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ON THE LOCALIZATION OF FACTORED FOURIER SERIES

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ABSTRACT. In this paper, a general theorem dealing with the local property of $\mid \bar{N}, p_n, \theta_n \mid_k$ summability of factored Fourier series has been proved , which generalizes some known results.

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1.INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k, k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{3}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{4}$$

In the special case $p_n = 1$ for all values of n, $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability. Also, if we take $k = 1$ and $p_n = 1/(n+1)$, then summability $\mid \bar{N}, p_n \mid_k$ is equivalent to the summability $\mid R, \log n, 1 \mid$. Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty. \tag{5}$$

If we take $\theta_n = \frac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $\mid C, 1 \mid_k$ summability.

Furthermore, if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ (see [4]) summability. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{6}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (b_n \cos nt + c_n \sin nt) = \sum_{n=1}^{\infty} A_n(t), \tag{7}$$

where (b_n) and (c_n) denote the Fourier coefficients. It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function f(t) (i. e., it depends only on the behaviour of f in a arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f(t) (see [10]).

2.Known result

Mohanty [8] has demonstrated that the summability $\mid R, \log n, 1 \mid$ of

$$\sum A_n(t)/\log(n+1), \tag{8}$$

at $t = x$, is a local property of the generating function of $\sum A_n(t)$. Later on Matsumoto [7] improved this result by replacing the series (8) by

$$\sum A_n(t)/\log\log(n+1)^{1+\epsilon}, \epsilon > 0. \tag{9}$$

Generalizing the above result Bhatt [1] proved the following theorem.

Theorem A. If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $\mid R, \log n, 1 \mid$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.
Bor [4] has proved Theorem A in a more general form which includes of the above results as special cases. Also it should be noted that the conditions on the sequence (λ_n) in that theorem, are somewhat more general than in Theorem A. His theorem is as follows.

Theorem B. Let $k \geq 1$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, then the summability $\mid \bar{N}, p_n \mid_k$ of the series $\sum A_n(t)\lambda_n P_n$ at a point is a local property of the generating function $f(t)$.

3.The main result

The aim of the present paper is to generalize Theorem B for $\mid \bar{N}, p_n, \theta_n \mid_k$ summability under suitable conditions. We shall prove the following theorem.

Theorem . Let $k \geq 1$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent and (θ_n) is any sequence of positive constants such that

$$\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} P_v\Delta\lambda_v = O(1) \quad as \quad m \to \infty, \tag{10}$$

$$\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} p_v\lambda_v = O(1) \quad as \quad m \to \infty, \tag{11}$$

$$\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} p_{v+1}\lambda_{v+1} = O(1) \quad as \quad m \to \infty, \tag{12}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_nP_{n-1}} = O\left\{\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\frac{1}{P_v}\right\}, \tag{13}$$

then the summability $\mid \bar{N}, p_n, \theta_n \mid_k$ of the series $\sum A_n(t)\lambda_n P_n$ at a point is a local property of the generating function $f(t)$.

It should be noted that if we take $\theta_n = \frac{P_n}{p_n}$, then we get Theorem B. In this case conditions (10)-(12) are obvious and condition (13) reduces to

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O\left\{\frac{1}{P_v}\right\}, \tag{14}$$

which always holds .Also,if we take $k = 1$ and $p_n = 1/(n+1)$,then we obtain Theorem A.

We need the following lemmas for the proof of our theorem.

Lemma 1([5]). If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, then $P_n\lambda_n = O(1)$ as $n \to \infty$ and $\sum P_n\Delta\lambda_n < \infty$.

Lemma 2 . Let (s_n)= $a_1 + a_2 + ... + a_n = O(1)$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent and the conditions (10)-(13) are satisfied , then the series $\sum a_n\lambda_n P_n$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$.

Proof. *Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n P_n$. Then, by definition, we have*

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r P_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v P_v.$$

Then, for $n \geq 1$, we have that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}P_v a_v\lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v P_v s_v \Delta\lambda_v - \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v s_v p_v \lambda_v \\ &- \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v p_{v+1} s_v \lambda_{v+1} + s_n p_n \lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say. \end{aligned}$$

By Minkowski's inequality for $k > 1$, to complete the proof of the Lemma 2 , it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{15}$$

Now, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$ and $k>1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_{n,1}\mid^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\mid s_v\mid^k P_vP_v\Delta\lambda_v\right\}\\
&\times \{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\\
&= O(1)\sum_{v=1}^{m}P_vP_v\Delta\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{\theta_np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= =O(1)\sum_{v=1}^{m}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}P_v\Delta\lambda_v=O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the Lemma 2 and Lemma 1. Again

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_{n,2}\mid^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\mid s_v\mid^k (P_v\lambda_v)^kp_v\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v)^kp_v\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^kp_v\sum_{n=v+1}^{m+1}\left(\frac{\theta_np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^k\frac{p_v}{P_v}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^{k-1}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}p_v\lambda_v\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}p_v\lambda_v=O(1)\quad as\quad m\to\infty,
\end{aligned}$$

in view of the hypotheses of the Lemma 2 and Lemma 1 . Using the fact that $P_v<P_{v+1}$, similarly we have that

$$\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_{n,3}\mid^k=O(1)\sum_{v=1}^{m}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}p_{v+1}\lambda_{v+1}=O(1)\quad as\quad m\to\infty,$$

Finally, we have that

$$\begin{aligned}\sum_{n=1}^{m}\theta_n^{k-1}\mid T_{n,4}\mid^k &= \sum_{n=1}^{m}\theta_n^{k-1}\mid s_n\mid^k p_n^k\lambda_n^k\\ &= O(1)\sum_{n=1}^{m}\theta_n^{k-1}p_n^{k-1}p_n\lambda_n^{k-1}\lambda_n\frac{P_n^{k-1}}{P_n^{k-1}}\\ &= O(1)\sum_{n=1}^{m}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}(P_n\lambda_n)^{k-1}p_n\lambda_n\\ &= O(1)\sum_{n=1}^{m}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}p_n\lambda_n = O(1)\quad as\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of the Lemma 2 and Lemma 1 . Therefore, we get that

$$\sum_{n=1}^{m}\theta_n^{k-1}\mid T_{n,r}\mid^k = O(1)\quad as\quad m\to\infty, for\quad r=1,2,3,4.$$

This completes the proof of the Lemma 2.

4.Proof of the Theorem

Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only , hence the truth of the Theorem is a necessarry consequence of Lemma 2. If we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we have a new local property result dealing with $\mid C,1\mid_k$ summability. Also, if we take $\theta_n = n$, then we obtain another new local property result for $\mid R,p_n\mid_k$ summability.

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