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INTEGRAL REPRESENTATION FORMULA FOR MAXIMAL SURFACES IN THE GROUP OF RIGID MOTIONS $E(1,1)$

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Abstract. In this paper, we describe a method to derive a Weierstrass-type representation formula for simply connected immersed maximal surfaces in $E(1,1)$. We consider the left invariant metric and use some results of Levi-Civita connection. Furthermore, we show that any harmonic map of a simply connected coordinate region D into $E(1,1)$ can be represented a form.

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1. Introduction

For the first time Weierstrass representation for conformal immersion of surface into $\mathbb{R}^3$ appeared in the result of variational problem on search of minimal surface restricted by the some curve [19]. Generalization of Weierstrass formulae for surfaces with mean curvature $H \neq 0$ was proposed by Eisenhart in 1909 [7].

It has been shown [12] that Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [19]. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory [8], chemical physics, fluid dynamics and membranes [16], minimal surfaces play an essential role. More recently it is worth mentioning that works by Kenmotsu [10], Hoffmann [9], Osserman [15], Budinich [5], Konopelchenko [6,11] and Bobenko [3, 4] have made very significant contributions to constructing minimal surfaces in a systematic way and to understanding their intrinsic geometric properties as well as their integrable dynamics. The type of extension of the Weierstrass representation which has been useful in three-dimensional applications to multidimensional spaces will continue to generate many additional applications to physics and mathematics. According to [13] integrable deformations of surfaces are generated by the Davey–Stewartson hierarchy of 2+1 dimensional soliton equations. These deformations of surfaces inherit all the remarkable properties of soliton equations. Geometrically such deformations are characterised by the invariance of an infinite set of functionals over surfaces, the simplest being the Willmore functional.

Direct approaches to describe surfaces always have been of great interest. The classical Weierstrass formulae for minimal surfaces immersed in the three-dimensional Euclidean space $\mathbb{R}^3$ is the best known example of such an approach. Recently the Weierstrass formulae have been generalized to the case of generic surfaces in $\mathbb{R}^3$. During the last two years the generalized Weierstrass formulae have been used intensively to study both global properties of surfaces in $\mathbb{R}^3$ and their integrable deformations. Analytic methods to study surfaces and their properties are of great interest both in mathematics and in physics. A classical example of such an approach is given by the Weierstrass representation for minimal surfaces [7]. This representation allows us to construct any minimal surface in the three-dimensional Euclidean space $\mathbb{R}^3$ via two holomorphic functions. It is the most powerful tool for the analysis of minimal surfaces.

D. A. Berdinski and I. A. Taimanov gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors and Dirac operators [1].

In this paper, we describe a method to derive a Weierstrass-type representation formula for simply connected immersed minimal surfaces in $E(1,1)$. We consider the left invariant metric and use some results of Levi-Civita connection. Furthermore, we show that any harmonic map of a simply connected coordinate region D into $E(1,1)$ can be represented a form.

2. Preliminaries

Let $E(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $E(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$E(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x, y, z)\,,$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x},\ \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z},\ \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3, \ [\mathbf{X}_2, \mathbf{X}_3] = 0, \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x, \ x^2 = \frac{1}{2}(y+z), \ x^3 = \frac{1}{2}(y-z).$$

Then,

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1}, \ \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right), \ \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right). \quad (2.1)$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3, \ [\mathbf{X}_2, \mathbf{X}_3] = 0, \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2. \quad (2.2)$$

We consider left-invariant Lorentzian metricswhich has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric, given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2. \quad (2.3)$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1, \ \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3, \ \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}_3 & \mathbf{0} & -\mathbf{X}_1 \\ -\mathbf{X}_2 & -\mathbf{X}_1 & \mathbf{0} \end{pmatrix}, \quad (2.4)$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{X}_i}\mathbf{X}_j$ for our basis

$$\{\mathbf{X}_k, k = 1,2,3\} = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}.$$

Its Ricci tensor vanishes except $R_{11} = -2$. Obviously, the Lorentzian metric g is not Einstein [14].

