a solution $g : I \to X$ of the Volterra integral equation

$$g(x) = h(x) + \lambda \int_a^x G(x, y, g(y))dy, \quad \forall x \in I, \tag{1.5}$$

such that

$$\|f(x) - g(x)\| \le \delta(\epsilon) \quad \forall x \in I, \tag{1.6}$$

where $\delta(\epsilon)$ is an expression of ϵ only. If the above statement is also true when we replace ϵ and $\delta(\epsilon)$ by $\phi(x)$ and $\Phi(x)$, where $\phi, \Phi : I \to [0, \infty)$ are functions not depending on f and g explicitly, then we say that the corresponding Volterra integral equation has the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability).

2. Preliminaries

For a nonempty set X, we recall the definition of the generalized metric on X. A function $d : X \times X \to [0, +\infty]$ is called a generalized metric on X if and only if d satisfies

(M1) $d(x, y) = 0$ if and only if $x = y$,
(M2) $d(x, y) = (y, x)$ for all $x, y \in X$,
(M3) $d(x, z) \le (x, y) + (y, z)$ for all $x, y, z \in X$.

We observe that the only one difference of the generalized metric from the usual metric is that the range of the former is allowed to include the infinity.

We now recall one of fundamental results of fixed point theory. For the proof, we refer to [7].

Theorem 2.1. (The alternative of fixed point) *Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping $\Lambda : X \to X$, with the Lipschitz constant L.*

Then, for each given point $x \in X$, either

$(A_1) : d(\Lambda^n x, \Lambda^{n+1} x) = +\infty, \quad \forall n \ge 0,$

or

(A_2) : *there exists a nonnegative integer k_0 such that:*

(i) $d(\Lambda^n x, \Lambda^{n+1} x) < \infty$ *for all natural number* $n \geq k_0$.
(ii) The sequence $\{\Lambda^n x\}$ *converges to a fixed point* y_* *of* Λ*;*
(iii) y_* *is the unique fixed point of* Λ *in the set*

$$Y = \{y \in X : d(\Lambda^{k_0} x, y) < \infty\};$$

(iv) If $y \in Y$*, then*

$$d(y, y_*) \leq \frac{1}{1-L} d(\Lambda y, y).$$

V. Radu [24] and L. Cădariu and V. Radu [6] have used this alternative fixed point theorem to study the stability for the Cauchy functional equation and the Jensen functional equation; and they present proofs for their Hyers-Ulam-Rassias stability. By their work, they unified the results of Hyers, Rassias and Gajda [8]. We point out that the stability of these equations have been studied by S.-M. Jung [13], W. Jian [12] and other authors.

Subsequently, certain authors have adopted fixed point methods to study the stability of some functional equations.

In a recent paper, S.-M. Jung in [17] has used the fixed point approach to prove the stability of ceratin differential equations of first order.

The aim of this paper is to use Theorem 2.1 above to establish the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the class of nonlinear Volterra integral equations in Banach spaces defined by (1.3).

3. Hyers-Ulam stability

In this section, we establish the Hyers-Ulam stability of the nonlinear Volterra integral equation (1.3) under some natural conditions. Our first main result reads as follows.

Theorem 3.1. *Let* a, b *are given real numbers such that* $a < b$ *and set* $I := [a, b]$*. Let* X *be a Banach space over the (real or complex) field* $\mathbf{K}$*. Let* $\lambda \in \mathbf{K}$*. Let* L *be a positive constant with* $0 < L|\lambda|(b-a) < 1$*. Assume that* $G : I \times I \times X \to X$ *is a continuous function which satisfies the following Lipschitz condition:*

$$\|G(t, s, y) - G(t, s, z)\| \leq L\|y - z\|, \quad \forall t, s \in I, \forall y, z \in X. \tag{3.1}$$

Suppose that a continuous function $f : I \to X$ *satisfies*

$$\|f(t) - h(t) - \lambda \int_a^t G(t, s, f(s)) ds\| \leq \epsilon, \quad \forall t \in I, \tag{3.2}$$

for some positive number ϵ.

Then there exists a unique continuous function $g_0 : I \to X$ *such that*

$$g_0(t) = h(t) + \lambda \int_a^t G(t, s, g_0(s))ds, \quad \forall t \in I, \tag{3.3}$$

(consequently, g_0 *is a solution to the equation (1.1)) and*

$$\|f(t) - g_0(t)\| \leq \frac{\epsilon}{1 - L|\lambda|(b-a)}, \tag{3.4}$$

for all $t \in [a, b]$.