3.Integral Representation Formula in E(1,1)

In this section, we obtain an integral representation formula for spacelike maximal surfaces in $E(1,1)$.

We denote with $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a simply connected domain with a complex coordinate $z = u + iv$, $u, v \in \mathbb{R}$. Also we will use the standard notations for complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{3.1}$$

For X, denote by $\mathsf{ad}(X)^*$ the adjoint operator of $\mathsf{ad}(X)$, i.e., it satisfies the equation

$$g\left([X,Y], Z\right) = g\left(Y, \mathsf{ad}(X)^*(Z)\right), \tag{3.2}$$

for any Y, Z. Let U be the symmetric bilinear operator on Lie algebra of $E(2)$ defined by

$$U(X,Y) := \frac{1}{2}\left\{\mathsf{ad}(X)^*(Y) + \mathsf{ad}(Y)^*(X)\right\}. \tag{3.3}$$

Lemma 3.1. *Let $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ be the orthonormal basis for an orthonormal basis for Lie algebra of $E(1,1)$ defined in (2.3). Then*

$$\begin{aligned}
U(\mathbf{X}_1, \mathbf{X}_3) &= U(\mathbf{X}_3, \mathbf{X}_1) = \frac{1}{2}\mathbf{X}_2, \\
U(\mathbf{X}_2, \mathbf{X}_1) &= U(\mathbf{X}_1, \mathbf{X}_2) = \frac{1}{2}\mathbf{X}_3, \\
U(\mathbf{X}_3, \mathbf{X}_2) &= U(\mathbf{X}_2, \mathbf{X}_3) = -\mathbf{X}_1, \\
U(\mathbf{X}_1, \mathbf{X}_1) &= U(\mathbf{X}_3, \mathbf{X}_3) = U(\mathbf{X}_2, \mathbf{X}_2) = 0.
\end{aligned}$$

Proof. Using (3.2) and (3.3), we have

$$2g\left(U(X,Y), Z\right) = g\left([X,Z], Y\right) + g\left([Y,Z], X\right).$$

Thus, direct computations lead to the table of U above. Lemma 3.1 is proved.

Lemma 3.2. *(see [10]) Let D be a simply connected domain. A smooth map $\varphi : D \longrightarrow E(1.1)$ is harmonic if and only if*

$$\left(\varphi^{-1}\varphi_u\right)_u + \left(\varphi^{-1}\varphi_v\right)_v - \mathsf{ad}\left(\varphi^{-1}\varphi_u\right)^*\left(\varphi^{-1}\varphi_u\right) - \mathsf{ad}\left(\varphi^{-1}\varphi_v\right)^*\left(\varphi^{-1}\varphi_v\right) = 0 \tag{3.4}$$

holds.

Let $z = u + iv$. Then in terms of complex coordinates z, $\bar{z}$, the harmonic map equation (3.4) can be written as

$$\frac{\partial}{\partial \bar{z}}\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) - 2U\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}, \varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) = 0. \tag{3.5}$$

Let $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$. Then, (3.5) is equivalent to

$$A_{\bar{z}} + \bar{A}_z = 2U\left(A, \bar{A}\right). \tag{3.6}$$

The Maurer–Cartan equation is given by

$$A_{\bar{z}} - \bar{A}_z = \left[A, \bar{A}\right]. \tag{3.7}$$

(3.6) and (3.7) can be combined to a single equation

$$A_{\bar{z}} = U\left(A, \bar{A}\right) + \frac{1}{2}\left[A, \bar{A}\right]. \tag{3.8}$$

(3.8) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$ and the condition for φ to be a harmonic map.

Let $D(z, \bar{z})$ be a simply connected domain and $\varphi : D \longrightarrow E(1,1)$ a smooth map. If we write $\varphi(z) = \left(x^1(z), x^2(z), x^3(z)\right)$, then by direct calculation

$$A = -x^1_z \mathbf{X}_1 + \left[e^{-x^1}x^2_z + e^{x^1}x^3_z\right]\mathbf{X}_2 + \left[e^{-x^1}x^2_z - e^{x^1}x^3_z\right]\mathbf{X}_3. \tag{3.9}$$

Theorem 3.3. $\varphi : D \longrightarrow E(1,1)$ *is harmonic if and only if the following equations hold:*

$$\begin{aligned} -x^1_{z\bar{z}} &+ \left[e^{-x^1}x^2_z + e^{x^1}x^3_z\right]\left[e^{-x^1}x^2_{\bar{z}} - e^{x^1}x^3_{\bar{z}}\right] \\ &+ \left[e^{-x^1}x^2_z - e^{x^1}x^3_z\right]\left[e^{-x^1}x^2_{\bar{z}} + e^{x^1}x^3_{\bar{z}}\right] = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\left[-x^1_{\bar{z}}e^{-x^1}x^2_z + e^{-x^1}x^2_{z\bar{z}} + x^1_{\bar{z}}e^{x^1}x^3_z + e^{x^1}x^3_{z\bar{z}}\right] + \left[-x^1_z e^{-x^1}x^2_{\bar{z}} + e^{-x^1}x^2_{\bar{z}z} + x^1_z e^{x^1}x^3_{\bar{z}}\right. \\ &\left. + e^{x^1}x^3_{\bar{z}z}\right] + x^1_z\left[e^{-x^1}x^2_{\bar{z}} - e^{x^1}x^3_{\bar{z}}\right] + x^1_{\bar{z}}\left[e^{-x^1}x^2_z - e^{x^1}x^3_z\right] = 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\left[-x^1_{\bar{z}}e^{-x^1}x^2_z + e^{-x^1}x^2_{z\bar{z}} - x^1_{\bar{z}}e^{x^1}x^3_z - e^{x^1}x^3_{z\bar{z}}\right] + \left[-x^1_z e^{-x^1}x^2_{\bar{z}} + e^{-x^1}x^2_{\bar{z}z} - x^1_z e^{x^1}x^3_{\bar{z}}\right. \\ &\left. - e^{x^1}x^3_{\bar{z}z}\right] + x^1_z\left[e^{-x^1}x^2_{\bar{z}} + e^{x^1}x^3_{\bar{z}}\right] + x^1_{\bar{z}}\left[e^{-x^1}x^2_z + e^{x^1}x^3_z\right] = 0. \end{aligned} \tag{3.12}$$

Proof. From (3.9), we have

$$\bar{A} = -x^1_{\bar{z}}\mathbf{X}_1 + \left[e^{-x^1}x^2_{\bar{z}} + e^{x^1}x^3_{\bar{z}}\right]\mathbf{X}_2 + \left[e^{-x^1}x^2_{\bar{z}} - e^{x^1}x^3_{\bar{z}}\right]\mathbf{X}_3. \tag{3.13}$$

Using (3.9) and (3.13), we obtain

$$\begin{aligned} U\left(A,\bar{A}\right) &= -\frac{1}{2}x^1_z\left[e^{-x^1}x^2_{\bar{z}} + e^{x^1}x^3_{\bar{z}}\right]\mathbf{X}_3 - \frac{1}{2}x^1_z\left[e^{-x^1}x^2_{\bar{z}} - e^{x^1}x^3_{\bar{z}}\right]\mathbf{X}_2 \\ &\quad -\frac{1}{2}x^1_{\bar{z}}\left[e^{-x^1}x^2_z + e^{x^1}x^3_z\right]\mathbf{X}_3 - \left[e^{-x^1}x^2_z + e^{x^1}x^3_z\right]\left[e^{-x^1}x^2_{\bar{z}} - e^{x^1}x^3_{\bar{z}}\right]\mathbf{X}_1 \\ &\quad -\frac{1}{2}x^1_{\bar{z}}\left[e^{-x^1}x^2_z - e^{x^1}x^3_z\right]\mathbf{X}_2 - \left[e^{-x^1}x^2_z - e^{x^1}x^3_z\right]\left[e^{-x^1}x^2_{\bar{z}} + e^{x^1}x^3_{\bar{z}}\right]\mathbf{X}_1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} A_{\bar{z}} &= -x^1_{z\bar{z}}\mathbf{X}_1 + \left[-x^1_{\bar{z}}e^{-x^1}x^2_z + e^{-x^1}x^2_{z\bar{z}} + x^1_{\bar{z}}e^{x^1}x^3_z + e^{x^1}x^3_{z\bar{z}}\right]\mathbf{X}_2 \\ &\quad + \left[-x^1_{\bar{z}}e^{-x^1}x^2_z + e^{-x^1}x^2_{z\bar{z}} - -x^1_{\bar{z}}e^{x^1}x^3_z - e^{x^1}x^3_{z\bar{z}}\right]\mathbf{X}_3, \end{aligned}$$