Proof. Le $E := \mathcal{C}(I, X)$ be the set of all continuous functions from I to X. For $f, g \in E$, we set

$$d(f, g) := \inf\{C \in [0, \infty] : \|f(t) - g(t)\| \leq C, \forall t \in I\}. \tag{3.5}$$

It is easy to see that (E, d) is a complete generalized metric space.

Now, consider the operator $\Lambda : E \to E$ defined by

$$(\Lambda f)(t) := h(t) + \lambda \int_a^t G(t, s, f(s))ds, \quad \forall t \in I. \tag{3.6}$$

We prove that Λ is strictly contractive on the space E. Let $f, g \in E$ and let $C(f, g) \in [0, \infty]$ be an arbitrary constant such that $d(f, g) \leq C(f, g)$. Then, by 3.6, we have

$$\|f(t) - g(t)\| \leq C(f, g), \forall t \in I. \tag{3.7}$$

For any $t \in I$, we have

$$\begin{aligned}
\|(\Lambda f)(t) - (\Lambda g)(t)\| &= |\lambda| \left\| \int_a^t (G(t, s, f(s)) - G(t, s, g(s)))ds \right\| \\
&\leq |\lambda| \int_a^t \|G(t, s, f(s)) - G(t, s, g(s))\|ds \\
&\leq |\lambda| L \int_a^t \|f(s) - g(s)\|ds \\
&\leq |\lambda| L C(f, g)(t - a) \\
&\leq |\lambda| L C(f, g)(b - a),
\end{aligned}$$

for all $t \in I$. Hence we have $d(\Lambda f, \Lambda g) \leq |\lambda| L(b - a)C(f, g)$. We conclude that

$$d(\Lambda f, \Lambda g) \leq |\lambda| L(b - a)d(f, g), \quad \forall f, g \in E. \tag{3.9}$$

Since, by assumption, we have $|\lambda|L(b-a) < 1$, then Λ is strictly contractive.

Let g be any arbitrary element in E. By continuity of the mappings $h, g, \Lambda g$ and ϕ on the compact set I, there exists a constant $C \in (0, \infty)$ such that

$$\|\Lambda g(t) - g(t)\| = \left\| h(t) + \lambda \int_a^t G(t, s, g(s))ds - g(t) \right\| \leq C,$$

for all $t \in I$.

We deduce that

$$d(g, \Lambda g) < +\infty, \quad \forall g \in E.$$

Let $f_0 \in E$ be given, then by virtue of Theorem 2.1, there exists a continuous function g_0 in E such that the sequence $\{\Lambda^n f_0\}$ converges to g_0 and $\Lambda g_0 = g_0$, that is g_0 is a solution to the equation (1.3).

We observe that d is actually a metric. Therefore, $g_0 : I \to X$ is the unique continuous function such that

$$g_0(x) = h(x) + \lambda \int_a^x G(x, y, g_0(y))dy, \quad \forall x \in I.$$

By assumption (3.2), we deduce that $d(f, \Lambda f) \leq \epsilon$, thus by virtue of (iv) of Theorem 2.1, we get the following estimate

$$d(f, g_0) \leq \frac{\epsilon}{1 - |\lambda|L(b-a)},$$

which implies that

$$\|f(t) - g_0(t)\| \leq \frac{\epsilon}{1 - |\lambda|L(b-a)}.$$

Also, by (ii) of Theorem 2.1, the sequence of iterates $\{\Lambda^n f\}$ converges to g_0 in the metric space (E, d). This completes the proof.

4. Hyers-Ulam-Rassias stability of Volterra integral equations

In this section, by using the fixed point method, we will study the Hyers–Ulam–Rassias stability of the Volterra integral equation (1.3).