$$\begin{aligned} \bar{A}_z &= -x^1_{\bar{z}z}\mathbf{X}_1 + \left[-x^1_z e^{-x^1}x^2_{\bar{z}} + e^{-x^1}x^2_{\bar{z}z} + x^1_z e^{x^1}x^3_{\bar{z}} + e^{x^1}x^3_{\bar{z}z}\right]\mathbf{X}_2 \\ &\quad + \left[-x^1_z e^{-x^1}x^2_{\bar{z}} + e^{-x^1}x^2_{\bar{z}z} - x^1_z e^{x^1}x^3_{\bar{z}} - e^{x^1}x^3_{\bar{z}z}\right]\mathbf{X}_3. \end{aligned}$$

Hence, using (3.6) we obtain (3.10)-(3.12). This completes the proof of the Theorem.

The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial}, \quad \partial = \frac{\partial}{\partial z}dz, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}d\bar{z}, \tag{3.14}$$

with respect to the conformal structure of D.

Let

$$\begin{gathered} \wp^1 = x^1_z dz, \quad \wp^2 = \left[e^{-x^1}x^2_z + e^{x^1}x^3_z\right]dz, \\ \wp^3 = \left[e^{-x^1}x^2_z - e^{x^1}x^3_z\right]dz. \end{gathered} \tag{3.15}$$

Theorem 3.4. *The triplet* $\{\wp^1, \wp^2, \wp^3\}$ *of* $(1,0)$*-forms satisfies the following differential system:*

$$\bar{\partial}\wp^1 = \wp^2 \wedge \overline{\wp^3} + \overline{\wp^2} \wedge \wp^3, \tag{3.16}$$

$$\bar{\partial}\wp^2 + \partial\overline{\wp^2} = -\wp^1 \wedge \overline{\wp^3} - \overline{\wp^1} \wedge \wp^3, \tag{3.17}$$

$$\bar{\partial}\wp^3 + \partial\overline{\wp^3} = -\wp^1 \wedge \overline{\wp^2} - \overline{\wp^1} \wedge \wp^2. \tag{3.18}$$

Theorem 3.5. *Let $\{\wp^1, \wp^2, \wp^3\}$ be a solution to (3.16)-(3.18) on a simply connected coordinate region D. Then*

$$\varphi(z, \bar{z}) = 2\mathrm{Re} \int_{z_0}^{z} \left(\wp^1, \left(\wp^2 + \wp^3\right) e^{x^1}, \left(\wp^2 - \wp^3\right) e^{-x^1}\right) \tag{3.19}$$

is a harmonic map into $E(1,1)$.

Conversely, any harmonic map of D into $E(1,1)$ can be represented in this form.

Proof. By theorem (3.3) we see that $\varphi(z, \bar{z})$ is a harmonic curve if and only if $\varphi(z, \bar{z})$ satisfy (3.10)-(3.12).

From (3.15), we have

$$x^1(z, \bar{z}) = 2\mathrm{Re} \int_{z_0}^{z} \wp^1, \; x^2(z, \bar{z}) = 2\mathrm{Re} \int_{z_0}^{z} \left(\wp^2 + \wp^3\right) e^{x^1},$$

$$x^3(z, \bar{z}) = 2\mathrm{Re} \int_{z_0}^{z} \left(\wp^2 - \wp^3\right) e^{-x^1},$$

which proves the theorem.

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