Theorem 4.1. *Let a, b are given real numbers such that $a < b$ and set $I := [a, b]$. Let X be a Banach space over the (real or complex) field $\mathbf{K}$. Let $\lambda \in \mathbf{K}$. Let K, L be positive constants with $0 < |\lambda|KL < 1$. Let $\phi : I \to (0, \infty)$ be a continuous function such that*

$$\int_a^x \phi(y)dy \leq K\phi(x), \quad \forall x \in [a, b]. \tag{4.1}$$

Assume that $G : I\times I\times X \to X$ *is a continuous function which satisfies the following Lipschitz condition:*

$$\|G(t,s,y) - G(t,s,z)\| \le L\|y-z\|, \quad \forall t,s \in I,\ \forall y,z \in X. \tag{4.2}$$

Suppose that a continuous function $f : I \to X$ *satisfies*

$$\|f(t) - h(t) - \lambda \int_a^t G(t,s,f(s))ds\| \le \phi(t), \quad \forall t \in I. \tag{4.3}$$

Then there exists a unique continuous function $g_0 : I \to X$ *such that*

$$g_0(t) = h(t) + \lambda \int_a^t G(t,s,g_0(s))ds, \quad \forall t \in I, \tag{4.4}$$

(consequently, g_0 *is a solution to the equation (1.1)) and*

$$\|f(t) - g_0(t)\| \le \frac{1}{1-|\lambda|KL}\phi(t), \tag{4.5}$$

for all $t \in [a,b]$.

Proof. We consider the set $E := \mathcal{C}(I,X)$ of all continuous functions from I to X. For $f,g \in E$, we set

$$d(f,g) := \inf\{C \in [0,\infty] : \|f(t)-g(t)\| \le C\phi(t),\ \forall t \in I\}. \tag{4.6}$$

It is easy to see that (E,d) is a generalized metric space. Also, it is easy to see that (E,d) is complete.

Now, consider the operator $\Lambda : E \to E$ defined by

$$(\Lambda f)(t) := h(t) + \lambda \int_a^t G(t,s,f(s))ds, \quad \forall t \in I. \tag{4.7}$$

We prove that Λ is strictly contractive on the space E. Let $f,g \in E$ and let $C(f,g) \in [0,\infty]$ be an aribitrary constant such that $d(f,g) \le C(f,g)$. Then, by (4.6), we have

$$\|f(t) - g(t)\| \le C(f,g)\phi(t),\ \forall t \in I. \tag{4.8}$$

For any $t \in I$, we have

$$\|(\Lambda f)(t) - (\Lambda g)(t)\| = |\lambda| \left\| \int_a^t (G(t,s,f(s)) - G(t,s,g(s)))ds \right\|$$

$$\le |\lambda| \int_a^t \|G(t,s,f(s)) - G(t,s,g(s))\|ds$$

$$\leq |\lambda| L \int_a^t \|f(s) - g(s)\| ds$$

$$\leq |\lambda| LC(f,g) \int_a^t \phi(s) ds \leq |\lambda| LC(f,g) K\phi(t),$$

for all $t \in I$. Hence we have $d(\Lambda f, \Lambda g) \leq |\lambda| KLC(f,g)$. We conclude that

$$d(\Lambda f, \Lambda g) \leq |\lambda| KLd(f,g), \quad \forall f, g \in E. \tag{4.9}$$

Since, by assumption, we have $|\lambda| KL < 1$, then Λ is strictly contractive.

Let g be any arbitrary element in E. Since $\phi(I) \subset (0, +\infty)$, then by continuity of the mappings $h, g, \Lambda g$ and ϕ on the compact set I, there exists a finite constant $C \in (0, \infty)$ such that

$$\|\Lambda g(t) - g(t)\| = \left\| h(t) + \lambda \int_a^t G(t, s, g(s)) ds - g(t) \right\| \leq C\phi(t)$$

for all $t \in I$.

We deduce that $d(g, \Lambda g) < +\infty, \quad \forall g \in E$. Let $f_0 \in E$ be given, then by virtue of Theorem 2.1, there exists a continuous function g_0 in E such that the sequence $\{\Lambda^n f_0\}$ converges to g_0 and $\Lambda g_0 = g_0$, that is g_0 is a solution to the equation (1.3).

Since $\min\{\phi(t) : t \in I\} > 0$, then d is actually a metric. Therefore, $g_0 : I \to X$ is the unique continuous function such that

$$g_0(x) = h(x) + \lambda \int_a^x G(x, y, g_0(y)) dy, \quad \forall x \in I.$$

By assumption (4.3), we know that $d(f, \Lambda f) \leq 1$, thus by virtue of (iv) of Theorem 2.1, we get the following estimate

$$d(f, g_0) \leq \frac{1}{1 - |\lambda| KL},$$

which implies that

$$\|f(t) - g_0(t)\| \leq \frac{1}{1 - |\lambda| KL} \phi(t).$$

Also, by (ii) of Theorem 2.1, the sequence of iterates $\{\Lambda^n f\}$ converges to g_0 in the (generalized) metric space (E, d). This completes the proof.

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